Chapter 4

Trigonometry

Curriculum Expectations

Trigonometric Functions

Understanding and Applying Radian Measure

B1.1 recognize the radian as an alternative unit to the degree for angle measurement, define the radian measure of an angle as the length of the arc that subtends this angle at the centre of a unit circle, and develop and apply the relationship between radian and degree measure

B1.2 represent radian measure in terms of π (e.g., $\frac{\pi}{3}$ radians, 2π radians) and as a rational number (e.g., 1.05 radians, 6.28 radians)

B1.3 determine, with technology, the primary trigonometric ratios (i.e., sine, cosine, tangent) and the reciprocal trigonometric ratios (i.e., cosecant, secant, cotangent) of angles expressed in radian measure

B1.4 determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, and their multiples less than or equal to 2π

Solving Trigonometric Equations

B3.1 recognize equivalent trigonometric expressions [e.g., by using the angles in a right triangle to recognize that $\sin x$ and $\cos\left(\frac{\pi}{2} - x\right)$ are equivalent; by using transformations to recognize that $\cos\left(\frac{\pi}{2} + x\right)$ and $-\sin x$ are equivalent], and verify equivalence using graphing technology

B3.2 explore the algebraic development of the compound angle formulas (e.g., verify the formulas in numerical examples, using technology; follow a demonstration of the algebraic development [student reproduction of the development of the general case is not required]), and use the formulas to determine exact values of trigonometric ratios [e.g., determining the exact value of $\sin\left(\frac{\pi}{12}\right)$ by first rewriting it in terms of special angles as $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$]

B3.3 recognize that trigonometric identities are equations that are true for every value in the domain (i.e., a counter-example can be used to show that an equation is not an identity), prove trigonometric identities through the application of reasoning skills, using a variety of relationships (e.g., $\tan x = \frac{\sin x}{\cos x}$; $\sin^2 x + \cos^2 x = 1$; the reciprocal identities; the compound angle formulas), and verify identities using technology.

Technology Notes

- The expectations for this course mandate heavy use of technology. Strong support for technology use has been included in the lesson design.
- To avoid the necessity for multiple learning curves, the primary technologies in use are
  - graphing calculators, specifically the TI-83 Plus/TI-84 Plus series
  - computer algebra system (CAS), specifically for the TI-89/TI-89T series
  - *The Geometer's Sketchpad®*
- *The Geometer's Sketchpad®* is licensed in Ontario for use by students at home. Consider providing each student with a copy to install on a home computer. This greatly expands the kinds of homework you can assign, and gets around some of the access problems you may run into using school computers. Ensure that students without home computers have an alternative. One possibility is to pair up with a classmate who is willing to share a computer. Another is availability of the software on public use computers in the school, such as library computers.
- Activities involving technology include specific instructions. In addition, you can refer to the Technology Appendix for help or review.
- A scientific calculator provides strong support for students, and is intended to be a standard tool available to the student at all times.

Chapter 4 Planning Chart

Section Suggested Timing	Student Text Page(s)	Teacher's Resource Blackline Masters	Assessment	Tools
Chapter 4 Opener • 10 min	199			
Prerequisite Skills • 40–50 min	200–201	• G–1 Grid Paper • BLM 4–1 Prerequisite Skills		• grid paper • scientific calculator
4.1 Radian Measure • 60–75 min	202–210	• G–4 Protractor • T–2 *The Geometer's Sketchpad*® 4 • T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 4–2 Section 4.1 Practice	• BLM 4–3 Section 4.1 Achievement Check Rubric	• a large wheel, such as a bicycle wheel • a length of heavy string • masking tape • protractor • scissors • computer • *The Geometer's Sketchpad*® • scientific calculator • graphing calculator • computer algebra system
4.2 Trigonometric Ratios and Special Angles • 60–75 min	211–219	• G–1 Grid Paper • G–4 Protractor • T–2*The Geometer's Sketchpad*® 4 • T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 4–4 Summary of Special Angles and Trigonometric Ratios • BLM 4–5 Section 4.2 Practice		• scientific calculator • graphing calculator • computer algebra system • compasses • grid paper • protractor • computer • *The Geometer's Sketchpad*®
4.3 Equivalent Trigonometric Expressions • 60–75 min	220–227	• G–1 Grid Paper • G–2 Placemat • G–4 Protractor • T–2 *The Geometer's Sketchpad*® 4 • BLM 4–6 Summary of Trigonometric Identities • BLM 4–7 Section 4.3 Practice		• grid paper • graphing calculator • compasses and protractor OR • computer with *The Geometer's Sketchpad*®
4.4 Compound Angle Formulas • 60–75 min	228–235	• BLM 4–8 Section 4.4 Practice		• scientific calculator • graphing calculator
4.5 Prove Trigonometric Identities • 60–75 min	236–241	• BLM 4–9 Section 4.5 Practice		• graphing calculator
Extension: Use *The Geometer's Sketchpad*® to Sketch and Manipulate Three-Dimensional Structures in a Two-Dimensional Representation • 30–40 min	242–243	• T–2 *The Geometer's Sketchpad*® 4		• computer • *The Geometer's Sketchpad*®
Chapter 4 Review • 60–75 min	244–245	• BLM 4–10 Chapter 4 Review		• scientific calculator • graphing calculator
Chapter 4 Problem Wrap-Up • 20–30 min	245	• G–1 Grid Paper	• BLM 4–11 Chapter 4 Problem Wrap-Up Rubric	• grid paper • scientific calculator
Chapter 4 Practice Test • 40–50 min	246–247		• BLM 4–12 Chapter 4 Test	• scientific calculator • graphing calculator
Chapter 4 Task: Make Your Own Identity • 60–75 min	248	• BLM 4–14 BLM Answers	• BLM 4–13 Task: Make Your Own Identity Rubric	• graphing calculator

Chapter 4 Blackline Masters Checklist

	BLM	Title	Purpose
Prerequisite Skills			
	G–1	Grid Paper	Student Support
	BLM 4–1	Prerequisite Skills	Practice
4.1 Radian Measure			
	G–4	Protractor	Student Support
	T–2	*The Geometer's Sketchpad®* 4	Technology
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 4–2	Section 4.1 Practice	Practice
	BLM 4–3	Section 4.1 Achievement Check Rubric	Assessment
4.2 Trigonometric Ratios and Special Angles			
	G–1	Grid Paper	Student Support
	G–4	Protractor	Student Support
	T–2	*The Geometer's Sketchpad®* 4	Technology
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 4–4	Summary of Special Angles and Trigonometric Ratios	Student Support
	BLM 4–5	Section 4.2 Practice	Practice
4.3 Equivalent Trigonometric Expressions			
	G–1	Grid Paper	Student Support
	G–2	Placemat	Student Support
	G–4	Protractor	Student Support
	T–2	*The Geometer's Sketchpad®* 4	Technology
	BLM 4–6	Summary of Trigonometric Identities	Student Support
	BLM 4–7	Section 4.3 Practice	Practice
4.4 Compound Angle Formulas			
	BLM 4–8	Section 4.4 Practice	Practice
4.5 Prove Trigonometric Identities			
	BLM 4–9	Section 4.5 Practice	Practice
Extension: Use *The Geometer's Sketchpad®* to Sketch and Manipulate Three-Dimensional Structures in a Two-Dimensional Representation			
	T–2	*The Geometer's Sketchpad®* 4	Technology
Chapter 4 Review			
	BLM 4–10	Chapter 4 Review	Practice
Chapter 4 Problem Wrap-Up			
	G–1	Grid Paper	Student Support
	BLM 4–11	Chapter 4 Problem Wrap-Up Rubric	Assessment
Chapter 4 Practice Test			
	BLM 4–12	Chapter 4 Test	Summative Assessment
Chapter 4 Task: Make Your Own Identity			
	BLM 4–13	Task: Make Your Own Identity Rubric	Assessment
	BLM 4–14	BLM Answers	Answers

Prerequisite Skills

Student Text Pages
200 to 201

Suggested Timing
40–50 min

Tools
- grid paper
- scientific calculator

Related Resources
- G–1 Grid Paper
- BLM 4–1 Prerequisite Skills

Assessment
You may wish to use **BLM 4–1 Prerequisite Skills** as a diagnostic assessment. Refer students to the Skills Appendix for examples and further practice of topics.

Chapter Problem

- The Chapter Problem is introduced on page 201. Have students discuss their understanding of the topic. To stimulate interest, you can find demonstration video clips for Computer Generated Imagery (CGI) on the Internet. You can bookmark clips of interest to make them easy to retrieve in class. The Chapter Problem is revisited in Sections 4.1 (question 18), 4.2 (question 19), 4.3 (question 22), and 4.4 (question 17). These questions are designed to help students move toward the Chapter 4 Problem Wrap-Up on page 245. Alternatively, you may wish to assign the Chapter Problem questions and Chapter Problem Wrap-Up when students have completed the chapter, as part of a summative assessment.

4.1 Radian Measure

Student Text Pages
202 to 210

Suggested Timing
60–75 min

Tools
- a large wheel, such as a bicycle wheel
- a length of heavy string
- masking tape
- protractor
- scissors
- computer
- *The Geometer's Sketchpad®*
- scientific calculator
- graphing calculator
- computer algebra system

Related Resources
- G–4 Protractor
- T–2 *The Geometer's Sketchpad®* 4
- T–4 The Computer Algebra System (CAS) on the TI-89 Calculator
- BLM 4–2 Section 4.1 Practice
- BLM 4–3 Section 4.1 Achievement Check Rubric

Teaching Suggestions

- If sufficient wheels are not available for the **Investigate**, consider working through Method 1 as a class group, with students taking turns performing steps or making measurements.
- Technology tips for Method 2 of the **Investigate**:
  - To turn on the grid display, from the **Graph** menu choose **Show Grid**.
  - Adjust the scale on the axes by dragging the **Unit Point** on the x-axis.
  - When using *The Geometer's Sketchpad®* get rid of unwanted points, lines, and labels by using the **Hide** features from the **Display** menu.
  - Change the label of a point by double clicking on that letter.
  - To calculate the ratio of the length of the arc to the length of the radius, from the **Measure** menu choose **Calculate**. Select the measurement of the length of the arc by clicking on it (this will appear on the calculator), then the division sign followed by the measurement of the length of the radius.
- While working through the **Investigate**, some students may need a refresher on how to perform elementary operations with *The Geometer's Sketchpad®*. Refer students to the Technology Appendix on page 506, the detailed instructions found under the **Help** menu in *The Geometer's Sketchpad®*, or use **T–2 *The Geometer's Sketchpad®* 4**.
- When taking up the **Investigate**, consider demonstrating the use of the **Transform** menu to perform step 11 of Method 2. Set the **Preferences** to measure angles in radians. Then, start with point C on the positive x-axis and repeatedly rotate it about the origin in steps of one radian. Point out that six radians are not quite enough for one complete revolution about the origin, and relate this fact to 2π.
- Before working through **Example 1**, have students estimate the answer using the fact that one radian is a little less than 60º. Hence, an angle of 30º should be about half a radian.
- Method 2 of **Example 1** makes use of a computer algebra system (CAS). The instructions are detailed enough so that students who have not used a CAS before can perform the calculations. You can also direct students to the Technology Appendix on page 506. If needed, use **T–4 The Computer Algebra System (CAS) on the TI-89 Calculator** to support **Example 1**. If a CAS is not readily available at all times, consider waiting until you have encountered several CAS activities, and then have students perform them all at once.
- Technology tip for Method 2 of **Example 1**:
  - On the TI-89 calculator, change the **MODE** and ensure that the **Angle** measure is set to **RADIAN**. Scroll down to change the **Exact/Approx** so that it is set to **AUTO**. To save these settings, select **ENTER**. Students at times select **ESC**, which does not save the current settings but cancels the save.
- Before working through **Example 2**, have students estimate the answer using the fact that one radian is a little less than 60º. $\frac{\pi}{4}$ is about $\frac{3}{4}$ radians, or about 45º.
- Before introducing **Example 3**, use the bicycle wheel as a visual aid in demonstrating why the formula for arc length works.
- Before introducing **Example 4**, use the bicycle wheel to demonstrate angular velocity. Attach a piece of masking tape to the rim, and spin the wheel. Have students time five revolutions, and then make an estimate of the angular velocity in both degrees and radians.

Ongoing Assessment

Use **Assessment Masters A–1** to **A–7** to remind students about the Math Processes expectations and how you may be assessing their integrated use of them.

- As students consider the **Communicate Your Understanding** questions, have them make connections to physical objects, such as the bicycle wheel or a carousel.
- *The Geometer's Sketchpad*® (GSP) is a useful tool for students to have at home. Consider taking advantage of the home use licence provided by the Ministry of Education to allow students to do questions such as **question 12** at home.
- Before assigning **question 13**, consider spending a minute or two with a globe reviewing the terms *latitude* and *longitude* and visually demonstrating the meaning of a nautical mile.
- **Question 13** allows the students to reflect and clarify their understanding of this particular question while they apply reasoning skills to make conjectures and to justify conclusions concerning the radius of Earth versus the nautical mile. This question also gives the students the opportunity to select the tools necessary and then connect methods from their past learning to determine the radian measure and the length of the nautical mile needed. Finally, communicating the reasoning and the reflecting that was done is an important component of this question as well.
- You can add interest to **question 17** by playing a video clip of a turbine engine. Go to *www.mcgrawhill.ca/books/functions12* and follow the links to find out how the engine works.
- You can demonstrate the operation of a geostationary satellite, such as the one in **question 19**, using GSP. Draw a circle with the centre at the origin. Draw a point on the circle. Mark the origin as a centre. Dilate the point on the circle using a factor of 2:1. Animate the point on the circle. You can turn tracing on to display the trail left by the satellite. Go to *www.mcgrawhill.ca/books/functions12* and follow the links to this GSP file.
- **Question 19** allows students to reason and reflect concerning angular velocities of two different elements. The students are then asked to communicate these thoughts and make connections among mathematical concepts learned in their past concerning one complete revolution around Earth and the calculation of angular velocity.
- Before assigning **question 24**, consider a visual demonstration using a globe.
- For **question 26**, you can demonstrate a polar grid using GSP. From the Graph menu, select **Grid Form**, and then **Polar Grid**. Print a blank grid for student use.
- Use **BLM 4–2 Section 4.1 Practice** for remediation or extra practice.

Investigate Answers (pages 202–204)

METHOD 1

4. The sector angle appears to be about 60°. Using a protractor, a student should be able to determine a more accurate measurement of 57° plus or minus 1°.
5. Using the result from step 4, one radian is about 57°.
6. To go around the circle, slightly more than six arc lengths are required. A good estimate is 6.3, plus or minus 0.1.
7. Using a string, a student should be able to arrive at an answer very close to 6.3 arc lengths.
8. The circumference of a circle is given by $C = 2\pi r$. The circumference has a length of about 6.3 radii. This compares favourably with the measurement in step 7.

METHOD 2

9. To the nearest hundredth, one radian is approximately equal to 57.30°.
10. To go around the circle, slightly more than six arc lengths are required. A good estimate is 6.3, plus or minus 0.1.
11. To the nearest hundredth, you need 6.28 radii to go around the circle.
12. The circumference of a circle is given by $C = 2\pi r$. The circumference has a length of about 6.28 radii. This compares favourably with the measurement in step 11.

Differentiated Instruction

Use a **journal entry**. Give the topic as "What is a radian?"

- Students often find radian measure confusing, especially in the early stages.

R_x Coach students to think in terms of radians, without converting to degrees first. For example, ask students to demonstrate an angle such as $\frac{\pi}{3}$ using a thumb and index finger. When converting from degree measure to radian measure, or vice versa, train students to estimate the expected answer first.

Ongoing Assessment

Achievement Check, question 21, on student text page 210.

Communicate Your Understanding Responses (page 208)

C1. Since a complete revolution measures 2π radians, π radians are half of a revolution, or 180°.

C2. A right angle is $\frac{1}{4}$ of a complete revolution. So, the radian measure of a right angle is $\frac{2\pi}{4}$, or $\frac{\pi}{2}$.

C3. Radian measure does not depend on the division of a complete revolution into an arbitrary number of units, such as 360. It makes use of a natural quantity, the arc length, compared to the radius of the circle.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	
Reasoning and Proving	13, 19, 22–25
Reflecting	13
Selecting Tools and Computational Strategies	1–9, 13, 20
Connecting	10–23, 25
Representing	12, 14
Communicating	13, 19, 24

Achievement Check, question 21, student text page 210

This performance task is designed to assess the specific expectations covered in Section 4.1. The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting
- Problem Solving

Achievement Chart Category	Related Math Processes
Knowledge and Understanding	Selecting tools and computational strategies
Thinking	Problem solving Reasoning and proving Reflecting
Communication	Communicating, Representing
Application	Selecting tools and computational strategies Connecting

Sample Solution

Provide students with **BLM 4–3 Section 4.1 Achievement Check Rubric** to help them understand what is expected.

a) $\frac{5}{30}(2\pi) = \frac{\pi}{3}$

A passenger will travel $\frac{\pi}{3}$ rad in 5 min.

b) $a = r\theta$

$= 67.5\left(\frac{\pi}{3}\right)$

$\doteq 70.7$

A passenger travels about 70.7 m in 5 min.

c) It will take $30\left(\frac{2}{2\pi}\right)$, or about 9.5 min to travel 2 rad.

d) The angular velocity of a passenger is $\frac{2\pi}{30 \times 60}$, or $\frac{\pi}{900}$ rad/s.

e) The angular velocity of a passenger is 0.2°/s.

4.2 Trigonometric Ratios and Special Angles

Student Text Pages
211 to 219

Suggested Timing
60–75 min

Tools
- scientific calculator
- graphing calculator
- computer algebra system
- compasses
- grid paper
- protractor
- computer
- *The Geometer's Sketchpad®*

Related Resources
- G–1 Grid Paper
- G–4 Protractor
- T–2 *The Geometer's Sketchpad®* 4
- T–4 The Computer Algebra System (CAS) on the TI-89 Calculator
- BLM 4–4 Summary of Special Angles and Trigonometric Ratios
- BLM 4–5 Section 4.2 Practice

Teaching Suggestions

- While working through **Investigate 1**, Method 1, be prepared to deal with the differences in scientific calculators. There is no standard way of entering keystrokes for operations with trigonometric ratios. Students will need to become familiar with their own scientific calculator when it comes to changing the mode from degrees to radians and vice versa.
- **Investigate 1**, Method 3, makes use of a computer algebra system (CAS). Be sure that students know the difference between exact and approximate displays of the answer, since they may not have seen this before. You can refer to the Technology Appendix on page 506. If needed, use **T–4 The Computer Algebra System (CAS) on the TI-89 Calculator** to support this method.
- For **Investigate 2**, let students draw their own unit circle. Students will have had experience working with this tool in grade 11 when determining trigonometric ratios of special angles using degree measure. The objective is to become familiar enough with the wealth of information available from this tool that students will be able to quickly sketch and use it, even in a time-constrained setting such as a test or examination.
- The table of special angles and their trigonometric ratios will be used extensively in the rest of this chapter and the next one on trigonometric functions. Once it is completed, encourage students to have it available on their desktops at all times. Colour coding by quadrant may be of use to some students. Use **BLM 4–4 Summary of Special Angles and Trigonometric Ratios** to support **Investigate 2**.
- You can use *The Geometer's Sketchpad®* to determine trigonometric ratios for special angles as an alternative method for **Investigate 2**.
- Before working through **Example 1**, you can perform a simulation using a short length of string, a kite held up by a student, and a protractor. You can paste a printout of equivalent radian measures over the degree scale on a standard large wooden or plastic protractor. In preparation for **question 10** in **Connect and Apply**, ask students to predict what would happen if the string were lengthened or shortened between the two measurements, and then demonstrate.
- As students consider **Communicate Your Understanding** question C1, ensure that they understand how to use the unit circle to explain why the tangent ratio sometimes returns undefined results.
- You can add some interest to question C2 by reminding students of the "toothpicks and striped bed sheets" method of determining the value of π, also known as Buffon's Needle. Go to *www.mcgrawhill.ca/books/functions12* and follow the links to find out more.
- **Question 10** gives the students the opportunity to employ inductive reasoning by investigating and making conjectures about the horizontal movement of a kite and the change in its vertical distance. Students will also select the tools that are needed to develop the expressions required and will make connections to do so, showing that they can use learning from other areas of mathematics to understand the particular concepts required in this question.
- You can create a visual aid for **question 14**. Use *The Geometer's Sketchpad®* to print a faceplate in radian measure, and paste it over the existing face of an inexpensive alarm clock. This can be used to verify student answers.

- Consider employing the home use licence for *The Geometer's Sketchpad®* so that students can do questions such as **question 15** as homework.
- You can prepare students for **question 22** by asking them to compare the value of $\frac{\pi}{36}$ to $\sin\frac{\pi}{36}$.
- Use **BLM 4–5 Section 4.2 Practice** for remediation or extra practice.

Investigate Answers (pages 211–213)

Investigate 1

METHOD 2

1. b) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) 0.7071; this matches the answer in step 1b).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. $\sin 1.5 \doteq 0.9975$, $\cos 1.5 \doteq 0.0707$, $\tan 1.5 \doteq 14.1014$, $\csc 1.5 \doteq 1.0025$, $\sec 1.5 \doteq 14.1368$, $\cot 1.5 \doteq 0.0709$

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

METHOD 2

1. c) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) $\sin\frac{\pi}{4} \doteq 0.7071$. This matches the answer in step 1c).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. sin 1.5 R 0.9975, cos 1.5 R 0.0707, tan 1.5 R 14.1014, csc 1.5 R 1.0025, sec 1.5 R 14.1368, cot 1.5 R 0.0709

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

METHOD 2

1. c) $\sin 45^\circ = \frac{\sqrt{2}}{2}$

d) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) $\sin\frac{\pi}{4} \doteq 0.7071$. This matches the answer in step 1d).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

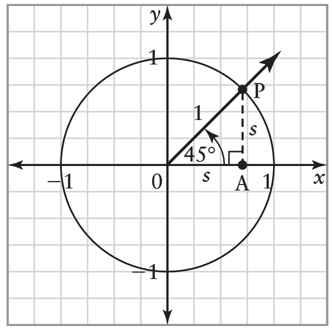
5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. $\sin 1.5 \doteq 0.9975$, $\cos 1.5 \doteq 0.0707$, $\tan 1.5 \doteq 14.1014$, $\csc 1.5 \doteq 1.0025$, $\sec 1.5 \doteq 14.1368$, $\cot 1.5 \doteq 0.0709$

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

Investigate 2

1. a)–d)



The triangle is isosceles. Let the side length be s.

$s^2 + s^2 = 1$

$2s^2 = 1$

$s^2 = \frac{1}{2}$

$s = \frac{1}{\sqrt{2}}$

e) $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = 1$

f) $45° = \frac{\pi}{4}$

1. g), 2.–5.

Special Angles and Trigonometric Ratios				
θ (°)	θ (radians)	sin θ	cos θ	tan θ
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	Undefined
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
180	π	0	-1	0
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	-1	0	Undefined

Differentiated Instruction

Use **jigsaw** to teach this section.

Common Errors

- The error explored in question C3, $\sin\frac{\pi}{2} = \frac{1}{2}\sin\pi$, is commonly made by students.

R_x Revisit this error from time to time, as appropriate, during lessons or take-up sessions.

Ongoing Assessment

Achievement Check, question 20, on student text page 219.

Special Angles and Trigonometric Ratios				
θ (°)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
360	2π	0	1	0

6. Results should match the table.

Communicate Your Understanding Responses (page 216)

C1. Evaluating $\tan\frac{\pi}{2}$ on a calculator results is an error. $\tan\frac{\pi}{2}$ is undefined. The denominator of the ratio is zero. $\tan\frac{3\pi}{2}$ is also undefined.

C2. $\frac{355}{113} \doteq 3.141\ 592\ 92$

$\sin(3.141\ 592\ 92) \doteq 0.000\ 000\ 267$

The answer should be zero. The approximation is not close enough to yield the correct value for the sine ratio within the limit of the accuracy of the calculator.

C3. $\sin\frac{\pi}{2} = 1$ $\quad\quad$ $\frac{1}{2}\sin\pi = \frac{1}{2} \times 0$

$= 0$

The two expressions are not equal. It is not possible to factor $\frac{1}{2}$ out of the argument of a sine ratio.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	15, 19, 23
Reasoning and Proving	10, 19–22
Reflecting	19
Selecting Tools and Computational Strategies	9–12, 15–22
Connecting	7–18, 20–22
Representing	15, 19
Communicating	10, 15, 20–22

Achievement Check, question 20, student text page 219

This performance task is designed to assess the specific expectations covered in Section 4.2.

The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting
- Reasoning and Proving
- Problem Solving

Sample Solution

a) $\text{L.S.} = 2\sin x\cos y$

$= 2\sin\frac{5\pi}{4}\cos\frac{3\pi}{4}$

$= 2\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)$

$= 1$

b) $\text{R.S.} = \sin(x + y) + \sin(x - y)$

$= \sin\left(\frac{5\pi}{4} + \frac{3\pi}{4}\right) + \sin\left(\frac{5\pi}{4} - \frac{3\pi}{4}\right)$

$= \sin 2\pi + \sin\frac{\pi}{2}$

$= 0 + 1$

$= 1$

c) Emilio is correct.

d) $\text{L.S.} = 2\sin x\cos y$

$= 2\sin\frac{5\pi}{6}\cos\frac{\pi}{6}$

$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2}$

$\text{R.S.} = \sin(x + y) + \sin(x - y)$

$= \sin\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$

$= \sin\pi + \sin\frac{2\pi}{3}$

$= 0 + \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2}$

Since **L.S.** = **R.S.**, $x = \frac{5\pi}{6}$ and $y = \frac{\pi}{6}$ is also a solution.

e) The answers to parts c) and d) cannot be used to conclude that the equation will work for all values of x and y. A formal proof is required.

Level 3 Notes

Look for the following:

- Trigonometric expressions *n* left side of equation are determined using exact values
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are determined using exact values
- Answer to part d) is supported with justification using method of part a) and part b)
- Answers in parts a), b), and c) are mostly accurate
- Valid reasoning/justification is provided to support answer in part e)
- Evidence of proper form occurs in most of the solution steps

What Distinguishes Level 2

- Trigonometric expressions in left side of equation are in decimal form
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are in decimal form
- Answer to part d) is somewhat justified
- Answers in parts a), b), and c) are somewhat accurate
- Valid reasoning/justification is provided to support answer in part e)
- Evidence of proper form occurs in some of the solution steps

What Distinguishes Level 4

- Trigonometric expressions in left side of equation are determined using exact values
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are determined using exact values
- Answer to part d) is supported with justification using method of part a) and part b)
- Answers in parts a), b), and c) are consistently accurate with minor errors
- Valid reasoning/justification is provided to support answer in part e)
- Solution steps show a high degree of proper form

4.3 Equivalent Trigonometric Expressions

Student Text Pages
220 to 227

Suggested Timing
60–75 min

Tools
- grid paper
- graphing calculator
- compasses and protractor

OR
- computer with *The Geometer's Sketchpad®*

Related Resources
- G–1 Grid Paper
- G–2 Placemat
- G–4 Protractor
- T–2 *The Geometer's Sketchpad®* 4
- BLM 4–6 Summary of Trigonometric Identities
- BLM 4–7 Section 4.3 Practice

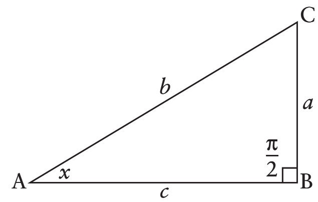
Teaching Suggestions

- Students will begin to assemble a table of trigonometric identities in **Investigate 1**. **Investigate 2** extends this table. Encourage students to continue to add to this table as the lesson progresses. Consider allowing students to use their own tables for tests and exams.
- If *The Geometer's Sketchpad®* (GSP) is used with **Investigate 2**, refer students needing assistance to the Technology Appendix on page 506, the detailed instructions found under the **Help** menu in GSP, or use **T–2 *The Geometer's Sketchpad®* 4**.
- Technology tips for **Investigate 2**:
  – When using GSP to determine equivalent trigonometric expressions, from the **Edit** menu choose **Preferences** and change the **Angle Units** from degrees to radians and vice versa.
  – To create a vertical line from point P to point Q on the x-axis, select point P and the x-axis, then from the **Construct** menu choose **Perpendicular Line**. Then select the perpendicular line and the x-axis and under **Construct** choose **Point of Intersection**. Finally, select points P and Q, then from the **Construct** menu, choose a **Line Segment**. Do the same for point Q and the origin.
  – To find the coordinates of points, select the point and from the **Measure** menu choose **Coordinates**.
- After working through **Example 1** and **Example 2**, take some time to reflect on how to choose a trigonometric identity involving $\frac{\pi}{2} - a$ or $\frac{\pi}{2} + a$ in a particular situation.
- In **Example 3**, students are often tempted to conclude that this method of verification constitutes a proof. Emphasize why it does not.
- Another approach to **Communicate Your Understanding** question C1 is to consider the CAST rule.
- **Questions 15** through **19** develop the remaining equivalent trigonometric expressions for students' summary sheets. Be sure to assign all of these questions, and ensure that the summary sheets contain the correct form for each pair of expressions. A summary table of the trigonometric identities from **Investigate 1**, **Investigate 2**, and **questions 15** through **19** is available as **BLM 4–6 Summary of Trigonometric Identities**.
- **Question 21** is a good example of how more than one identity can be used to obtain an answer.
- **Question 21** allows students to reflect and search for related content knowledge to help Charmaine find the sine of the required angle without using the SIN key on her calculator. They will also have to use their reasoning skills to solve this problem. The students will be required to select the necessary computational strategies and then make connections with past mathematical skills that have been learned.
- Although **question 22** refers to an aircraft, the same equation applies to leaning a bicycle or a motorcycle when making a turn on a road that is not banked.
- **Question 24** asks students to use their critical-thinking skills to evaluate the results of a question that is new to them. To solve this problem, they will have to use their reflecting and reasoning skills. They will also be required to select the appropriate tools necessary for solving this problem and then make connections with concepts that they have learned in their past.
- Use **BLM 4–7 Section 4.3 Practice** for remediation or extra practice.

Investigate Answers (pages 220–222)

Investigate 1

1.



2. $\angle C = \frac{\pi}{2} - x$

3. $\sin x = \frac{a}{b}$

4. $\sin C = \frac{c}{b}$, $\cos C = \frac{a}{b}$, $\tan C = \frac{c}{a}$; $\sin x = \cos\left(\frac{\pi}{2} - x\right)$

5. a) No, the relationship is true for all values of x. The graphs of $\sin x$ and $\cos\left(\frac{\pi}{2} - x\right)$ look the same.

b) $\frac{\pi}{2}$; complementary angles

6. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

7. $\csc x = \sec\left(\frac{\pi}{2} - x\right)$, $\sec x = \csc\left(\frac{\pi}{2} - x\right)$, $\cot x = \tan\left(\frac{\pi}{2} - x\right)$

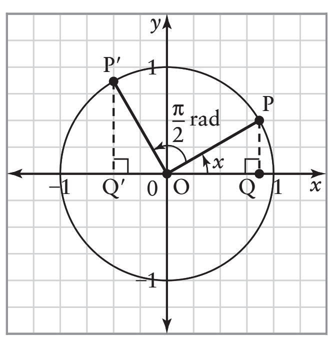
8.

Trigonometric Identities
$\sin x = \cos\left(\frac{\pi}{2} - x\right)$
$\cos x = \sin\left(\frac{\pi}{2} - x\right)$
$\tan x = \cot\left(\frac{\pi}{2} - x\right)$
$\csc x = \sec\left(\frac{\pi}{2} - x\right)$
$\sec x = \csc\left(\frac{\pi}{2} - x\right)$
$\cot x = \tan\left(\frac{\pi}{2} - x\right)$

9. They are cofunctions because their angle relationships are complementary.

Investigate 2

1.–4.



$P\left(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right)$ and $P'\left(\cos\left(x + \frac{\pi}{2}\right), \sin\left(x + \frac{\pi}{2}\right)\right)$

5. PO = OP′
$\angle POQ = \angle OP'Q'$
$\angle Q = \angle Q'$
Therefore, $\triangle PQO \cong \triangle OQ'P'$ (AAS).

Differentiated Instruction

Use a **matching game** to reinforce this section.

Use **placemat** to teach this section.

Common Errors

- Students often confuse an identity with an equation.

R_x Establish the meaning of an identity early in the lesson, preferably in the **Investigate**. Revisit the concept as appropriate throughout this chapter.

- Question C2 illustrates a common error, $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2} - \cos x$, made by students.

R_x Emphasize that this is not an identity. Revisit the concept from time to time throughout this chapter.

Ongoing Assessment

Achievement Check, question 23, on student text page 226.

6. Due to a rotation of $\frac{\pi}{2}$, (x, y) becomes $(-y, x)$.

$P'\left(\cos\left(x + \frac{\pi}{2}\right), \sin\left(x + \frac{\pi}{2}\right)\right) = P(-\sin x, \cos x)$

7. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

8. $\tan\left(x + \frac{\pi}{2}\right) = -\cot x$

9. $\csc\left(\frac{\pi}{2} + x\right) = \sec x$, $\sec\left(\frac{\pi}{2} + x\right) = -\csc x$, $\cot\left(\frac{\pi}{2} + x\right) = -\tan x$

10.

Trigonometric Identities Featuring $\frac{\pi}{2}$			
Cofunction Identities			
$\sin x = \cos\left(\frac{\pi}{2} - x\right)$	$\cos x = \sin\left(\frac{\pi}{2} - x\right)$	$\sin\left(x + \frac{\pi}{2}\right) = \cos x$	$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$
$\tan x = \cot\left(\frac{\pi}{2} - x\right)$	$\cot x = \tan\left(\frac{\pi}{2} - x\right)$	$\tan\left(x + \frac{\pi}{2}\right) = -\cot x$	$\cot\left(x + \frac{\pi}{2}\right) = -\tan x$
$\csc x = \sec\left(\frac{\pi}{2} - x\right)$	$\sec x = \csc\left(\frac{\pi}{2} - x\right)$	$\csc\left(x + \frac{\pi}{2}\right) = \sec x$	$\sec\left(x + \frac{\pi}{2}\right) = -\csc x$

11. $\cos(\pi - x) = -\cos x$

Communicate Your Understanding Responses (page 225)

C1. No. The graphs are reflections of each other in the x-axis. $-\cos(\pi - x) = \cos x$

C2. No. Since $\cos\left(\frac{\pi}{2}\right) = 0$, then $\cos\left(\frac{\pi}{2} - x\right) \neq \cos x$.

C3. No. The graphs are reflections of each other in the x-axis. $\cos\left(\frac{\pi}{2} - x\right) = \cos\left(x - \frac{\pi}{2}\right)$

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	21, 24–26
Reasoning and Proving	21–27
Reflecting	21, 22, 24–26
Selecting Tools and Computational Strategies	1–4, 20–22
Connecting	5–27
Representing	17–19
Communicating	

Achievement Check, question 23, student text page 226

This performance task is designed to assess the specific expectations covered in Section 4.3. The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting

Sample Solution

a) $\sec x = \csc\left(\frac{\pi}{2} - x\right)$

b) $x + \frac{\pi}{2} - x = \frac{\pi}{2}$

c) $\frac{\pi}{2} = a + 3a$

$\frac{\pi}{2} = 4a$

$a = \frac{\pi}{8}$

d) $\sec\frac{\pi}{8} = \csc\frac{3\pi}{8} \doteq 1.0823$

Level 3 Notes

Look for the following:

- Correct answers, with explanation where necessary
- Left side versus right side reasoning is used to check the solution in part d)
- Consistent use of proper mathematical form

What Distinguishes Level 2

- Somewhat correct answers, with some explanation where necessary
- Left side versus right side reasoning is not used to check the solution in part d)
- Some use of proper mathematical form

What Distinguishes Level 4

- Correct answers, with high degree of justification provided
- Left side versus right side reasoning is used to check the solution in part d)
- High degree and consistent use of proper mathematical form

4.4 Compound Angle Formulas

Student Text Pages
228 to 235

Suggested Timing
60–75 min

Tools
- scientific calculator
- graphing calculator

Related Resources
- BLM 4–8 Section 4.4 Practice

Teaching Suggestions

- While working through the **Investigate**, ensure that students make the distinction between a verification for specific values and a proof.
- *The Geometer's Sketchpad*® can also be used to verify that the compound angle formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$ in the **Investigate** is valid for all angles, as follows:
  - Construct a unit circle on a grid. Construct a point A on the unit circle and drag that point in the first quadrant. Construct a line segment from the origin to point A.
  - Construct a point B on the unit circle and drag that point in the fourth quadrant. Construct a line segment from the origin to point B.
  - Construct a line segment from the origin to the unit point. (The unit point can be labelled C.)
  - Measure the angles of ∠BOC and ∠AOC. Let b represent ∠BOC and let a represent ∠AOC.
  - Select line segments OA and OB. Choose **Rotate** from the **Transform** menu to rotate counterclockwise about the origin through angle b (∠BOC). You will have two new line segments. One of them is the terminal arm of angle $(a + b)$ with point A′ being the point where the line segment intersects the unit circle. The other is on the x-axis with point B′ being the point where this line segment intersects the unit circle.
  - **Construct** line segments AB and A′B′.
  - Now investigate the coordinates:
    A$(\cos a, \sin a)$
    A′$(\cos(a + b), \sin(a + b))$
    B′$(1, 0)$
    B$(\cos(2\pi - b), \sin(2\pi - b))$
- Before beginning the development of the Addition Formula for Cosine, you may want to revisit why the coordinates of a point on the unit circle can be written in the form $(\cos\theta, \sin\theta)$.
- When working through **Example 1**, emphasize proper form for this kind of verification question.
- **Example 2** requires the proper choice of special angles whose sum or difference is equal to the desired angle. Much of the time, the proper choice is apparent from inspection. However, sometimes some trial and error is needed.
- It is well worth spending some time on **Communicate Your Understanding** question C1, and demonstrating the concept using a unit circle. You can use *The Geometer's Sketchpad*® for this purpose, or you can make a unit circle with moveable arms using Bristol board and push pins.
- Before assigning **question 18**, consider spending some time with a globe demonstrating the concepts in the question. Many students have a poor understanding of these phenomena.
- **Question 18** encourages students to reflect and then reason out various situations that arise from different angles of latitude. In addition, in order to justify their answers they will have to make connections with concepts that they have learned previously and then communicate in writing to clarify their ideas.
- **Question 20** allows students to reflect upon and then reason out a selection of computational strategies which will help them develop new trigonometric expressions that they have not seen before. In doing so, they will also make connections with various trigonometric expressions that they have used previously.

DIFFERENTIATED INSTRUCTION

Use **timed retell** to reinforce this section.

Use a **word wall/information wall** to summarize this section.

COMMON ERRORS

- The assumption that the relation in question C2, $\sin(x + y) = \sin x + \sin y$, is correct is a common error.

R_x Ensure that students understand why it is not correct, and revisit the concept from time to time in this chapter.

ONGOING ASSESSMENT

Achievement Check, question 19, on student text page 234.

- The tangent identities in **questions 20** and **21**, the sum formula in **question 23**, and the half-angle formulas in **question 24** are not required to satisfy the expectations for this course but are useful to know for students contemplating the study of mathematics at a higher level.
- Use **BLM 4–8 Section 4.4 Practice** for remediation or extra practice.

Investigate Answers (page 228)

1. $\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\sin\frac{\pi}{3}$

$\cos\frac{\pi}{2} = \cos\frac{\pi}{6}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\sin\frac{\pi}{3}$

2. Yes.

3. b) Yes.

4. Yes.

5. No, three valid examples is not a proof. You must be able to prove algebraically for all angles.

6. Answers may vary. Substituting $x = \frac{\pi}{6}$ and $y = \frac{\pi}{3}$ shows that the left side does not equal the right side.

Communicate Your Understanding Responses (page 232)

C1. Drawing angle $(a + b)$ in the second quadrant would have no effect on the development of the compound angle formula, since the coordinates of the points will not change.

C2. Answers may vary. For example, by substituting $x = \frac{\pi}{6}$ and $y = \frac{\pi}{3}$, the left side equals 1 but the right side equals $\frac{1 + \sqrt{3}}{2}$. Sine is not a multiplication operator.

C3. Yes. Answers may vary. For example, by substituting $x = \frac{\pi}{2}$ and $y = 0$, both the left side and the right side equal 1.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	22
Reasoning and Proving	8–14, 17–24
Reflecting	17–24
Selecting Tools and Computational Strategies	1–7, 15, 16, 20–22
Connecting	2–24
Representing	16, 19
Communicating	18

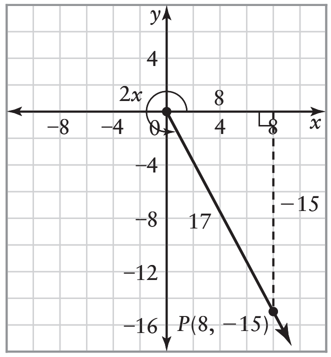
Achievement Check, question 19, student text page 234

This performance task is designed to assess the specific expectations covered in Section 4.4. The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting
- Reasoning and Proving

Sample Solution

a)



b) Angle x is in the second quadrant.

c) Use the formula $\cos 2x = 2\cos^2 x - 1$.

$$\frac{8}{17} = 2\cos^2 x - 1$$
$$\frac{25}{17} = 2\cos^2 x$$
$$\frac{25}{34} = \cos^2 x$$
$$\cos x = \pm\sqrt{\frac{25}{34}}$$
$$\cos x = \pm\frac{5}{\sqrt{34}}$$

Since angle x is in the second quadrant, $\cos x = -\frac{5}{\sqrt{34}}$.

d) The calculator returns a measure of approximately 0.54 rad. However, angle x is in the second quadrant, so $x = \pi - 0.54$, or 2.60 rad.

e) $\cos 2.60 \doteq -0.857$, which is approximately the same as the result in part c).

Level 3 Notes

Look for the following:

- Clearly labelled sketch provided in part a)
- Most answers are accurate
- Answer given in exact value or radians where required
- Consistent evidence of proper form

What Distinguishes Level 2

- Sketch in part a) is somewhat labelled
- Most answers are accurate
- Answer not given in exact value or radians where required
- Some evidence of proper form

What Distinguishes Level 4

- Clearly labelled and detailed sketch provided in part a)
- Answers may contain minor errors
- Answers given in exact value or radians where required
- High degree of proper form evident
- Answers supported with explanation

4.5 Prove Trigonometric Identities

Student Text Pages
236 to 241

Suggested Timing
60–75 min

Tools
- graphing calculator

Related Resources
- BLM 4–9 Section 4.5 Practice

Teaching Suggestions

- The **Investigate** revisits two important concepts. The first is the difference between an equation and an identity. The second is what constitutes a proof, and what does not constitute a proof.
- While working through **Example 1**, review proper form for a proof. Revisit the issue of form as appropriate.
- While working through **Examples 2**, **3**, and **4**, point out that it is not always necessary to begin with the left side, and work towards the right side. **Example 3** can be done either way. If time permits, show how **Example 3** can be done starting from the right side.
- Sometimes both sides of an identity are complex. A useful procedure is to simplify the left side, then the right side, and show that both are equal to the same expression.
- In **Example 4**, reasoning and then proving is involved: these two skills will help students make sense of this mathematical problem. To solve this problem the students will have to choose the appropriate tools and then make connections with trigonometric concepts that they have previously learned.
- While working through the **Communicate Your Understanding** questions, encourage students to use technology to confirm whether an equation might be an identity before attempting to prove it. This is a good approach for many of the exercise questions as well.
- If no particular approach suggests itself for attacking a proof, one possible approach is to use identities to render an expression solely in terms of sines and cosines, working on both the left side and the right side, such as in **question 9a**).
- **Question 17** encourages the students to use deductive reasoning to assess the validity of a given conjecture. It will then be necessary for them to solve the identity problem and, if needed, to use a counterexample to disprove the given conjecture (i.e., this is another way in which their reasoning skills will be put to use). In solving this trigonometric problem, computational strategies will have to be used by the students to represent this identity graphically and, in addition, connections will have to be made with previous work learned by the students.
- Although the double angle formula for tangents in **question 22** is not required to satisfy the expectations for this course, it is useful knowledge for students proceeding to the study of mathematics at a higher level.
- Hint for **question 21**: write $\sin^6 x$ as $(\sin^2 x)^3$. Then, use the Pythagorean identity, and expand using the binomial theorem.
- Use **BLM 4–9 Section 4.5 Practice** for remediation or extra practice.

Differentiated Instruction

Use **concept attainment** to introduce this section.

Use **what-so what double entry** to teach strategy.

Build a **decision tree** to help students prove identities.

Use a **journal entry** entitled "My Favourite Identity and Why."

Common Errors

- Some students will rearrange an identity algebraically, similar to solving an equation, in an attempt to simplify it before beginning the proof.

R_x Ensure that this error is discussed while working through the lesson or taking up the homework.

Ongoing Assessment

Achievement Check, question 19, on student text page 241.

Investigate Answers (page 236–237)

1. Yes.

2. No. One valid example does not prove that the equation holds for all values of x for which the functions are defined.

3. Answers may vary. For example, $x = \frac{\pi}{4}$.

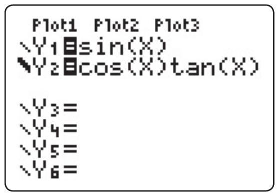
5. a) They are different.

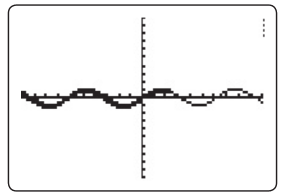
b) The equation is true at the points where the graphs cross.

6. a) Yes.

b) No counterexample.

c)





7. Yes. The graphs appear to be the same.

Communicate Your Understanding Responses (page 240)

C1. A general equation is true for some values of the variable. An identity is proven for both sides of the equation and is true for all values of the variable for which the functions are defined.

C2. a)

$$\sin^2 2\left(\frac{\pi}{4}\right) = 4\sin\frac{\pi}{4}\cos\frac{\pi}{4} - 1$$

$$\sin^2\frac{\pi}{2} = 4\sin\frac{\pi}{4}\cos\frac{\pi}{4} - 1$$

$$1^2 = 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - 1$$

$$1 = 2 - 1$$

$$1 = 1$$

b) Test other values. If a counterexample is found, it is not an identity. When $x = \frac{\pi}{3}$ is substituted, the left side of the equation does not equal the right side, so this equation is not an identity.

C3. Answers may vary.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	9–13, 15–17, 19–21
Reasoning and Proving	9–13, 15–22
Reflecting	
Selecting Tools and Computational Strategies	1–8, 14, 17, 19, 22
Connecting	1–22
Representing	9, 17, 19, 22
Communicating	14, 18

Achievement Check, question 19, student text page 241

This performance task is designed to assess the specific expectations covered in Section 4.5. The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting
- Reasoning and Proving

Sample Solution

a) $3x = 2x + x$

b) $\sin 3x = \sin(2x + x)$

c) $\sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$

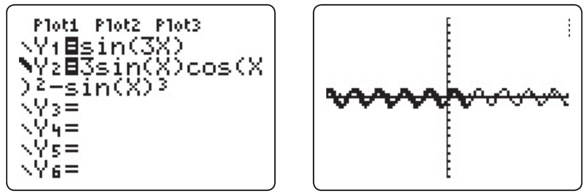
d) $2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x = 3\sin x \cos^2 x - \sin^3 x$

e) Let $x = \pi$.

$\sin(2\pi + \pi) = \sin 3\pi = 0$

$3\sin\pi\cos^2\pi - \sin^3\pi = 3(0)(-1)^2 - 0^3 = 0$

f)



Level 3 Notes

Look for the following:

- Evidence of fairly accurate solution
- Correct choice of addition formula to expand part c)
- Correct choice of double angle formula to expand part d)
- Use of formulas in part c) and part d) is fairly accurate
- Mostly simplified answer in part d)
- Appropriate verification of answer in part e) demonstrating algebraic justification
- Fairly accurate illustrations of identity
- Consistent evidence of proper form

What Distinguishes Level 2

- Evidence of somewhat accurate solution
- Correct choice of addition formula to expand part c)
- Correct choice of double angle formula to expand part d)
- Use of formulas in part c) and part d) is somewhat accurate
- Somewhat simplified answer in part d)
- Some verification of answer in part e)
- Somewhat accurate illustration of identity
- Some evidence of proper form

What Distinguishes Level 4

- High degree of accuracy in solution
- Correct choice of addition formula to expand part c)
- Correct choice of double angle formula to expand part d)
- Formulas in part c) and part d) are applied accurately
- Highly simplified answer in part d)
- Verification of answer in part e) demonstrating high degree of reasoning and justification
- Accurate illustration of identity
- Answers consistently supported with explanation
- High degree of proper form evident

Extension

Student Text Pages
242 to 243

Suggested Timing
30–40 min

Tools
- computer
- *The Geometer's Sketchpad®*

Related Resources
- T–2 *The Geometer's Sketchpad®* 4

Use *The Geometer's Sketchpad®* to Sketch and Manipulate Three-Dimensional Structures in a Two-Dimensional Representation

Teaching Suggestions

- This extension allows students to get a feel for the production of Computer Generated Imagery (CGI) that is prevalent in the entertainment industry, using a familiar software package.
- *The Geometer's Sketchpad®* (GSP) allows students to create fairly sophisticated representations of motion in three dimensions while confined to a two-dimensional workspace.
- Ensure that students understand that the complex calculations involving trigonometric expressions are being done by the software behind the scenes, and that they are instructing the software as to what kinds of motions they want to create.
- This is an open-ended activity with tremendous potential for extension, including the use of colour, morphing, and other enhancements. Some students can extend this activity into a full-fledged project.
- Use **T–2 *The Geometer's Sketchpad®* 4** to support this activity, if needed.
- If you have little experience in using action buttons, you may wish to spend a little time in the GSP Help file to ensure that you understand what these do. Look under Animate, Action Buttons, Presentation, and Movement.

Review

Student Text Pages
244 to 245

Suggested Timing
60–75 min

Tools
- scientific calculator
- graphing calculator

Related Resources
- BLM 4–10 Chapter 4 Review

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

Question	Section(s)	Refer to
1	4.1	Example 1 (page 205)
2	4.1	Example 2 (page 206)
3	4.1	Example 1 (page 205)
4	4.1	Example 2 (page 206)
5	4.1	Example 4 (page 207)
6	4.1	Example 4 (page 207)
7	4.2	Investigate 1 (pages 211–213)
8	4.2	Investigate 2 (page 213)
9	4.2	Example 1 (page 214)
10	4.3	Example 1 (pages 222–223)
11	4.3	Example 1 (pages 222–223)
12	4.3	Example 2 (page 223)
13	4.3	Example 1 (pages 222–223)
14	4.4	Example 1 (pages 230–231)
15	4.4	Questions 8–11 (page 233)
16	4.4	Questions 12–13 (page 233)
17	4.4	Example 2 (page 231)
18	4.5	Example 2 (page 238)
19	4.5	Example 1 (page 238), Example 4 (page 239)
20	4.5	Example 4 (page 239)
21	4.5	Investigate (pages 236–237)
22	4.5	Investigate (pages 236–237)
23	4.5	Example 3 (page 239)

Problem Wrap-Up

Student Text Page
245

Suggested Timing
20–30 min

Tools
- grid paper
- scientific calculator

Related Resources
- G–1 Grid Paper
- BLM 4–11 Chapter 4 Problem Wrap-Up Rubric

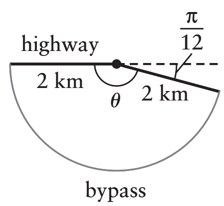
Summative Assessment
- Use **BLM 4–11 Chapter 4 Problem Rubric** to assess student achievement.

Using the Chapter Problem

- If you have elected to work on the Chapter Problem as it was introduced throughout the chapter, students will have the skills that they need to complete the Chapter Problem Wrap-Up.
- If you have elected to leave all of the Chapter Problem to the end of the chapter, students will need to be given 40–50 min to work through the various parts of the problem introduced in each section. Alternatively, you can assign this part as homework.
- Consider allowing students to use technology such as *The Geometer's Sketchpad*® to prepare diagrams and make measurements.

Level 3 Sample Response

a) i)



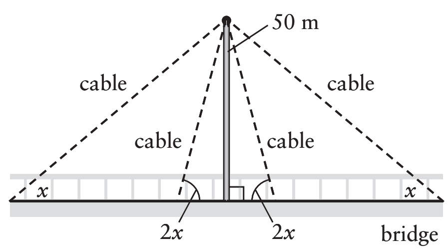
ii) Arc length $= r\theta$

$$= 2 \times \frac{11\pi}{12}$$

$$\doteq 5.76$$

The length of the bypass is about 5.76 km.

b) i)



ii) Let the length of the inner cable be represented by l_1, and the length of the outer cable be represented by l_2.

$$\sin 2x = \frac{l_1}{50} \qquad \sin x = \frac{l_2}{50}$$

$$l_1 = \frac{50}{\sin 2x} \qquad l_2 = \frac{50}{\sin x}$$

$$l_1 + l_2 = \frac{50}{\sin 2x} + \frac{50}{\sin x}$$

$$l = 50\left(\frac{1}{\sin 2x} + \frac{1}{\sin x}\right)$$

$$l = 50\left(\frac{1}{\sin 2x} + \frac{2\cos x}{2\sin x\cos x}\right)$$

$$l = 50\left(\frac{1}{\sin 2x} + \frac{2\cos x}{\sin 2x}\right)$$

$$l = \frac{50(1 + 2\cos x)}{\sin 2x}$$

Level 3 Notes

- Neat, labelled diagrams include all information for parts a) and b)
- For part a), accurate calculation of the arc length, including the formula used and concluding statement with correct units
- For part b), a complete development of the derivation asked for in the question
- For technology use, the inclusion of at least five of the features listed in the question, smooth operation of the animation, and obvious start, stop, and reset controls

What Distinguishes Level 2

- Diagrams are mostly complete but sloppily drawn and with some information missing for parts a) and b)
- For part a), arithmetic errors in the calculation of the arc length, the formula is not present, either no concluding statement or one with incorrect or missing units
- For part b), missing, unexplained, or unneeded steps in the development of the derivation asked for in the question
- For technology use, includes only three or four of the features listed in the question, jerky or otherwise improper operation of the animation, missing start, stop, or reset controls

What Distinguishes Level 4

- Diagrams drawn using technology for parts a) and b)
- For part b), a derivation uses reciprocal trigonometric ratios to avoid fractions until the last line
- For technology use, includes two or more features not listed in the question that enhance the animation, multiple animations, either simultaneous or sequential, imaginative subject material

Practice Test

Student Text Pages
246 to 247

Suggested Timing
40–50 min

Tools
- scientific calculator
- graphing calculator

Related Resources
- BLM 4–12 Chapter 4 Test

Summative Assessment
- You may wish to use **BLM 4–12 Chapter 4 Test** as a summative assessment.

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

Question	Section(s)	Refer to
1	4.1	Example 1 (page 205)
2	4.1	Example 2 (page 206)
3	4.2	Investigate 1 (pages 211–213)
4	4.3	Investigate 2 (pages 221–222)
5	4.2	Investigate 2 (page 213)
6	4.2	Example 2 (page 215)
7	4.4	Example 1 (pages 230–231)
8	4.1	Example 3 (page 206), Example 4 (page 207)
9	4.2	Investigate 2 (page 213)
10	4.2	Example 1 (page 214)
11	4.3	Example 2 (page 223)
12	4.4	Example 2 (page 231)
13	4.4	Questions 8–11 (page 233)
14	4.1	Example 4 (page 207)
15	4.2	Example 1 (page 214)
16	4.5	Example 4 (page 239)
17	4.5	Example 4 (page 239)
18	4.5	Investigate (pages 236–237)
19	4.5	Example 3 (page 239)
20	4.2	Example 1 (page 214)
21	4.3	Example 3 (page 224)

Can students do each of the following?

- Convert degree measure to radian measure, in exact and approximate formats
- Convert radian measure to degree measure, in exact and approximate formats
- Determine arc lengths
- Determine angular velocity using both degree measure and radian measure
- Determine exact values for trigonometric ratios of special angles
- Use technology to determine values for trigonometric ratios
- Use equivalent trigonometric expressions to simplify calculations
- Apply the compound angle formulas for sine and cosine
- Solve problems involving compound angle formulas
- Prove trigonometric identities

Task

Student Text Page
248

Suggested Timing
60–75 min

Tools
- graphing calculator

Related Resources
- BLM 4–13 Task: Make Your Own Identity Rubric

Ongoing Assessment
- Use **BLM 4–13 Task: Make Your Own Identity Rubric** to assess student achievement.

Make Your Own Identity

Teaching Suggestions

Have the students read the question in pairs, but then go back to their desks and work on their own. This will help ESL and ELL students as well as make certain students have another pair of eyes interpreting the questions so there is no mistaking what is being asked for. The Task could be assigned as an in-class assignment or as an independent assignment to be completed outside of class.

Assess student responses for the level of mathematical understanding they represent. As you assess each response, consider the following questions:

- Has the student used understanding of trigonometric identities?
- Did the student comprehend the given information?
- Has the student provided written step by step solution to the proof?
- Has the student provided good hints for the solution to the proof?

Level 3 Sample Response

I will start with $\sin x = \sin x$ and then make it progressively more complicated both on the L.S. and the R.S.

$$\begin{aligned}\sin x &= \sin x\\ \tan x\cos x &= \frac{1}{\csc x}\\ \tan x\left(\frac{1}{\sec x}\right) &= \frac{1}{\cot x\sec x}\end{aligned}$$

When I graphed the L.S. of the equation it matched the graph of the R.S. of the equation.

My question to trade is: Prove $\tan x\left(\frac{1}{\sec x}\right) = \frac{1}{\cot x\sec x}$.

My check is as follows:

$$\begin{aligned}\textbf{L.S.} &= \tan x\left(\frac{1}{\sec x}\right)\\ &= \frac{\sin x}{\cos x}\left(\frac{1}{\frac{1}{\cos x}}\right)\\ &= \frac{\sin x}{\cos x}(\cos x)\\ &= \sin x\end{aligned}$$

$$\begin{aligned}\textbf{R.S.} &= \frac{1}{\cot x(\sec x)}\\ &= \frac{1}{\frac{\cos x}{\sin x}\left(\frac{1}{\cos x}\right)}\\ &= \frac{1}{\frac{1}{\sin x}}\\ &= \sin x\end{aligned}$$

Hints for my partner proving my identity:

1) consider the quotient identity
2) consider reciprocal identities