Chapter 4

Trigonometry

You may be surprised to learn that scientists, engineers, designers, and other professionals who use angles in their daily work generally do not measure the angles in degrees. In this chapter, you will investigate another method of measuring angles known as radian measure. You will extend the use of radian measure to trigonometric ratios and gain an appreciation for the simpler representations that occur when using radian measure. You will develop trigonometric formulas for compound angles and investigate equivalent trigonometric expressions using a variety of approaches. You will develop techniques for identifying and proving trigonometric identities.

By the end of this chapter, you will

- recognize the radian as an alternative unit to the degree for angle measurement, define the radian measure of an angle as the length of the arc that subtends this angle at the centre of a unit circle, and develop and apply the relationship between radian and degree measure
- represent radian measure in terms of π and as a rational number
- determine, with technology, the primary trigonometric ratios and the reciprocal trigonometric ratios of angles expressed in radian measure
- determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, and their multiples less than or equal to 2π
- recognize equivalent trigonometric expressions, and verify equivalence using graphing technology
- explore the algebraic development of the compound angle formulas and use the formulas to determine exact values of trigonometric ratios
- recognize that trigonometric identities are equations that are true for every value in the domain; prove trigonometric identities through the application of reasoning skills, using a variety of relationships; and verify identities using technology

Prerequisite Skills

Primary Trigonometric Ratios and the CAST Rule

1. An exact value for a trigonometric ratio is given for each angle. Determine the exact values of the other two primary trigonometric ratios.

a) $\sin\theta = \frac{3}{5}$, $0° \le \theta \le 90°$

b) $\cos\theta = \frac{5}{13}$, $270° \le \theta \le 360°$

c) $\tan x = -\frac{7}{24}$, $90° \le x \le 180°$

d) $\sin x = -\frac{8}{17}$, $180° \le x \le 270°$

2. Use the CAST rule to determine the sign of each value. Then, use a calculator to evaluate each trigonometric ratio, rounded to four decimal places.

a) $\sin 15°$ **b)** $\cos 56°$

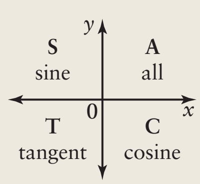
c) $\tan 75°$ **d)** $\sin 100°$

e) $\cos 157°$ **f)** $\tan 250°$

g) $\sin 302°$ **h)** $\cos 348°$

CONNECTIONS

Recall that the memory device CAST shows which trigonometric ratios are positive in each quadrant.



3. Use a calculator to determine angle x, rounded to the nearest degree.

a) $\sin x = 0.65$ **b)** $\cos x = \frac{5}{12}$

c) $\tan x = 8.346$ **d)** $\cos x = -0.45$

Reciprocal Trigonometric Ratios

Note: If you have not worked with reciprocal trigonometric ratios before, refer to the Skills Appendix on page XXX.

4. Determine the reciprocal of each number.

a) 2 **b)** $\frac{1}{3}$ **c)** $\frac{3}{5}$ **d)** $\frac{\sqrt{3}}{2}$

5. An exact value for a reciprocal trigonometric ratio is given for each angle. Determine the exact values of the other two reciprocal trigonometric ratios.

a) $\csc x = \frac{5}{4}$, $0° \le x \le 90°$

b) $\sec\theta = \frac{13}{12}$, $270° \le \theta \le 360°$

c) $\cot x = -\frac{24}{7}$, $90° \le x \le 180°$

d) $\csc\theta = -\frac{17}{15}$, $180° \le \theta \le 270°$

6. Use a calculator to evaluate each trigonometric ratio, rounded to four decimal places.

a) $\csc 35°$

b) $\sec 216°$

c) $\cot 25°$

d) $\csc 122°$

e) $\sec 142°$

f) $\cot 223°$

g) $\csc 321°$

h) $\sec 355°$

7. Use a calculator to determine angle x, rounded to the nearest degree.

a) $\csc x = 1.25$

b) $\sec x = \frac{12}{7}$

c) $\cot x = 3.1416$

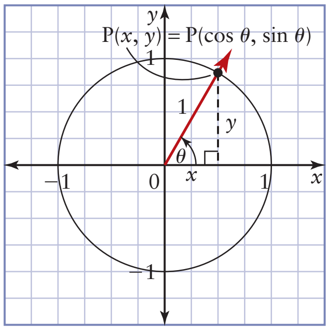
d) $\sec x = -1.32$

Exact Trigonometric Ratios of Special Angles

8. Use appropriate special right triangles to determine exact values for the primary trigonometric ratios.

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
a)	30°			
b)	45°			
c)	60°			

9. a) Use a unit circle, similar to the one shown, to represent an angle of 45° in standard position with point P(x, y) as the point of intersection of the terminal arm and the unit circle.



Create a right triangle that will allow you to determine the primary trigonometric ratios of 45°. Label the coordinates of point P using the exact values for (cos 45°, sin 45°).

b) Determine exact expressions for the lengths of the sides of the triangle.

c) Use the side lengths to determine exact values for the reciprocal trigonometric ratios of 45°.

10. Extend your diagram from question 9 to help you determine exact primary and reciprocal trigonometric ratios for 135°, 225°, and 315°.

Distance Between Two Points

11. Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to determine the distance between the points in each pair.

a) A(10, 8) and B(6, 5)

b) C(−5, −3) and D(7, 2)

c) E(−8, 4) and F(7, 12)

d) G(−3, −6) and H(3, 2)

Product of Two Binomials

12. Expand and simplify each product.

a) $(a + b)(a + b)$

b) $(c + d)(c - d)$

c) $(2x + y)(3x - 2y)$

d) $(\sin x + \cos y)(\sin x + \cos y)$

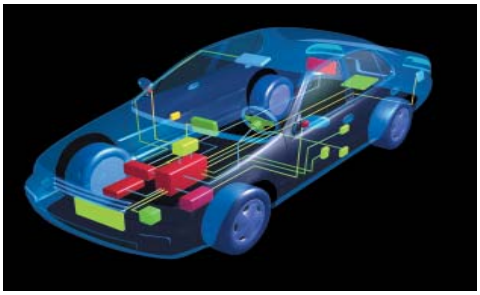
Trigonometric Identities

13. Use a calculator to verify that the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ is true for $x = 20°$, $x = 130°$, $x = 200°$, and $x = 305°$.

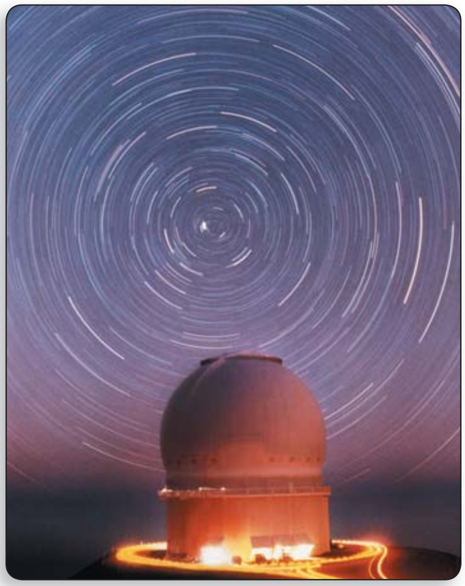
14. Use a calculator to verify that the quotient identity $\tan x = \frac{\sin x}{\cos x}$ is true for $x = 40°$, $x = 110°$, $x = 232°$, and $x = 355°$.

CHAPTER PROBLEM

What do a civil engineer designing highway on-ramps, a pilot turning an aircraft onto the final approach for landing, and a software designer developing the code for engineering design software have in common? All of them use angles and trigonometry to help solve the problems they encounter. The engineer applies trigonometry to determine the length and the angle of inclination of the on-ramp. The pilot selects the angle of bank for the aircraft from tables based on the tangent ratio to ensure a proper alignment with the runway at the end of the turn. The software designer employs trigonometry to realistically render a three-dimensional world onto a two-dimensional computer screen. As you work through this chapter, you will apply trigonometry to solve problems relating to the transportation industry.



4.1 Radian Measure



Ancient Babylonian astronomers are credited with inventing degree measure and for choosing 360 as the number of degrees in a complete turn. They noted that the stars in the night sky showed two kinds of movement. They would "rise" and "set" during the course of a single night, moving in arcs centred on the North Star. However, they also noted a change in the position of a given star from night to night of about $\frac{1}{360}$ of a circle, taking about a year to complete one revolution. This long-term motion of the stars is thought to be the basis for choosing 360° as one complete revolution.

In this section, you will investigate and learn to use another way of measuring angles, known as radian measure.

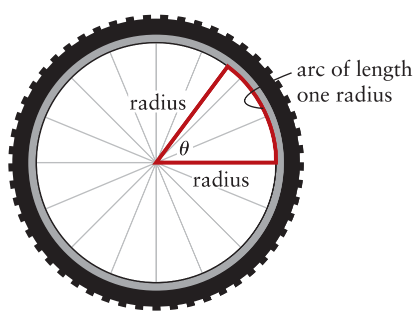
Investigate — How can you determine the meaning of radian measure?

Tools

- a large wheel, such as a bicycle wheel
- a length of heavy string
- masking tape
- protractor
- scissors

Method 1: *Use a Wheel and a String*

1. Measure and cut a length of string equal to the radius of the wheel.
2. Lay the string along the outside circumference of the wheel to form an arc. Use tape to hold the string on the wheel.
3. Measure and cut two more lengths, each equal to the radius. Tape these lengths of string from each end of the arc to the centre of the wheel. The two radii and the arc form an area known as a **sector**.
4. The angle θ formed at the centre of the wheel by the two radii is the **central angle**. The arc **subtends**, or is opposite to, this central angle. Estimate the measure of the central angle, in degrees. Then, use a protractor to measure the angle. Record the measurement.



5. One **radian** is the measure of the angle subtended at the centre of a circle by an arc that has the same length as the radius of the circle. Since you used an arc with the same length as the radius, the angle θ measures 1 radian. Estimate how many degrees are equivalent to 1 radian.

6. Estimate the number of arcs of length one radius that it would take to go once around the outside circumference of the wheel. Explain how you made the estimate.

7. Use a length of string to make a more accurate measure of the number of arcs of length one radius that it takes to go once around the wheel. If you have a fraction of an arc left over, estimate the fraction, and convert to a decimal.

8. **Reflect** What is the relationship between the radius of a circle and its circumference? How does this relationship compare to your measurement in step 7?

Method 2: ***Use The Geometer's Sketchpad®***

Tools

- computer with *The Geometer's Sketchpad®*

1. Open *The Geometer's Sketchpad®*. Turn on the grid display. Adjust the scale on the axes so that 1 unit in each direction fills most of the workspace, as shown.

2. Construct a circle with centre at the origin A and radius 1 unit.

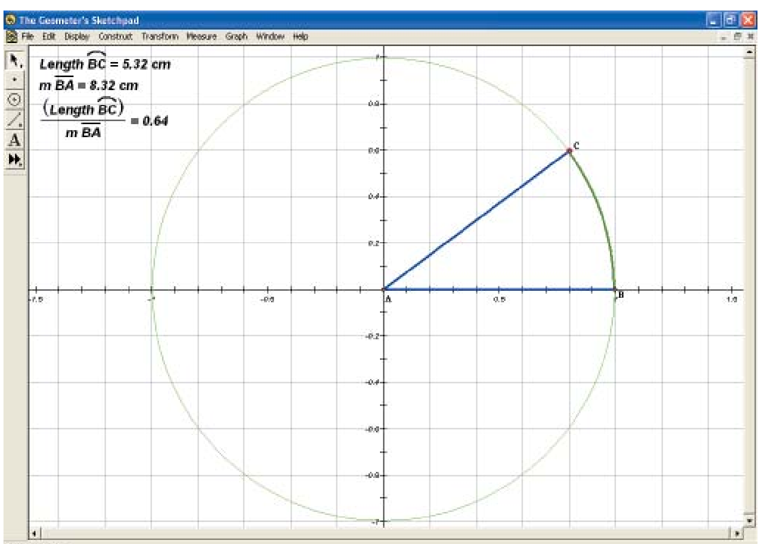
3. Select the circle. From the **Construct** menu, choose **Point On Circle** to create point B. Move point B to (1, 0). Construct a line segment AB from the origin to (1, 0). Use the **Display** menu to adjust the line width to thick.

4. Select the circle. From the **Construct** menu, choose **Point On Circle** to create point C. Move point C on the circle to the first quadrant, if necessary. Join C to A with a thick line segment.

5. Select points B and C, in that order, and the circle. From the **Construct** menu, choose **Arc On Circle**.

6. CA, AB, and the arc BC form an area known as a **sector**. Measure the length of the arc and the length of the radius.

7. Calculate the ratio of the length of the arc to the length of the radius.



8. The angle formed at the centre of the circle by the two radii is the **central angle**. Drag point C along the circle until the ratio from step 7 equals 1. When the length of the arc is equal to the radius, the central angle **subtended** by the arc, ∠BAC, measures 1 **radian**.

9. Measure ∠BAC. How many degrees make up 1 radian? Record the measurement.

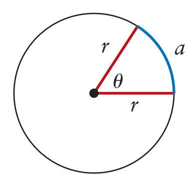
10. Estimate the number of arcs of length one radius that it would take to go once around the circumference of the circle. Explain how you made the estimate.

11. Make a more accurate measure of the number of arcs of length one radius that it takes to go once around the circumference by continuing to drag point C around the circle until you have completed one revolution.

12. **Reflect** What is the relationship between the radius of a circle and its circumference? How does this relationship compare to your measurement in step 11?

13. Save your sketch for later use.

The radian measure of an angle θ is defined as the length, a, of the arc that subtends the angle divided by the radius, r, of the circle.



$$\theta = \frac{a}{r}$$

For one complete revolution, the length of the arc equals the circumference of the circle, $2\pi r$.

$$\theta = \frac{2\pi r}{r} = 2\pi$$

One complete revolution measures 2π radians.

You can determine the relationship between radians and degrees.

Radians to Degrees

$2\pi \text{ rad} = 360°$

$1 \text{ rad} = \left(\frac{360}{2\pi}\right)°$ Divide both sides by 2π.

$1 \text{ rad} = \left(\frac{180}{\pi}\right)°$ Simplify.

Degrees to Radians

$360° = 2\pi \text{ rad}$ The abbreviation rad means radian.

$1° = \frac{2\pi}{360} \text{ rad}$ Divide both sides by 360.

$1° = \frac{\pi}{180} \text{ rad}$ Simplify.

One radian is $\left(\frac{180}{\pi}\right)°$, or approximately 57.3°. One degree is $\frac{\pi}{180}$ rad, or approximately 0.0175.

In practice, the term *radian* or its abbreviation, rad, is often omitted. An angle with a degree symbol, such as 30°, is understood to be measured in degrees. An angle with no symbol, such as 6.28, is understood to be measured in radians.

Exact angles in radians are usually written in terms of π. For example, a straight angle is referred to as π radians.

Example 1 Degree Measure to Radian Measure

Determine an exact and an approximate radian measure, to the nearest hundredth, for an angle of 30°.

Solution

To convert a degree measure to a radian measure, multiply the degree measure by $\frac{\pi}{180}$ radians.

Method 1: ***Use Pencil, Paper, and Scientific Calculator***

$$30° = 30 \times \frac{\pi}{180}$$
$$= \frac{\pi}{6}$$

30° is exactly $\frac{\pi}{6}$ radians.

Use a calculator to determine the approximate radian measure.

$$\frac{\pi}{6} \doteq 0.52$$

30° is approximately 0.52 radians.

Note: Whenever possible, you should leave angle measures in exact form, to preserve accuracy.

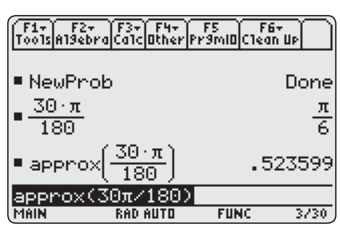
Method 2: ***Use a Computer Algebra System (CAS)***

On a TI-89 calculator, press MODE and ensure that the **Angle** measure is set to **RADIAN**. While in the **MODE** menu, scroll down to **Exact/Approx** and ensure that it is set to **AUTO**.

In the Home screen, type 30 × π ÷ 180, and press ENTER. Note that the CAS will display the exact answer.

30° is exactly $\frac{\pi}{6}$ radians.

From the **F2** menu, select **5:approx)**.
Type 30 × π ÷ 180, and press) ENTER.
Notice that the CAS returns the approximate answer.



30° is approximately 0.52 radians.

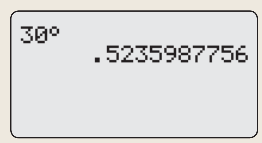
CONNECTIONS

$\frac{\pi}{6}$ is usually read "pi over six."

Technology Tip

You can use a TI-83 Plus or TI-84 Plus graphing calculator to convert degree measure to approximate radian measure. Ensure that the calculator is in Radian mode.
Type the angle, in degrees.
- For example, type 30.
- Press 2nd APPS to access the **ANGLE** menu.
- Select **1:°**.
- Press ENTER.

The measure of the angle is displayed in radians.



Technology Tip

Instead of using the **approx(** operation of a CAS, you can press MODE and set the accuracy to **APPROXIMATE** or press ◆ ENTER for ≈, which is an approximation symbol.

Technology Tip

You can use a TI-83 Plus or TI-84 Plus graphing calculator to convert radian measure to degree measure.
Ensure that the calculator is in Degree mode.
Type the angle, in radians.
- For example, type (π ÷ 4).
- Press 2nd APPS to access the **ANGLE** menu.
- Select **3:**^{r}.
- Press ENTER.

The measure of the angle is displayed in degrees.

```
(π/4)r
                45
```

Example 2 Radian Measure to Degree Measure

Determine the degree measure, to the nearest tenth, for each radian measure.

a) $\frac{\pi}{4}$

b) 5.86

Solution

a) To convert radian measure to degree measure, multiply the radian measure by $\left(\frac{180}{\pi}\right)^\circ$.

$$\frac{\pi}{4} = \frac{\pi}{4} \times \left(\frac{180}{\pi}\right)^\circ$$
$$= 45^\circ$$

$\frac{\pi}{4}$ radians is exactly 45°.

b) $5.86 = 5.86 \times \left(\frac{180}{\pi}\right)^\circ$
$$\doteq 335.8^\circ$$

5.86 radians is approximately 335.8°.

Example 3 Arc Length for a Given Angle

Suzette chooses a camel to ride on a carousel. The camel is located 9 m from the centre of the carousel. If the carousel turns through an angle of $\frac{5\pi}{6}$, determine the length of the arc travelled by the camel, to the nearest tenth of a metre.

Solution

The radius, r, is 9 m, and the measure of angle θ is $\frac{5\pi}{6}$ radians.

$$\text{Angle measure in radians} = \frac{\text{length of arc subtended by the angle}}{\text{radius}}$$
$$\theta = \frac{a}{r}$$
$$a = r\theta$$
$$a = 9 \times \frac{5\pi}{6}$$
$$a \doteq 23.6$$

The camel travels approximately 23.6 m.

CONNECTIONS

The invention of radian measure is credited to Roger Cotes as early as 1714, although he did not use the term *radian*. This term first appeared in an examination paper set by James Thomson at Queen's College, Belfast, in 1873.

Example 4 Angular Velocity of a Rotating Object

The angular velocity of a rotating object is the rate at which the central angle changes with respect to time. The hard disk in a personal computer rotates at 7200 rpm (revolutions per minute). Determine its angular velocity, in

a) degrees per second

b) radians per second

Solution

a) The hard disk rotates through an angle of $7200 \times 360°$, or 2 592 000°, in 1 min.

$$\text{Angular velocity} = \frac{2\ 592\ 000°}{60\text{ s}}$$
$$= 43\ 200°/\text{s}$$

The angular velocity of the hard disk is 43 200°/s.

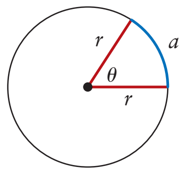
b) The hard disk rotates through an angle of $7200 \times 2\pi$, or $14\ 400\pi$ rad, in 1 min.

$$\text{Angular velocity} = \frac{14\ 400\pi\text{ rad}}{60\text{ s}}$$
$$= 240\pi\text{ rad/s}$$

The angular velocity of the hard disk is 240π rad/s.

KEY CONCEPTS

- The radian measure of angle θ is defined as the length, a, of the arc that subtends the angle divided by the radius, r, of the circle: $\theta = \frac{a}{r}$.



- 2π rad = 360° or π rad = 180°.
- To convert degree measure to radian measure, multiply the degree measure by $\frac{\pi}{180}$ radians.
- To convert radian measure to degree measure, multiply the radian measure by $\left(\frac{180}{\pi}\right)^{\circ}$.

Communicate Your Understanding

C1 Determine mentally the degree measure of an angle of π radians. Justify your answer.

C2 Determine mentally the radian measure of a right angle. Justify your answer.

C3 As described in the introduction, the Babylonian method of measuring angles in degrees is based on the division of a complete revolution into an arbitrary number of units. Why is radian measure a more suitable method of measuring angles than degree measure?

A Practise

For help with questions 1 to 6, refer to Example 1.

1. Determine mentally the exact radian measure for each angle, given that 30° is exactly $\frac{\pi}{6}$ rad.

a) 60° **b)** 90° **c)** 120° **d)** 150°

2. Determine mentally the exact radian measure for each angle, given that 30° is exactly $\frac{\pi}{6}$ rad.

a) 15° **b)** 10° **c)** 7.5° **d)** 5°

3. Determine mentally the exact radian measure for each angle, given that 45° is exactly $\frac{\pi}{4}$ rad.

a) 90° **b)** 135° **c)** 180° **d)** 225°

4. Determine mentally the exact radian measure for each angle, given that 45° is exactly $\frac{\pi}{4}$ rad.

a) 22.5° **b)** 15° **c)** 9° **d)** 3°

5. Determine the exact radian measure for each angle.

a) 40° **b)** 10° **c)** 315°

d) 210° **e)** 300° **f)** 75°

6. Determine the approximate radian measure, to the nearest hundredth, for each angle.

a) 23° **b)** 51° **c)** 82°

d) 128° **e)** 240° **f)** 330°

For help with questions 7 and 8, refer to Example 2.

7. Determine the exact degree measure for each angle.

a) $\frac{\pi}{5}$ **b)** $\frac{\pi}{9}$ **c)** $\frac{5\pi}{12}$

d) $\frac{5\pi}{18}$ **e)** $\frac{3\pi}{4}$ **f)** $\frac{3\pi}{2}$

8. Determine the approximate degree measure, to the nearest tenth, for each angle.

a) 2.34 **b)** 3.14 **c)** 5.27

d) 7.53 **e)** 0.68 **f)** 1.72

For help with question 9, refer to Example 3.

9. A circle of radius 25 cm has a central angle of 4.75 radians. Determine the length of the arc that subtends this angle.

B Connect and Apply

For help with question 10, refer to Example 4.

10. Kumar rides his bicycle such that the back wheel rotates 10 times in 5 s. Determine the angular velocity of the wheel in

a) degrees per second

b) radians per second

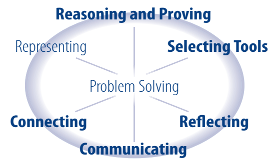
11. The measure of one of the equal angles in an isosceles triangle is twice the measure of the remaining angle. Determine the exact radian measures of the three angles in the triangle.

12. **Use Technology** Refer to *The Geometer's Sketchpad*® sketch from Method 2 of the Investigate. Right-click on point C, and select **Animate.** Use a stopwatch to determine the length of time required for five complete revolutions of C about the origin. Calculate the angular velocity of point C about the origin in

a) degrees per second

b) radians per second

13. Aircraft and ships use nautical miles for measuring distances. At one time, a nautical mile was defined as one minute of arc, or $\frac{1}{60}$ of a degree, along a meridian of longitude, following Earth's surface.



a) Determine the radian measure of one minute of arc, to six decimal places.

b) The radius of Earth is about 6400 km. Determine the length of a nautical mile, using the old definition, to the nearest metre.

c) The length of a nautical mile is no longer calculated in this way. In 1929, the International Extraordinary Hydrographic Conference adopted a definition of exactly 1852 m. Suggest reasons why a calculation using the radius of Earth might not be the best way to define the nautical mile.

14. A milliradian (mrad) is $\frac{1}{1000}$ of a radian. Milliradians are used in artillery to estimate the distance to a target.

a) Show that an arc of length 1 m subtends an angle of 1 mrad at a distance of 1 km.

b) A target scope shows that a target known to be 2 m high subtends an angle of 0.25 mrad. How far away is the target, in kilometres?

15. The Moon has a diameter of about 3480 km and an orbital radius of about 384 400 km from the centre of Earth. Suppose that the Moon is directly overhead. What is the measure of the angle subtended by the diameter to the Moon as measured by an astronomer on the surface of Earth? Answer in both radians and degrees.

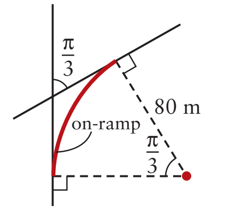
16. David made a swing for his niece Sarah using ropes 2.4 m long, so that Sarah swings through an arc of length 1.2 m. Determine the angle through which Sarah swings, in both radians and degrees.

17. An engine on a jet aircraft turns at about 12 000 rpm. Find an exact value, as well as an approximate value, for the angular velocity of the engine in radians per second.

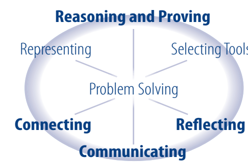
18. Chapter Problem Two highways meet at an angle measuring $\frac{\pi}{3}$ rad, as shown. An on-ramp in the shape of a circular arc is to be built such that the arc has a radius of 80 m.

a) Determine an exact expression for the length of the on-ramp.

b) Determine the length of the on-ramp, to the nearest tenth of a metre.



19. Satellites that are used for satellite television must stay over the same point on Earth at all times, so that the receiving antenna is always pointed at the satellite. This is known as a geostationary orbit and requires an orbit at a distance of approximately 35 900 km above Earth's surface.



a) Explain how such a satellite could remain over the same point on Earth as it orbits the planet.

b) How long would such an orbit require for one complete revolution of Earth?

c) What is the angular velocity of such a satellite, in radians per second?

d) How does the angular velocity of the satellite compare to the angular velocity of the point on Earth that it is hovering over? Justify your answer.

20. When the metric system was developed in France, it was suggested to change the measure of one revolution from 360° to 400 grads. This was known as gradian measure. Gradian measure has not enjoyed wide popularity, but most calculators can handle angles measured in gradians. Look for a key marked DRG on your scientific calculator. Alternatively, look for a MODE key, as it may be an option in the mode menu.

a) What is the measure of a right angle in gradians?

b) Determine the radian measure of an angle of 150 grads.

Achievement Check

21. The London Eye is a large Ferris wheel located on the banks of the Thames River in London, England. Each sealed and air-conditioned passenger capsule holds about 25 passengers. The diameter of the wheel is 135 m, and the wheel takes about half an hour to complete one revolution.

a) Determine the exact angle, in radians, that a passenger will travel in 5 min.

b) How far does a passenger travel in 5 min?

c) How long would it take a passenger to travel 2 radians?

d) What is the angular velocity of a passenger, in radians per second?

e) What is the angular velocity of a passenger, in degrees per second?

C Extend and Challenge

22. Refer to question 19. Determine the orbital speed of a geostationary satellite.

23. In the days of sailing ships, the speed of a ship was determined by tying knots in a rope at regular intervals, attaching one end of the rope to a log, and tossing the log overboard. A sailor would count how many knots followed the log in 30 s, giving the speed of the ship in knots. If the speed in knots is the same as the speed in nautical miles per hour, how far apart would the knots in the rope need to be tied?

CONNECTIONS

The speed of the ship was recorded in a book, which came to be known as the ship's log.

24. Consider your angular velocity due to the rotation of Earth. How does your angular velocity compare to that of an African living on the equator? an Inuit living on the Arctic Circle? Justify your answers.

25. The following proportion is true:

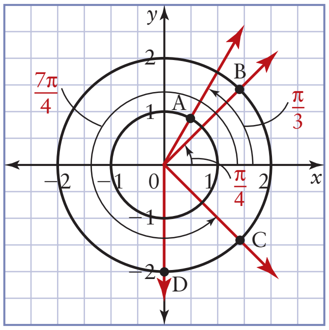
$$\frac{\text{sector area}}{\text{area of circle}} = \frac{\text{central angle}}{\text{one revolution angle}}$$

a) Use this proportion to derive a formula for the sector area, A, in terms of the central angle θ and the radius, r, of the circle.

b) Determine the area of a sector with a central angle of $\frac{\pi}{5}$ and a radius of 12 cm.

26. **Math Contest** You can locate a point in a plane using the Cartesian coordinate system and a unique ordered pair (x, y). You can also locate a point in a plane by using its distance, r, from the origin and its direction angle, θ, from the positive x-axis. This form of coordinates, (r, θ), is known as polar coordinates.

a) Give the polar coordinates of the points A, B, C, and D for $0 \le \theta \le 2\pi$.



b) Determine the polar coordinates of each point.

i) $(1, 1)$

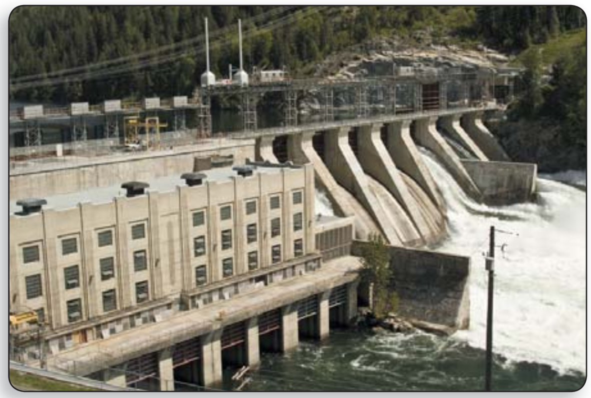
ii) $(-3, 4)$

iii) $(0, -5)$

4.2

Trigonometric Ratios and Special Angles

Scientists, engineers, and other professionals work with radians in many ways. For example, the variety of sounds possible in electronic music is created using trigonometric models. The electricity produced by a power plant follows a trigonometric relation. In this section, you will learn how to use technology to determine trigonometric ratios of angles expressed in radian measure and how to determine exact trigonometric ratios of special angles.



Investigate 1 How can you use technology to evaluate trigonometric ratios for angles measured in radians?

Tools

- scientific calculator

Method 1: *Use a Scientific Calculator*

1. The angle mode is usually displayed on the screen of a scientific calculator. Possible displays are DEG, RAD, GRAD, or D, R, G. Often, there is a key marked DRG that can be pressed to cycle through the angle modes.

a) Set the angle mode to degrees.

b) Evaluate sin 45°. Record the answer, to four decimal places.

2. a) What is the exact equivalent of 45° in radian measure?

b) Set the angle mode to radians.

c) To evaluate $\sin \frac{\pi}{4}$, press the key sequence [π] [÷] 4 [=] [SIN]. How does the answer compare to the answer from step 1b)?

3. Evaluate $\cos \frac{\pi}{4}$ and $\tan \frac{\pi}{4}$.

4. You can evaluate the reciprocal trigonometric ratios using the reciprocal key, which is usually marked x^{-1} or $\frac{1}{x}$. To find $\csc \frac{\pi}{4}$, press the key sequence for $\sin \frac{\pi}{4}$ from step 2c). Then, press the reciprocal key.

5. Evaluate $\sec \frac{\pi}{4}$ and $\cot \frac{\pi}{4}$.

6. Consider an angle of 1.5 radians. Determine the six trigonometric ratios for this angle.

7. Reflect Compare sin 1.5 and sin 1.5°. Explain why they are not equal.

Technology Tip

Scientific calculators will vary. If the instructions shown do not match your calculator, refer to your calculator manual.

Tools

- graphing calculator

Technology Tip

Using a calculator in the wrong angle mode is a common source of error in assignments and on tests and exams. It is wise to develop the habit of checking the mode before using any calculator for calculations involving angles.

Method 2: *Use a Graphing Calculator*

1. a) Press [MODE]. Ensure that the calculator is set to Degree mode.

b) Return to the calculation screen.

c) Evaluate sin 45°. Record the answer, to four decimal places.

2. a) What is the exact equivalent of 45° in radian measure?

b) Set the angle mode to radians.

c) To evaluate $\sin\frac{\pi}{4}$, press the key sequence [SIN] [2nd] [π] [÷] 4 [)] [ENTER]. How does the answer compare to the answer from step 1c)?

3. Evaluate $\cos\frac{\pi}{4}$ and $\tan\frac{\pi}{4}$.

4. You can evaluate the reciprocal trigonometric ratios using the reciprocal key, [x^{-1}]. To find $\csc\frac{\pi}{4}$, press the sequence for $\sin\frac{\pi}{4}$ from step 2c). Then, press [x^{-1}] [ENTER].

5. Evaluate $\sec\frac{\pi}{4}$ and $\cot\frac{\pi}{4}$.

6. Consider an angle of 1.5 radians. Determine the six trigonometric ratios for this angle.

7. Reflect Compare sin 1.5 and sin 1.5°. Explain why they are not equal.

Tools

- computer algebra system

Technology Tip

Notice that the CAS displays an equivalent expression for a radical expression that does not have a radical in the denominator. This is because the denominator has been rationalized as follows:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Method 3: *Use a Computer Algebra System (CAS)*

1. a) Press [MODE]. Ensure that the **Angle** mode is set to **DEGREE**.

b) Return to the Home screen.

c) Evaluate sin 45°. Notice that the CAS returns the exact answer of $\frac{\sqrt{2}}{2}$.

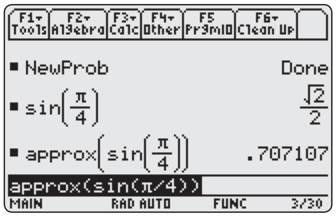
d) To obtain the approximate decimal equivalent, use the **approx(** operation from the **F2** menu. Record the answer, to four decimal places.

2. a) What is the exact equivalent of 45° in radian measure?

b) Set the angle mode to radians.

c) To evaluate $\sin\frac{\pi}{4}$, press the key sequence [2nd] [SIN] [2nd] [π] [÷] 4 [)]. Note that the CAS returns the exact answer. Use the **approx(** operation to obtain the approximate decimal equivalent. How does the answer compare to the answer from step 1d)?

3. Evaluate $\cos\frac{\pi}{4}$ and $\tan\frac{\pi}{4}$.



4. You can evaluate the reciprocal trigonometric ratios by using the exponent key, [^], followed by −1. To find $\csc\frac{\pi}{4}$, press the sequence for $\sin\frac{\pi}{4}$ from step 2c). Then, press [^] [(−)] **1** [ENTER]. Note that the CAS returns the exact answer. Use the **approx(** operation to obtain the approximate decimal equivalent.

5. Evaluate $\sec \frac{\pi}{4}$ and $\cot \frac{\pi}{4}$.

6. Consider an angle of 1.5 radians. Determine the six trigonometric ratios for this angle.

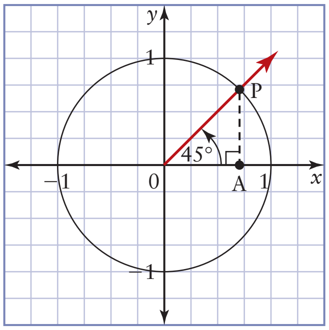
7. **Reflect** Compare sin 1.5 and sin 1.5°. Explain why they are not equal.

Investigate 2 How can you determine trigonometric ratios for special angles without using technology?

Tools

- compasses
- grid paper
- protractor

1. a) Draw a unit circle on grid paper. Draw the terminal arm for an angle $\theta = 45°$ in standard position.
   b) Mark the intersection of the terminal arm and the circle as point P.
   c) Create a triangle by drawing a vertical line from point P to point A on the x-axis.
   d) Consider the special relationships among the sides of △POA to determine an exact value for sides PA and OA. Alternatively, recall that the terminal arm of an angle θ in standard position intersects the unit circle at $P(x, y) = P(\cos\theta, \sin\theta)$.



   e) Use the measures from part d) to write exact values for sin 45°, cos 45°, and tan 45°.
   f) What is the exact equivalent of 45° in radian measure?
   g) Record your results in a table like this one.

Special Angles and Trigonometric Ratios				
θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°				
30°				
45°				
60°				
90°				

2. Use your unit circle to help you complete the table from step 1.

3. Consider the angles 120°, 135°, 150°, and 180°. Use the unit circle, the CAST rule, and the results from step 2 to extend the table for these angles.

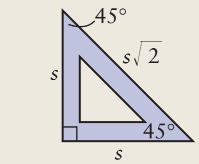
4. Consider the angles 210°, 225°, 240°, and 270°. Use the unit circle, the CAST rule, and the results from step 2 to extend the table for these angles.

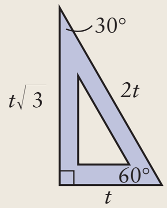
5. Consider the angles 300°, 315°, 330°, and 360°. Use the unit circle, the CAST rule, and the results from step 2 to extend the table for these angles.

6. **Reflect** Select two special angles from each quadrant. Use a calculator set to Radian mode to evaluate the three trigonometric ratios for each angle. Compare the results to those in your table.

CONNECTIONS

The triangles found in a geometry set are a 45°-45°-90° triangle and a 30°-60°-90° triangle. These triangles can be used to construct similar triangles with the same special relationships among the sides.





Example 1 Apply Trigonometric Ratios for Special Angles

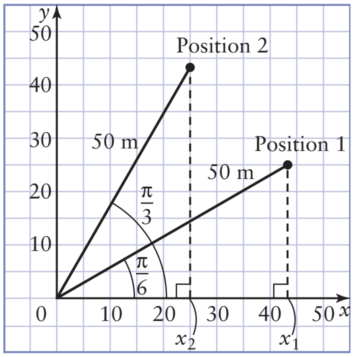
Ravinder is flying his kite at the end of a 50-m string. The sun is directly overhead, and the string makes an angle of $\frac{\pi}{6}$ with the ground. The wind speed increases, and the kite flies higher until the string makes an angle of $\frac{\pi}{3}$ with the ground.

a) Determine an exact expression for the horizontal distance that the shadow of the kite moves between the two positions of the kite.

b) Determine the distance in part a), to the nearest tenth of a metre.

Solution

a) Draw a diagram to represent the situation. Place Ravinder at the origin.



Use the exact values of the trigonometric ratios for $\frac{\pi}{6}$ and $\frac{\pi}{3}$ to write expressions for the two x-coordinates.

Position 1

$$\cos\frac{\pi}{6} = \frac{x_1}{50}$$
$$\frac{\sqrt{3}}{2} = \frac{x_1}{50}$$
$$\frac{50\sqrt{3}}{2} = x_1$$
$$x_1 = 25\sqrt{3}$$

Position 2

$$\cos\frac{\pi}{3} = \frac{x_2}{50}$$
$$\frac{1}{2} = \frac{x_2}{50}$$
$$\frac{50}{2} = x_2$$
$$x_2 = 25$$

Subtract the x-coordinates to determine the horizontal distance, d, that the shadow moves.

$$d = 25\sqrt{3} - 25$$
$$= 25(\sqrt{3} - 1)$$

An exact expression for the distance that the shadow of the kite moves is $25(\sqrt{3} - 1)$ m.

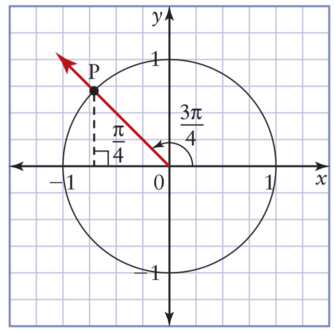
b) $25(\sqrt{3} - 1) \doteq 18.3$

The shadow of the kite moves approximately 18.3 m.

Example 2 Trigonometric Ratios for a Multiple of a Special Angle

Use the unit circle to determine exact values of the six trigonometric ratios for an angle of $\frac{3\pi}{4}$.

Solution



Sketch the angle in standard position. Let P be the point of intersection of the terminal arm and the unit circle. Create a triangle by drawing a perpendicular from point P to the x-axis. The angle between the terminal arm and the x-axis is $\frac{\pi}{4}$. The terminal arm of an angle of $\frac{\pi}{4}$ intersects the unit circle at a point with coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Since the terminal arm of an angle of $\frac{3\pi}{4}$ is in the second quadrant, the coordinates of the point of intersection are $P\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$\sin\frac{3\pi}{4} = y = \frac{1}{\sqrt{2}}$$

$$\cos\frac{3\pi}{4} = x = -\frac{1}{\sqrt{2}}$$

$$\tan\frac{3\pi}{4} = \frac{y}{x} = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$$

$$\csc\frac{3\pi}{4} = \frac{1}{y} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\sec\frac{3\pi}{4} = \frac{1}{x} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

$$\cot\frac{3\pi}{4} = \frac{x}{y} = \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$$

KEY CONCEPTS

- You can use a calculator to calculate trigonometric ratios for an angle expressed in radian measure by setting the angle mode to radians.
- You can determine the reciprocal trigonometric ratios for an angle expressed in radian measure by first calculating the primary trigonometric ratios and then using the reciprocal key on a calculator.
- You can use the unit circle and special triangles to determine exact values for the trigonometric ratios of the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.
- You can use the unit circle along with the CAST rule to determine exact values for the trigonometric ratios of multiples of the special angles.

Communicate Your Understanding

C1 Use a calculator to evaluate $\tan \frac{\pi}{2}$. Explain why the calculator returns this answer. Predict another trigonometric ratio that will return the same answer. Check your prediction with a calculator.

C2 Ancient mathematicians tried to represent π as a rational number. A fraction that is very close to π is $\frac{355}{113}$. Use a calculator to determine the decimal approximation of this fraction. Is it close enough to π to yield the same value for $\sin \pi$, to the limit of the accuracy of your calculator? Use a calculator to check.

C3 Does $\sin \frac{\pi}{2} = \frac{1}{2} \sin \pi$? Explain why or why not.

A Practise

For help with questions 1 to 6, refer to Investigate 1.

1. a) Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **i)** $\sin 25°$ **ii)** $\cos 72°$
- **iii)** $\tan 115°$ **iv)** $\sin 165°$

b) Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **i)** $\sin 0.436$ **ii)** $\cos 1.257$
- **iii)** $\tan 2.007$ **iv)** $\sin 2.880$

c) Note any similarities between the answers to parts a) and b). Explain why they occur.

2. a) Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **i)** $\sin 1.21$ **ii)** $\cos 2.53$
- **iii)** $\tan 3.05$ **iv)** $\cos 4.72$

b) Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **i)** $\sin 69°$ **ii)** $\cos 145°$
- **iii)** $\tan 175°$ **iv)** $\cos 270°$

c) Note any similarities between the answers to parts a) and b). Explain why they occur.

3. Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **a)** $\sin \frac{\pi}{4}$ **b)** $\cos \frac{\pi}{7}$
- **c)** $\tan \frac{5\pi}{6}$ **d)** $\tan \frac{9\pi}{8}$

4. Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **a)** $\csc 15°$ **b)** $\sec 52°$
- **c)** $\cot 124°$ **d)** $\csc 338°$

5. Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **a)** $\csc 0.97$ **b)** $\sec 1.84$
- **c)** $\cot 3.21$ **d)** $\sec 5.95$

6. Use a calculator to evaluate each trigonometric ratio, to four decimal places.

- **a)** $\csc \frac{7\pi}{6}$ **b)** $\sec \frac{3\pi}{11}$
- **c)** $\cot \frac{13\pi}{5}$ **d)** $\cot \frac{17\pi}{9}$

For help with questions 7 and 8, refer to Example 2.

7. Use the unit circle to determine exact values of the primary trigonometric ratios for each angle.

- **a)** $\frac{2\pi}{3}$ **b)** $\frac{5\pi}{6}$
- **c)** $\frac{3\pi}{2}$ **d)** $\frac{7\pi}{4}$

8. Use the unit circle to determine exact values of the six trigonometric ratios for each angle.

- **a)** $\frac{7\pi}{6}$ **b)** $\frac{4\pi}{3}$
- **c)** $\frac{5\pi}{4}$ **d)** π

B Connect and Apply

For help with questions 9 and 10, refer to Example 1.

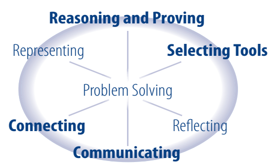
9. Lynda is flying her kite at the end of a 40-m string. The string makes an angle of $\frac{\pi}{4}$ with the ground. The wind speed increases, and the kite flies higher until the string makes an angle of $\frac{\pi}{3}$ with the ground.

a) Determine an exact expression for the horizontal distance that the kite moves between the two positions.

b) Determine an exact expression for the vertical distance that the kite moves between the two positions.

c) Determine approximate answers for parts a) and b), to the nearest tenth of a metre.

10. Refer to question 9. Suppose that the length of string is 40 m at an angle of $\frac{\pi}{4}$, but then Lynda lets out the string to a length of 60 m as the angle changes to $\frac{\pi}{3}$.



a) Determine an exact expression for the horizontal distance that the kite moves between the two positions.

b) Does the kite move closer to Lynda, horizontally, or farther away? Justify your answer.

c) Determine an exact expression for the vertical distance that the kite moves between the two positions. Does the altitude of the kite above the ground increase or decrease? Justify your answer.

d) Determine approximate answers for parts a) and c), to the nearest tenth of a metre.

11. a) Determine an exact value for each expression.

i) $\dfrac{\sin\frac{\pi}{3}\tan\frac{\pi}{6}}{\cos\frac{\pi}{4}}$ **ii)** $\tan\frac{\pi}{4} + \tan\frac{\pi}{6}\tan\frac{\pi}{3}$

b) Use a calculator to check your answers to part a).

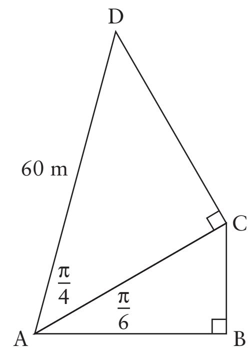
12. a) Determine an exact value for each expression.

i) $\dfrac{\cos\frac{4\pi}{3}\tan\frac{5\pi}{6}}{\sin\frac{3\pi}{4}}$

ii) $\cot\frac{5\pi}{4} + \tan\frac{11\pi}{6}\tan\frac{5\pi}{3}$

b) Use a calculator to check your answers to part a).

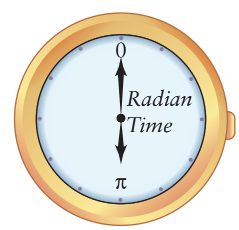
13. Two triangular building lots, ABC and ACD, have side AC in common. ∠ABC and ∠ACD are right angles. $\angle CAB = \frac{\pi}{6}$ and $\angle DAC = \frac{\pi}{4}$. Side DA measures 60 m.



a) Determine an exact expression for the length of side AC.

b) Determine an exact expression for the length of side AB.

14. When Sunita received her bachelor's degree in mathematics, her friends presented her with a "radian watch." Her friends repainted the face of the watch. Instead of the usual numbers around the face, they replaced the 12 with 0 and the 6 with π. The hour and minute hands of the watch run in the usual clockwise direction. A radian time of π is shown in the diagram.



a) What radian time corresponds to 3:00?

b) What radian time corresponds to 4:00?

c) What normal time corresponds to a radian time of $\frac{3\pi}{2}$?

d) What normal time corresponds to a radian time of $\frac{11\pi}{6}$?

e) What radian time corresponds to 7:30?

15. **Use Technology** You can set the angle units in *The Geometer's Sketchpad*® to radians and then calculate trigonometric ratios for angles measured in radians.

a) Open *The Geometer's Sketchpad*®. From the **Edit** menu, choose **Preferences….** Click on the **Units** tab. Set the angle units to radians and the precision to thousandths for all measurements. Click on **OK**.

b) Construct a right △ABC such that ∠B is a right angle. Measure ∠B. If it does not appear to be a right angle, adjust your diagram until it is. How is radian measure expressed in *The Geometer's Sketchpad*®?

c) Measure ∠A and ∠C. Taking care not to change ∠B, adjust the triangle until $\angle A = \frac{\pi}{6}$ and $\angle C = \frac{\pi}{3}$.

d) Use the **Measure** menu to calculate values for sin ∠A, cos ∠A, and tan ∠A. Compare these calculations with your Special Angles and Trigonometric Ratios table from Investigate 2.

Technology Tip

When it is in radian mode, *The Geometer's Sketchpad*® uses directed angles (angles with signs). If an angle measurement is negative, select the points that define the angle in the opposite order.

16. a) Determine an exact value for each expression.

i) $\sin\frac{4\pi}{3}\cos\frac{5\pi}{3} + \sin\frac{5\pi}{3}\cos\frac{4\pi}{3}$

ii) $\sin\frac{5\pi}{4}\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}\cos\frac{5\pi}{4}$

b) Use a calculator to verify your answers to part a).

17. a) Determine an exact value for each expression.

i) $\cos\frac{11\pi}{6}\cos\frac{\pi}{6} - \sin\frac{11\pi}{6}\sin\frac{\pi}{6}$

ii) $\cos\frac{7\pi}{4}\cos\frac{\pi}{4} + \sin\frac{7\pi}{4}\sin\frac{\pi}{4}$

b) Use a calculator to verify your answers to part a).

18. a) Determine an exact value for each expression.

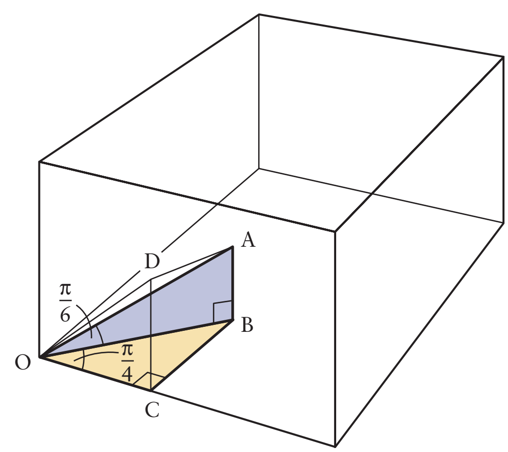
i) $\dfrac{\tan\frac{\pi}{4} + \tan\frac{3\pi}{4}}{1 - \tan\frac{\pi}{4}\tan\frac{3\pi}{4}}$

ii) $\dfrac{\tan\frac{5\pi}{3} - \tan\frac{7\pi}{6}}{1 + \tan\frac{5\pi}{3}\tan\frac{7\pi}{6}}$

b) Use a calculator to verify your answers to part a).

19. **Chapter Problem** The engineering software used to design modern vehicles must include routines to render models of three-dimensional objects onto a two-dimensional computer screen. The software constructs the model in a three-dimensional world, represented by a box, and then projects the model onto the computer screen, represented by the front of the box.

To get an idea of how this works, consider a three-dimensional model that includes line segment OA, as shown. OA lies along an angle of $\frac{\pi}{4}$ from the plane of the screen, and upward at an angle of $\frac{\pi}{6}$ from the bottom plane. The line segment OD is the rendering of OA in the plane of the screen. If OA is 10 cm long, show that the length of OC is given by the expression $10\cos\frac{\pi}{6}\cos\frac{\pi}{4}$.



Achievement Check

20. Consider the equation
$2\sin x\cos y = \sin(x + y) + \sin(x - y)$.
Emilio claims that substituting $x = \frac{5\pi}{4}$ and $y = \frac{3\pi}{4}$ makes the equation true.

a) Determine an exact value for the left side using Emilio's solution.

b) Determine an exact value for the right side using Emilio's solution.

c) Is Emilio correct?

d) Is $x = \frac{5\pi}{6}$ and $y = \frac{\pi}{6}$ a solution? Justify your answer.

e) Based on the answers to parts c) and d), is it reasonable to conclude that the equation is true no matter what angles are substituted for x and y? Justify your answer.

C Extend and Challenge

21. Another way to measure angles is to use gradians. These units were intended to replace degree measure when the metric system was developed in France in the 1790s. A complete revolution is 400 grads.

a) Determine the six trigonometric ratios of an angle measuring 150 grads.

b) Explain why, in general, conversions of angle measures from degrees to gradians or from radians to gradians is difficult.

22. **Use Technology** For small angles, the measure of an angle in radians and the sine of that angle are approximately equal. Investigate the domain over which this approximation is accurate to two decimal places. The instructions below assume the use of a graphing calculator. You can also use a spreadsheet or a CAS.

a) Set a graphing calculator to Radian mode, and set the accuracy to two decimal places.

b) Clear all lists. In list **L1**, enter the numbers 0, 0.1, 0.2, …, 1.0.

c) In list **L2**, enter the formula sin(L1).

d) Over what domain does the approximation seem to be accurate to two decimal places?

e) Modify this method to determine the domain over which the approximation is valid to three decimal places.

23. a) Determine an expression using trigonometric ratios of special angles that simplifies to an answer of $\frac{1}{2}$. You must use three different angles and three different ratios.

b) Trade expressions with a classmate and simplify.

c) Trade solutions and resolve any concerns.

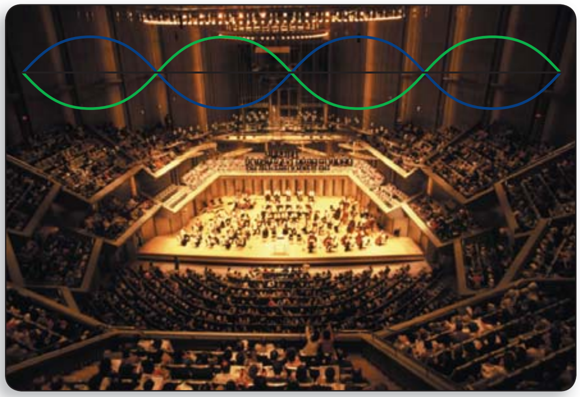
24. **Math Contest** If $\sin\theta = \frac{-5}{13}$, then determine $\sin\frac{\theta}{2}$.

A $\frac{\sqrt{26}}{26}$ **B** $\frac{\sqrt{3}}{3}$ **C** $\frac{2\sqrt{2}}{13}$ **D** $\frac{\sqrt{3}}{12}$

25. **Math Contest** The term *inverse sine* is confusing to some students who have studied functions and their inverses as reflections in the line $y = x$. In this case, inverse sine indicates that you are looking for the angle that gives the indicated ratio. Another term that is sometimes used to mean the same as $\sin^{-1}$ is *arcsin*. Determine the exact value of $\tan\left(\arcsin\left(\frac{-4}{5}\right)\right)$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

A $\frac{5}{4}$ **B** $\frac{4}{3}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{\sqrt{3}}{2}$

4.3 Equivalent Trigonometric Expressions

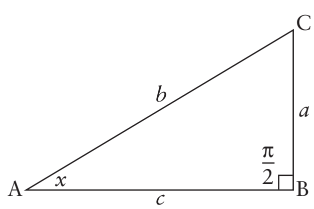


When trigonometry is used to model a real-world application, such as a system of gears for an automobile transmission or the acoustic response of a concert hall, the expressions and equations generated can become very complicated. These expressions can often be simplified by substituting **equivalent trignometric expressions**, which are expressions that yield the same value for all values of the variable. In this section, you will learn how to show that certain trigonometric expressions are equivalent, and you will assemble a table of equivalent expressions for use in subsequent sections.

Investigate 1 How can you use a right triangle to determine equivalent trigonometric expressions?

1. Sketch △ABC such that ∠B is a right angle. Mark ∠B as $\frac{\pi}{2}$.

2. Mark ∠A as x. Use the sum of the interior angles of a triangle to derive an expression for the measure of ∠C in terms of x.



3. Determine an expression for sin x in terms of the sides of the triangle, a, b, and c.

4. Determine expressions for the sine, cosine, and tangent of ∠C in terms of the sides of the triangle. Which of these is equal to sin x?

5. Reflect

a) You have shown that $\sin x$ and $\cos\left(\frac{\pi}{2} - x\right)$ are equivalent trigonometric expressions. Does the relationship between these equivalent expressions depend on the value of x? Justify your answer.

b) What is the sum of x and $\left(\frac{\pi}{2} - x\right)$? What name is given to a pair of angles such as x and $\left(\frac{\pi}{2} - x\right)$?

6. Use △ABC to determine equivalent trigonometric expressions for $\cos x$ and $\tan x$.

7. Use △ABC to determine equivalent trigonometric expressions for $\csc x$, $\sec x$, and $\cot x$.

8. Summarize the six relations among trigonometric functions in a table like the one shown. The first line has been entered for you.

Trigonometric Identities
$\sin x = \cos\left(\frac{\pi}{2} - x\right)$

9. **Reflect** An **identity** is an equation that is true for all values of the variable for which the expressions on both sides of the equation are defined. An identity involving trigonometric expressions is called a **trigonometric identity**. The trigonometric identities in step 8 are known as the cofunction identities. Why are they called *cofunction* identities? Suggest an easy way to remember these identities.

CONNECTIONS

Sometimes the cofunction identities are referred to as the *correlated angle identities*.

Investigate 2

How can you use transformations to determine equivalent trigonometric expressions?

Tools

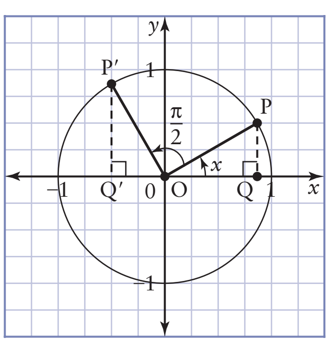
- compasses and protractor
  OR
- computer with *The Geometer's Sketchpad®*

1. Draw a unit circle with centre O.

2. Draw the terminal arm for an angle x in standard position in the first quadrant, $x \in \left[0, \frac{\pi}{2}\right]$. Let the intersection of the unit circle and the terminal arm of angle x be represented by point P. Label the coordinates of point P in terms of the appropriate trigonometric functions of angle x.

3. Transform P to P′ by applying a rotation of $\frac{\pi}{2}$ counterclockwise about the origin. Label the coordinates of P′ in terms of the appropriate trigonometric functions of angle $\left(x + \frac{\pi}{2}\right)$.

4. Create △PQO by drawing a vertical line from point P to point Q on the x-axis. Similarly, create △OQ′P′.



CONNECTIONS

The symbol ≅ means "is congruent to." Congruent geometric figures have exactly the same shape and size.

CONNECTIONS

After a rotation of $\frac{\pi}{2}$, or 90°, counterclockwise about the origin, the coordinates of the point (x, y) become $(-y, x)$.

5. Explain why $\triangle PQO \cong \triangle OQ'P'$.

6. Use the coordinates of points P and P′ to show that $\cos\left(x + \frac{\pi}{2}\right)$ and $-\sin x$ are equivalent trigonometric expressions.

7. Determine an equivalent trigonometric expression of angle x for $\sin\left(x + \frac{\pi}{2}\right)$.

8. Determine an equivalent trigonometric expression of angle x for $\tan\left(x + \frac{\pi}{2}\right)$.

9. Determine an equivalent trigonometric expression of angle x for each of $\csc\left(x + \frac{\pi}{2}\right)$, $\sec\left(x + \frac{\pi}{2}\right)$, and $\cot\left(x + \frac{\pi}{2}\right)$.

10. Extend your table from step 8 of Investigate 1 to summarize these six relations among trigonometric functions.

11. Reflect You now have 12 trigonometric identities in your table, which all feature the angle $\frac{\pi}{2}$. Similar identities are possible using other angles, such as π, $\frac{3\pi}{2}$, and 2π. Use the unit circle to derive an equivalent trigonometric expression for $\cos(\pi - x)$ in terms of angle x.

Example 1 Use Equivalent Trigonometric Expressions to Evaluate Primary Trigonometric Expressions

Given that $\sin\frac{\pi}{5} \doteq 0.5878$, use equivalent trigonometric expressions to evaluate the following, to four decimal places.

a) $\cos\frac{3\pi}{10}$ **b)** $\cos\frac{7\pi}{10}$

Solution

a) Since an angle of $\frac{3\pi}{10}$ lies in the first quadrant, it can be expressed as a difference between $\frac{\pi}{2}$ and an angle a. Find the measure of angle a.

$$\frac{3\pi}{10} = \frac{\pi}{2} - a$$
$$a = \frac{\pi}{2} - \frac{3\pi}{10}$$
$$a = \frac{5\pi}{10} - \frac{3\pi}{10}$$
$$a = \frac{2\pi}{10}$$
$$a = \frac{\pi}{5}$$

Now apply a cofunction identity.

$$\cos\frac{3\pi}{10} = \cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$$
$$= \sin\frac{\pi}{5}$$
$$\doteq 0.5878$$

b) Since an angle of $\frac{7\pi}{10}$ lies in the second quadrant, it can be expressed as a sum of $\frac{\pi}{2}$ and an angle a.

$$\begin{aligned} a + \frac{\pi}{2} &= \frac{7\pi}{10} \\ a &= \frac{7\pi}{10} - \frac{\pi}{2} \\ a &= \frac{7\pi}{10} - \frac{5\pi}{10} \\ a &= \frac{2\pi}{10} \\ a &= \frac{\pi}{5} \end{aligned}$$

Now apply a trigonometric identity.

$$\begin{aligned} \cos \frac{7\pi}{10} &= \cos\left(\frac{\pi}{5} + \frac{\pi}{2}\right) \\ &= -\sin \frac{\pi}{5} \\ &\doteq -0.5878 \end{aligned}$$

Example 2 Use an Equivalent Trigonometric Expression to Evaluate a Reciprocal Trigonometric Expression

Given that $\csc \frac{2\pi}{7} \doteq 1.2790$, use an equivalent trigonometric expression to determine $\sec \frac{3\pi}{14}$, to four decimal places.

Solution

An angle of $\frac{3\pi}{14}$ lies in the first quadrant. Express $\frac{3\pi}{14}$ as a difference between $\frac{\pi}{2}$ and an angle a. Then, apply a cofunction identity.

$$\begin{aligned} \frac{3\pi}{14} &= \frac{\pi}{2} - a \\ a &= \frac{\pi}{2} - \frac{3\pi}{14} \\ a &= \frac{7\pi}{14} - \frac{3\pi}{14} \\ a &= \frac{4\pi}{14} \\ a &= \frac{2\pi}{7} \end{aligned}$$

Therefore,

$$\begin{aligned} \sec \frac{3\pi}{14} &= \sec\left(\frac{\pi}{2} - \frac{2\pi}{7}\right) \\ &= \csc \frac{2\pi}{7} \\ &\doteq 1.2790 \end{aligned}$$

Example 3 Use Graphing Technology to Verify Equivalent Trigonometric Expressions

Use a graphing calculator to verify that $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

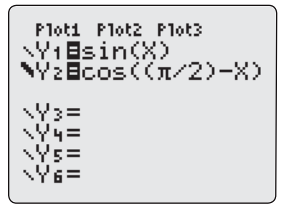
Solution

Ensure that your calculator is set to Radian mode and that all stat plots are turned off.

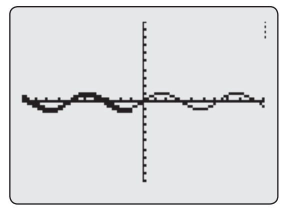
Use the **Y=** editor to enter $\sin x$ in **Y1** and $\cos\left(\frac{\pi}{2} - x\right)$ in **Y2**.

Change the line display for **Y2** to heavy.

- Cursor left to the slanted line beside **Y2**.
- Press ENTER to change the line style.



From the **ZOOM** menu, select **6:ZStandard**. The graph of $\sin x$ will be drawn first. Then, the graph of $\cos\left(\frac{\pi}{2} - x\right)$ will be drawn with a heavier line. You can pause the plot by pressing ENTER. Pressing ENTER again will resume the plot.



CONNECTIONS

You have graphed trigonometric functions in degree measure before. You can also graph these functions in radian measure.

Technology Tip

If you want to watch the plot again, you cannot just press GRAPH. The graphing calculator will simply display the completed plot. To see the plot again, return to the **Y=** editor, clear **Y1**, and then re-enter $\sin x$. When you press GRAPH, both curves will be re-plotted.

KEY CONCEPTS

- You can use a right triangle to derive equivalent trigonometric expressions that form the cofunction identities, such as $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.
- You can use the unit circle along with transformations to derive equivalent trigonometric expressions that form other trigonometric identities, such as $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$.
- Given a trigonometric expression of a known angle, you can use equivalent trigonometric expressions to evaluate trigonometric expressions of other angles.
- You can use graphing technology to demonstrate that two trigonometric expressions are equivalent.

Trigonometric Identities Featuring $\frac{\pi}{2}$			
Cofunction Identities			
$\sin x = \cos\left(\frac{\pi}{2} - x\right)$	$\cos x = \sin\left(\frac{\pi}{2} - x\right)$	$\sin\left(x + \frac{\pi}{2}\right) = \cos x$	$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$
$\tan x = \cot\left(\frac{\pi}{2} - x\right)$	$\cot x = \tan\left(\frac{\pi}{2} - x\right)$	$\tan\left(x + \frac{\pi}{2}\right) = -\cot x$	$\cot\left(x + \frac{\pi}{2}\right) = -\tan x$
$\csc x = \sec\left(\frac{\pi}{2} - x\right)$	$\sec x = \csc\left(\frac{\pi}{2} - x\right)$	$\csc\left(x + \frac{\pi}{2}\right) = \sec x$	$\sec\left(x + \frac{\pi}{2}\right) = -\csc x$

Communicate Your Understanding

C1 Is $\cos(x + \pi)$ an equivalent trigonometric expression for $\cos x$? Justify your answer using the unit circle. Check your answer with graphing technology.

C2 Does $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2} - \cos x$? Investigate using the unit circle. Check your answer with a calculator.

C3 Does $\cos\left(\frac{\pi}{2} - x\right) = -\cos\left(x - \frac{\pi}{2}\right)$? Investigate using the unit circle. Check your answer with a calculator.

A Practise

For help with questions 1 and 2, refer to Investigate 1.

1. Given that $\sin\frac{\pi}{6} = \frac{1}{2}$, use an equivalent trigonometric expression to show that $\cos\frac{\pi}{3} = \frac{1}{2}$.

2. Given that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use an equivalent trigonometric expression to show that $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

For help with questions 3 and 4, refer to Investigate 2.

3. Given that $\sin\frac{\pi}{6} = \frac{1}{2}$, use an equivalent trigonometric expression to show that $\cos\frac{2\pi}{3} = -\frac{1}{2}$.

4. Given that $\csc\frac{\pi}{4} = \sqrt{2}$, use an equivalent trigonometric expression to show that $\sec\frac{3\pi}{4} = -\sqrt{2}$.

For help with questions 5 to 10, refer to Examples 1 and 2.

5. Given that $\cos\frac{\pi}{7} = \sin y$, first express $\frac{\pi}{7}$ as a difference between $\frac{\pi}{2}$ and an angle, and then apply a cofunction identity to determine the measure of angle y.

6. Given that $\cot\frac{4\pi}{9} = \tan z$, first express $\frac{4\pi}{9}$ as a difference between $\frac{\pi}{2}$ and an angle, and then apply a cofunction identity to determine the measure of angle z.

7. Given that $\cos\frac{13\pi}{18} = \sin y$, first express $\frac{13\pi}{18}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle y.

8. Given that $\cot\frac{13\pi}{14} = \tan z$, first express $\frac{13\pi}{14}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle z.

9. Given that $\cos\frac{3\pi}{11} \doteq 0.6549$, use equivalent trigonometric expressions to evaluate the following, to four decimal places.

a) $\sin\frac{5\pi}{22}$ **b)** $\sin\frac{17\pi}{22}$

10. Given that $\tan\frac{2\pi}{9} \doteq 0.8391$, use equivalent trigonometric expressions to evaluate the following, to four decimal places.

a) $\cot\frac{5\pi}{18}$ **b)** $\cot\frac{13\pi}{18}$

B Connect and Apply

11. Given that $\csc a = \sec 1.45$ and that a lies in the first quadrant, use a cofunction identity to determine the measure of angle a, to two decimal places.

12. Given that $\sec b = \csc 0.64$ and that b lies in the first quadrant, use a cofunction identity to determine the measure of angle b, to two decimal places.

13. Given that $\csc a = \sec 0.75$ and that a lies in the second quadrant, determine the measure of angle a, to two decimal places.

14. Given that $\sec b = \csc 1.34$ and that b lies in the second quadrant, determine the measure of angle b, to two decimal places.

For help with questions 15 to 19, refer to Investigate 2.

15. Use the unit circle to investigate equivalent expressions involving the six trigonometric functions of $(\pi - x)$, where x lies in the first quadrant. Add these to your trigonometric identities table from Investigate 2.

16. Use the unit circle to investigate equivalent expressions involving the six trigonometric functions of $(x + \pi)$, where x lies in the first quadrant. Add these to your trigonometric identities table from Investigate 2.

17. Use the unit circle to investigate equivalent expressions involving the six trigonometric functions of $\left(\frac{3\pi}{2} - x\right)$, where x lies in the first quadrant. Add these to your trigonometric identities table from Investigate 2.

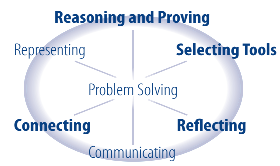
18. Use the unit circle to investigate equivalent expressions involving the six trigonometric functions of $\left(x + \frac{3\pi}{2}\right)$, where x lies in the first quadrant. Add these to your trigonometric identities table from Investigate 2.

19. Use the unit circle to investigate equivalent expressions involving the six trigonometric functions of $(2\pi - x)$, where x lies in the first quadrant. Add these to your trigonometric identities table from Investigate 2.

For help with question 20, refer to Example 3.

20. **Use Technology** Select one trigonometric identity from each of questions 15 to 19. Use graphing technology to verify the identities.

21. Charmaine finds that the SIN key on her calculator is not working. Determine two different ways that she can find $\sin \frac{9\pi}{13}$ without using the SIN key.



22. **Chapter Problem**

a) When a pilot banks an aircraft moving at a speed v, in metres per second, at an angle θ, the radius of the turn that results is given by the formula $r = \frac{v^2}{g}\tan\left(\frac{\pi}{2} - \theta\right)$, where g is the acceleration due to gravity, or 9.8 m/s^2. Use an appropriate equivalent trigonometric expression to show that this formula can be simplified to $r = \frac{v^2}{g\tan\theta}$.

b) A pilot is flying an aircraft at a speed of 50 m/s, and banks at an angle of $\frac{\pi}{4}$. Determine the radius of the turn, to the nearest metre.

Achievement Check

23. Determine an exact value for angle a such that $\sec a = \csc(3a)$.

a) Write the cofunction identity related to this problem.

b) In the cofunction identity, what is the sum of the arguments on the left side and the right side?

c) Write an equation relating the sum in part b) to the sum of the arguments on the left side and the right side of the given equation. Then, solve for a.

d) Check your solution.

C Extend and Challenge

24. a) Determine an exact value of b such that $\csc\left(6b + \frac{\pi}{8}\right) = \sec\left(2b - \frac{\pi}{8}\right)$.

b) Check your answer.

25. a) Determine an exact value of c such that $\cot\left(4c - \frac{\pi}{4}\right) + \tan\left(2c + \frac{\pi}{4}\right) = 0$.

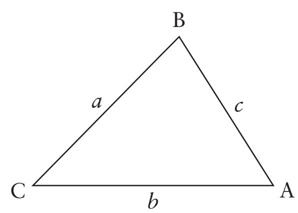
b) Check your answer.

26. a) Use one of the equivalent trigonometric expressions from questions 15 to 19 to develop a problem similar to question 24 or 25.

b) Exchange problems with a classmate. Solve each other's problem.

c) Return the solutions, and check. Resolve any concerns.

27. Given △ABC with sides a, b, and c, show that the area of the triangle is given by $A = \frac{a^2 \sin B \sin C}{2\sin(B + C)}$.



28. Math Contest Refer to Section 4.1, question 27.

a) Determine the polar coordinates, (r, θ), of each point given in Cartesian coordinates. Use the domain $-\pi \le \theta \le \pi$.

i) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **ii)** $\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$

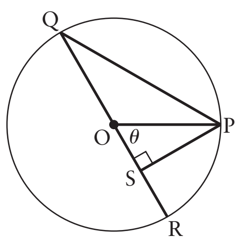
b) Determine the Cartesian coordinates, (x, y), of each point given in polar form.

i) $\left(3, \frac{\pi}{6}\right)$ **ii)** $(4, 3\pi)$

c) Give the general form of all terminal rays that pass from O (the origin) through point A, where

i) $\theta = \frac{\pi}{6}$ **ii)** $\theta = -\frac{\pi}{3}$

29. Math Contest Use the unit circle shown to prove that $\tan\frac{1}{2}\theta = \frac{\sin\theta}{1 + \cos\theta}$.



CAREER CONNECTION

Rapid advances in computer technology are mainly a result of the research efforts of computer hardware engineers. Lucas completed a 4-year bachelor's degree in computer engineering at McMaster University. This program gave him the skills to research, design, develop, and test computer equipment for scientific use. In his job as a computer hardware engineer, he does research in the field of neural networks and assists in the design of robots that mimic the behaviour of the human brain. Lucas needs to have an expert understanding of how all parts of a computer work. A good knowledge of trigonometry is vital when designing an efficient robot.

4.4

Compound Angle Formulas

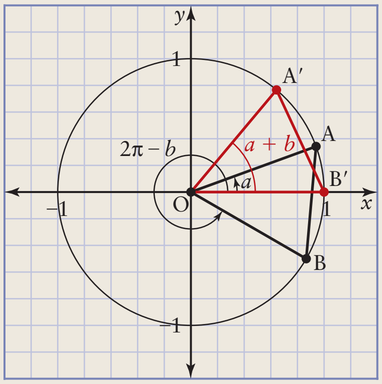
The electromagnetic waves that are used to broadcast radio and television signals can be modelled using trigonometric expressions. Since these waves are moving away from the broadcasting antenna, they depend on both position x and time t. A possible modelling expression looks like $\sin(x + 2\pi ft)$. A trigonometric expression that depends on two or more angles is known as a **compound angle expression**. In this section, you will see how compound angle formulas are developed and learn how to apply them.

Investigate — How can you verify a compound angle formula?

1. The compound angle addition formula for cosine is $\cos(x + y) = \cos x \cos y - \sin x \sin y$. Suppose that $x = \frac{\pi}{6}$ and $y = \frac{\pi}{3}$. Substitute these angles into the formula. Simplify the expression for $x + y$.
2. Use your Special Angles and Trigonometric Ratios table from Section 4.2 to evaluate the trigonometric ratios and simplify both sides. Is the formula true for these values of x and y?
3. a) Suppose that $x = \frac{\pi}{5}$ and $y = \frac{\pi}{7}$. Substitute these angles into the formula given in step 1.
   b) Use a calculator to evaluate each side of the formula. Is the formula true for these values of x and y?
4. Repeat step 3 with $x = \frac{2\pi}{9}$ and $y = \frac{\pi}{8}$.
5. **Reflect** Based on your results, can you conclude that the formula is valid for any choice of angles? Justify your answer.
6. **Reflect** It is tempting to conjecture that $\cos(x + y) = \cos x + \cos y$. Show that this is not generally true for angles x and y.

Addition Formula for Cosine

The unit circle can be used to show that the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$ is valid for all angles. Consider the unit circle shown.



Point A is on the terminal arm of angle a, and point B is on the terminal arm of angle $2\pi - b$.

Rotate OA and OB counterclockwise about the origin through an angle b. Point A′ is on the terminal arm of angle $a + b$, and point B′ is on the x-axis.

Join A to B and A′ to B′.

The coordinates of the four points are
A$(\cos a, \sin a)$
A′$(\cos (a + b), \sin (a + b))$
B′$(1, 0)$
B$(\cos (2\pi - b), \sin (2\pi - b))$

You can use equivalent trigonometric expressions from Section 4.3 to write the coordinates of point B as
B$(\cos b, -\sin b)$

Since lengths are preserved under a rotation, A′B′ = AB.

Apply the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y^2 - y_1)^2}$ to A′B′ and AB.

$$\sqrt{[\cos (a + b) - 1]^2 + [\sin (a + b) - 0]^2} = \sqrt{(\cos a - \cos b)^2 + [\sin a - (-\sin b)]^2}$$

$$[\cos(a + b) - 1]^2 + [\sin(a + b) - 0]^2 = (\cos a - \cos b)^2 + [\sin a - (-\sin b)]^2$$ Square both sides.

$$\cos^2(a + b) - 2\cos (a + b) + 1 + \sin^2(a + b) = \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a + 2\sin a \sin b + \sin^2 b$$ Expand the binomials.

$$\sin^2(a + b) + \cos^2(a + b) + 1 - 2\cos (a + b) = \sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b - 2\cos a \cos b + 2\sin a \sin b$$ Rearrange the terms.

$$1 + 1 - 2\cos (a + b) = 1 + 1 - 2\cos a \cos b + 2\sin a \sin b$$ Apply the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$.

$$-2\cos (a + b) = -2\cos a \cos b + 2\sin a \sin b$$

$$\cos (a + b) = \cos a \cos b - \sin a \sin b$$ Divide both sides by –2.

The addition formula for cosine is usually written as $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

Subtraction Formula for Cosine

The subtraction formula for cosine can be derived from the addition formula for cosine.

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x + (-y)) = \cos x \cos (-y) - \sin x \sin (-y)$$ Substitute $-y$ for y.

$$\cos (x - y) = \cos x \cos (2\pi - y) - \sin x \sin (2\pi - y)$$ From the unit circle, angle $-y$ is the same as angle $(2\pi - y)$.

$$\cos (x - y) = \cos x \cos y - \sin x(-\sin y)$$ $\cos (2\pi - y) = \cos y$; $\sin (2\pi - y) = -\sin y$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

Addition Formula for Sine

Recall the cofunction identities $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$.

Apply these and the subtraction formula for cosine.

$$\sin(x + y) = \cos\left[\frac{\pi}{2} - (x + y)\right] \quad \text{Apply a cofunction identity.}$$

$$= \cos\left[\left(\frac{\pi}{2} - x\right) - y\right] \quad \text{Regroup the terms in the argument.}$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y \quad \text{Apply the subtraction formula for cosine.}$$

$$= \sin x \cos y + \cos x \sin y \quad \text{Apply cofunction identities.}$$

Subtraction Formula for Sine

The subtraction formula for sine can be derived from the addition formula for sine, following the approach used for the subtraction formula for cosine.

$$\sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y) \quad \text{Substitute } -y \text{ for } y.$$

$$\sin(x - y) = \sin x \cos y + \cos x(-\sin y)$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Example 1 Test Compound Angle Formulas

a) Show that the formula $\cos(x - y) = \cos x \cos y + \sin x \sin y$ is true for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

b) Show that the formula $\sin(x + y) = \sin x \cos y + \cos x \sin y$ is true for $x = \frac{\pi}{2}$ and $y = \frac{3\pi}{4}$.

Solution

a) Consider the left side and right side separately and verify that they are equal.

$$\text{L.S.} = \cos(x - y) = \cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{6} - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\text{R.S.} = \cos x \cos y + \sin x \sin y = \cos\frac{\pi}{3}\cos\frac{\pi}{6} + \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

L.S. = R.S.

The formula is valid for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

b)

$$\begin{aligned}\text{L.S.} &= \sin(x + y) \\ &= \sin\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) \\ &= \sin\left(\frac{2\pi}{4} + \frac{3\pi}{4}\right) \\ &= \sin\left(\frac{5\pi}{4}\right) \\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \sin x \cos y + \cos x \sin y \\ &= \sin\frac{\pi}{2}\cos\frac{3\pi}{4} + \cos\frac{\pi}{2}\sin\frac{3\pi}{4} \\ &= 1 \times \left(-\frac{1}{\sqrt{2}}\right) + 0 \times \frac{1}{\sqrt{2}} \\ &= -\frac{1}{\sqrt{2}} + 0 \\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

$$\text{L.S.} = \text{R.S.}$$

The formula is valid for $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$.

Example 2 Determine Exact Trigonometric Ratios for Angles Other Than Special Angles

a) Use an appropriate compound angle formula to determine an exact value for $\sin\frac{\pi}{12}$.

b) Check your answer using a calculator.

Solution

a) First, express the non-special angle as a sum or difference of two special angles. In this case,

$$\begin{aligned}\frac{\pi}{3} - \frac{\pi}{4} &= \frac{4\pi}{12} - \frac{3\pi}{12} \\ &= \frac{\pi}{12}\end{aligned}$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$ Write the given angle as a difference.

$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$ Apply the subtraction formula for sine.

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$ Simplify.

b) Use a calculator to determine that $\sin\frac{\pi}{12} \doteq 0.2588$ and $\frac{\sqrt{3} - 1}{2\sqrt{2}} \doteq 0.2588$. So, the answer in part a) checks.

KEY CONCEPTS

- You can develop compound angle formulas using algebra and the unit circle.
- Once you have developed one compound angle formula, you can develop others by applying equivalent trigonometric expressions.
- The compound angle, or addition and subtraction, formulas for sine and cosine are

  $\sin(x + y) = \sin x \cos y + \cos x \sin y$

  $\sin(x - y) = \sin x \cos y - \cos x \sin y$

  $\cos(x + y) = \cos x \cos y - \sin x \sin y$

  $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- You can apply compound angle formulas to determine exact trigonometric ratios for angles that can be expressed as sums or differences of special angles.

Communicate Your Understanding

C1 In the proof on page XXX, angle $(a + b)$ was drawn in the first quadrant. Would it make any difference to the development of the compound angle formula if the angle was in the second quadrant? If so, explain what difference it would make. If not, explain why not.

C2 It is sometimes tempting to conclude that $\sin(x + y) = \sin x + \sin y$. Identify one pair of angles x and y that shows that it is not generally true. Explain why it is not generally true.

C3 Are there any angles x and y that satisfy $\sin(x + y) = \sin x + \sin y$? If so, determine one example, and show that it works. If not, explain why not.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. Use an appropriate compound angle formula to express as a single trigonometric function, and then determine an exact value for each.

a) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

b) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} - \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

c) $\cos\frac{\pi}{4}\cos\frac{\pi}{12} - \sin\frac{\pi}{4}\sin\frac{\pi}{12}$

d) $\cos\frac{\pi}{4}\cos\frac{\pi}{12} + \sin\frac{\pi}{4}\sin\frac{\pi}{12}$

2. Use an appropriate compound angle formula to express as a single trigonometric function, and then determine an exact value for each.

a) $\sin\frac{3\pi}{5}\cos\frac{\pi}{15} + \cos\frac{3\pi}{5}\sin\frac{\pi}{15}$

b) $\sin\frac{7\pi}{5}\cos\frac{\pi}{15} - \cos\frac{7\pi}{5}\sin\frac{\pi}{15}$

c) $\cos\frac{2\pi}{9}\cos\frac{5\pi}{18} - \sin\frac{2\pi}{9}\sin\frac{5\pi}{18}$

d) $\cos\frac{10\pi}{9}\cos\frac{5\pi}{18} + \sin\frac{10\pi}{9}\sin\frac{5\pi}{18}$

3. Apply a compound angle formula, and then determine an exact value for each.

a) $\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ b) $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

c) $\cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$ d) $\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$

For help with questions 4 to 7, refer to Example 2.

4. Use an appropriate compound angle formula to determine an exact value for each.

a) $\sin\frac{7\pi}{12}$ b) $\sin\frac{5\pi}{12}$

5. Use an appropriate compound angle formula to determine an exact value for each.

a) $\cos\frac{11\pi}{12}$ b) $\cos\frac{5\pi}{12}$

6. Use an appropriate compound angle formula to determine an exact value for each.

a) $\sin\frac{13\pi}{12}$ b) $\cos\frac{17\pi}{12}$

7. Use an appropriate compound angle formula to determine an exact value for each.

a) $\sin\frac{19\pi}{12}$ b) $\cos\frac{23\pi}{12}$

B Connect and Apply

8. Angles x and y are located in the first quadrant such that $\sin x = \frac{3}{5}$ and $\cos y = \frac{5}{13}$.

a) Determine an exact value for $\cos x$.

b) Determine an exact value for $\sin y$.

9. Refer to question 8. Determine an exact value for each of the following.

a) $\sin(x + y)$

b) $\sin(x - y)$

c) $\cos(x + y)$

d) $\cos(x - y)$

10. Angle x is in the second quadrant and angle y is in the first quadrant such that $\sin x = \frac{5}{13}$ and $\cos y = \frac{3}{5}$.

a) Determine an exact value for $\cos x$.

b) Determine an exact value for $\sin y$.

11. Refer to question 10. Determine an exact value for each of the following.

a) $\sin(x + y)$

b) $\sin(x - y)$

c) $\cos(x + y)$

d) $\cos(x - y)$

12. Use a compound angle formula for sine to show that $\sin 2\theta = 2\sin\theta\cos\theta$.

13. Use a compound angle formula for cosine to show that $\cos 2x = \cos^2 x - \sin^2 x$.

CONNECTIONS

The formulas in questions 12 and 13 are known as the double angle formulas.

14. Use the Pythagorean identity to show that the double angle formula for cosine can be written as

a) $\cos 2x = 1 - 2\sin^2 x$

b) $\cos 2x = 2\cos^2 x - 1$

15. Angle θ lies in the second quadrant and $\sin\theta = \frac{7}{25}$.

a) Determine an exact value for $\cos 2\theta$.

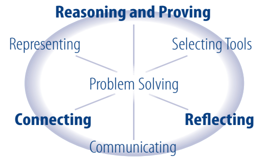
b) Determine an exact value for $\sin 2\theta$.

c) Use a calculator to determine an approximate measure for θ in radians.

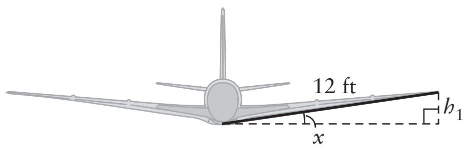
d) Use a calculator to check your answers to parts a) and b).

16. **Use Technology** Use a graphing calculator to check the formulas in questions 12, 13, and 14 by graphing the left side and then the right side. Refer to the method used in Section 4.3, Example 3.

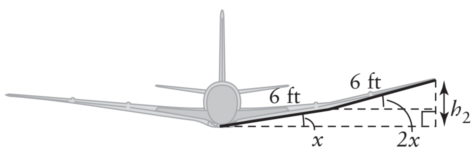
17. Chapter Problem The wings on an airplane are usually angled upward relative to the body, or fuselage, of the aircraft. This is known as a dihedral, and makes the aircraft more stable in flight, especially during turbulence.



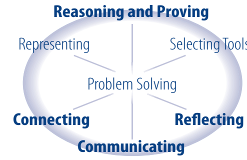
a) An aircraft wing is 12 ft long and has a dihedral angle x. Determine an expression for the vertical displacement, h_1, above the bottom of the fuse lage, in terms of x.



b) Some aircraft have a double dihedral. Suppose that a wing is designed as shown, with an initial dihedral angle of x for the first 6 ft, and an additional angle x for the next 6 ft. Show that $h_2 = 6\sin x(1 + 2\cos x)$.



18. At the winter solstice (on or about December 21), the power, in watts, received from the Sun on each square metre of Earth's surface can be modelled by the formula $P = 1000\sin(x + 113.5°)$, where x represents the angle of latitude in the northern hemisphere.



a) Determine the angle of latitude at which the power level drops to 0. Suggest a reason why this happens at this latitude. Justify your answer.

b) Determine the angle of latitude at which the power level is a maximum. Explain what is meant by the negative sign. Suggest a reason why this happens at this latitude. Justify your answer.

✓ Achievement Check

19. The angle $2x$ lies in the fourth quadrant such that $\cos 2x = \frac{8}{17}$.

a) Sketch the location of angle $2x$.

b) Which quadrant contains angle x?

c) Determine an exact value for $\cos x$.

d) Use a calculator to determine the measure of x, in radians.

e) Use a calculator to verify your answer for part c).

C Extend and Challenge

20. Recall the quotient identity $\tan x = \frac{\sin x}{\cos x}$.

a) Use the quotient identity and the compound angle formulas for $\sin(x + y)$ and $\cos(x + y)$ to write a formula for $\tan(x + y)$.

b) Show that the formula in part a) can be written as $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

c) Use your Special Angles and Trigonometric Ratios table from Section 4.2 to show that the formula in part b) is valid for $x = \frac{2\pi}{3}$ and $y = \frac{\pi}{6}$.

21. a) Use the formula in question 20, part b), and appropriate equivalent trigonometric expressions to show that $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$.

b) Use your Special Angles and Trigonometric Ratios table from Section 4.2 to show that the formula in part a) is valid for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

22. a) Use the formula from question 20, part b), to derive a formula for $\tan 2x$.

b) **Use Technology** Use a graphing calculator to check the formula in part a) by graphing the left side and then the right side.

c) Use a calculator to show that the formula in part a) is valid for $x = 0.52$.

23. The compound angle formulas can be used to derive many other formulas. One of these is the sum formula
$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

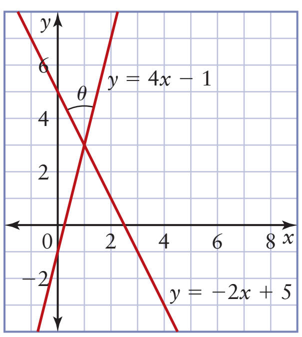
a) Show that the formula is valid for $x = \frac{2\pi}{3}$ and $y = \frac{\pi}{3}$.

b) Use appropriate equivalent trigonometric expressions to derive a similar formula for $\sin x - \sin y$.

24. a) Derive the half-angle formula for sine, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$, from a double angle formula for cosine.

b) Derive the half-angle formula for cosine, $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$, from a double angle formula for cosine.

25. Math Contest Find the acute angle θ of intersection between the lines $y = 4x - 1$ and $y = -2x + 5$.



26. Math Contest If A, B, and C are angles in a triangle, by expanding $\tan(A + B + C)$, show that $\tan A \tan B \tan C = \tan A + \tan B + \tan C$.

27. Math Contest An interesting area of mathematics revolves around the study of approximation. An example is the approximation of π in the form of a rational number, $\pi \doteq \frac{22}{7}$. Both sine and cosine have approximations in polynomial form:
$\sin\theta \doteq \theta - \frac{\theta^3}{6}$ and $\cos\theta \doteq 1 - \frac{\theta^2}{2}$, where θ is an angle measure in radians ($\theta \in \mathbb{R}$) and is a value near zero.

a) Calculate each value, and then copy and complete the table.

θ	0.01	0.05	0.10	0.15	0.25	0.35
$\theta - \frac{\theta^3}{6}$						
$\sin\theta$						

b) Calculate each value, and then copy and complete the table.

θ	0.01	0.05	0.10	0.15	0.25	0.35
$1 - \frac{\theta^2}{2}$						
$\cos\theta$						

28. Math Contest Trigonometric functions can be approximated by using an expanded power series, where θ is measured in radians and is close to zero. Determine an approximation in simplified form for $\sin\theta$, $\cos\theta$, and $\tan\theta$ given the equivalent power series for each.

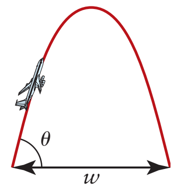
a) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

b) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$

c) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \cdots$

4.5 Prove Trigonometric Identities

To simulate a weightless environment to train astronauts, an aircraft is flown along a trajectory in the shape of a parabolic arc. The width of the parabola to be flown can be modelled by the relation $w = \frac{2v^2}{g}\sin\theta\cos\theta$, where v is the speed of the aircraft entering the parabola, in metres per second; g is the acceleration due to gravity, 9.8 m/s^2; and θ is the initial climb angle selected by the pilot of the aircraft.



This equation can be simplified using the double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$ to get $w = \frac{2v^2}{g}\sin 2\theta$.

This form is easier to work with. For example, suppose that the pilot wishes to maximize the width of the parabola flown. The maximum value of $\sin 2\theta$ is 1, when $2\theta = \frac{\pi}{2}$, or $\theta = \frac{\pi}{4}$. The pilot should use an initial climb angle of $\frac{\pi}{4}$ to maximize the width of the parabola flown.

CONNECTIONS

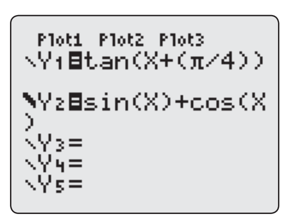
A company called Zero-G, based in Florida, offers the weightless experience to the general public on a modified Boeing 727 aircraft. Go to *www.mcgrawhill.ca/links/functions12* and follow the links to get more information on Zero-G.

Investigate — How can you determine whether an equation is an identity?

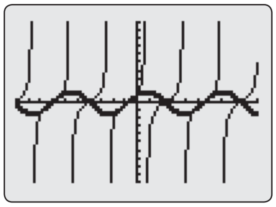
Tools

- graphing calculator

1. Consider the equation $\tan\left(x + \frac{\pi}{4}\right) = \sin x + \cos x$. Substitute $x = 0$ in both sides. Is the equation true for this value of x?
2. **Reflect** Can you conclude that the equation is an identity? Justify your answer.
3. One way to show that an equation is not an identity is to find a value of x for which the equation is not true. This is known as finding a **counterexample**. Use your Special Angles and Trigonometric Ratios table from Section 4.2 to determine a counterexample for the equation in step 1.
4. Another way to check is to graph each side of the equation individually using a graphing calculator.
   a) Ensure that your calculator is set to Radian mode and all stat plots are turned off.
   b) Use the **Y=** editor to enter $\tan\left(x + \frac{\pi}{4}\right)$ in **Y1** and $\sin x + \cos x$ in **Y2**.
   c) Change the line display for **Y2** to heavy.



d) From the **ZOOM** menu, select **6:ZStandard**. The graph of $\tan\left(x + \frac{\pi}{4}\right)$ will be drawn first with a light line. Then, the graph of $\sin x + \cos x$ will be drawn with a heavier line. You can pause the plot by pressing ENTER. Pressing ENTER again will resume the plot.



5. Reflect

a) How do the graphs demonstrate that the equation is not an identity?

b) Take note of the points where the graphs cross. What do these points signify?

6. a) Consider the equation $\sin x = \cos x \tan x$. Is the equation true for $x = \frac{\pi}{4}$?

b) Attempt to find a counterexample to show that the equation is not an identity. Report your findings.

c) Graph both sides of the equation on a graphing calculator.

7. Reflect Does the equation in step 6 appear to be an identity? Justify your answer.

Graphing both sides of an equation can make an equation appear to be an identity. However, this approach does not constitute a proof. Some graphs appear identical in a particular domain, but then diverge when the domain is extended. To prove an identity, show that one side of the equation is equivalent to the other side. Start by writing down the left side (**L.S.**) and right side (**R.S.**) separately. Then, transform the expression for one side into the exact form of the expression on the other side.

In the remainder of this section, you will use the basic trigonometric identities shown to prove other identities.

Pythagorean Identity

$\sin^2 x + \cos^2 x = 1$

Quotient Identity

$\tan x = \frac{\sin x}{\cos x}$

Reciprocal Identities

$\csc x = \frac{1}{\sin x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \cot x = \frac{1}{\tan x}$

Compound Angle Formulas

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

Example 1 Use a Compound Angle Formula to Prove a Double Angle Formula

a) Prove that $\sin 2x = 2\sin x \cos x$.

b) Illustrate the identity by graphing.

Solution

a) L.S. $= \sin 2x$ R.S. $= 2\sin x \cos x$

$= \sin(x + x)$

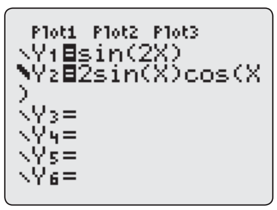
$= \sin x \cos x + \cos x \sin x$

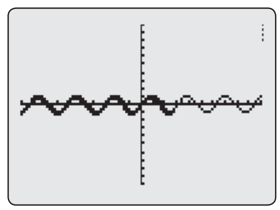
$= 2\sin x \cos x$

L.S. = R.S.

Therefore, $\sin 2x = 2\sin x \cos x$ is an identity.

b) Use the **Y=** editor of a graphing calculator to enter sin $2x$ in **Y1** and $2\sin x \cos x$ in **Y2**. Then, observe the graphs being drawn.





Example 2 Use a Compound Angle Formula to Prove a Cofunction Identity

Prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Solution

L.S. $= \cos\left(\frac{\pi}{2} - x\right)$ R.S. $= \sin x$

$= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 \times \cos x + 1 \times \sin x$

$= \sin x$

L.S. = R.S.

Therefore, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ is an identity.

Example 3 Prove an Identity Involving Reciprocal Trigonometric Ratios

Prove that $\csc 2x = \dfrac{\csc x}{2\cos x}$.

Solution

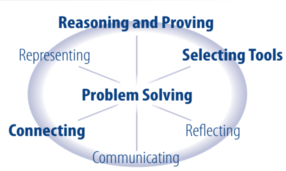
$$
\begin{aligned}
\text{L.S.} &= \csc 2x \\
&= \frac{1}{\sin 2x} && \text{Use a reciprocal identity.} \\
&= \frac{1}{2\sin x\cos x} && \text{Use the double angle formula from Example 1.} \\
&= \frac{1}{2} \times \frac{1}{\sin x} \times \frac{1}{\cos x} && \text{Write as separate rational expressions.} \\
&= \frac{1}{2} \times \csc x \times \frac{1}{\cos x} && \text{Use a reciprocal identity.} \\
&= \frac{\csc x}{2\cos x} && \text{Combine into one term.} \\
\text{R.S.} &= \frac{\csc x}{2\cos x}
\end{aligned}
$$

L.S. = R.S.

Therefore, $\csc 2x = \dfrac{\csc x}{2\cos x}$ is an identity.

Example 4 Prove a More Complex Identity

Prove $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$.



Solution

$$
\begin{aligned}
\text{L.S.} &= \cos(x + y)\cos(x - y) \\
&= (\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y) && \text{Use the compound angle formulas for cosine.} \\
&= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y && \text{Expand the binomials and collect like terms.} \\
&= \cos^2 x\cos^2 y - (1 - \cos^2 x)(1 - \cos^2 y) && \text{Use the Pythagorean identity.} \\
&= \cos^2 x\cos^2 y - (1 - \cos^2 y - \cos^2 x + \cos^2 x\cos^2 y) && \text{Expand the binomials.} \\
&= \cos^2 x\cos^2 y - 1 + \cos^2 y + \cos^2 x - \cos^2 x\cos^2 y \\
&= \cos^2 y + \cos^2 x - 1 && \text{Collect like terms.} \\
\text{R.S.} &= \cos^2 x + \cos^2 y - 1
\end{aligned}
$$

L.S. = R.S.

Therefore, $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$ is an identity.

KEY CONCEPTS

- A trigonometric identity is an equation with trigonometric expressions that is true for all angles in the domain of the expressions on both sides.
- One way to show that an equation is not an identity is to determine a counterexample.
- To prove that an equation is an identity, treat each side of the equation independently and transform the expression on one side into the exact form of the expression on the other side.
- The basic trigonometric identities are the Pythagorean identity, the quotient identity, the reciprocal identities, and the compound angle formulas. You can use these identities to prove more complex identities.
- Trigonometric identities can be used to simplify solutions to problems that result in trigonometric expressions. This is important in understanding solutions for problems in mathematics, science, engineering, economics, and other fields.

Communicate Your Understanding

C1 Explain the difference between a general equation that involves trigonometric expressions and an equation that is an identity.

C2 **a)** Show that $\sin^2 2x = 4\sin x\cos x - 1$ is true for $x = \frac{\pi}{4}$.

b) How can you determine whether the equation in part a) is an identity? Carry out your method, and report on your findings.

C3 Construct a trigonometric equation that is true for at least one value of x. Then, determine whether or not it is an identity. Justify your conclusion.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Prove that $\cos 2x = 2\cos^2 x - 1$.

2. Prove that $\cos 2x = 1 - 2\sin^2 x$.

For help with questions 3 to 6, refer to Example 2.

3. Prove that $\sin(x + \pi) = -\sin x$.

4. Prove that $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$.

5. Prove that $\cos(\pi - x) = -\cos x$.

6. Prove that $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$.

For help with questions 7 and 8, refer to Example 3.

7. Prove that $\cos x = \sin x \cot x$.

8. Prove that $1 + \sin x = \sin x(1 + \csc x)$.

B Connect and Apply

9. a) Prove that
$1 - 2\cos^2 x = \sin x \cos x (\tan x - \cot x)$.

b) **Use Technology** Illustrate the identity by graphing with technology.

10. a) Prove that $\csc^2 x = 1 + \cot^2 x$.

b) Prove that $\sec^2 x = 1 + \tan^2 x$.

11. Prove that $\dfrac{1 - \sin^2 x}{\cos x} = \dfrac{\sin 2x}{2\sin x}$.

12. Prove that $\dfrac{\csc^2 x - 1}{\csc^2 x} = 1 - \sin^2 x$.

13. Prove that $\dfrac{\csc x}{\cos x} = \tan x + \cot x$.

14. The software code that renders three-dimensional images onto a two-dimensional screen uses trigonometric expressions.

Any simplification that can be made to the calculations decreases processing time and smoothes the motion on the screen. Trigonometric identities can be used to simplify calculations. To get a feel for the usefulness of identities, you will time how long it takes to calculate each side of the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

a) Let $x = \dfrac{\pi}{12}$ and $y = \dfrac{\pi}{8}$. Ask a classmate to time how long it takes you to evaluate $\sin(x + y)$ using a calculator.

b) Determine how long it takes you to evaluate $\sin x \cos y + \cos x \sin y$.

c) Compare the times in parts a) and b). Express the time saved as a percent.

d) Let $x = \dfrac{3\pi}{4}$. How much time can you save using $\sin 2x$ rather than $2\sin x \cos x$?

15. Prove that $2\sin x \sin y = \cos(x - y) - \cos(x + y)$.

16. Prove that
$\sin 2x + \sin 2y = 2\sin(x + y)\cos(x - y)$.

17. a) **Use Technology** Use graphing technology to determine whether it is reasonable to conjecture that $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$ is an identity.

b) If the equation is an identity, prove that it is an identity. If it is not, determine one counterexample.

18. a) Show that $\sin x = \sqrt{1 - \cos^2 x}$ is not an identity.

b) Explain why it cannot be an identity.

Achievement Check

19. Use an appropriate compound angle formula to determine an expression for $\sin 3x$ in terms of $\sin x$ and $\cos x$.

a) Write $3x$ as the sum of two terms involving x.

b) Substitute the expression from part a) into $\sin 3x$.

c) Use an appropriate compound angle formula to expand the expression from part b).

d) Use an appropriate double angle formula to expand any terms involving $2x$ in the expanded expression from part c). Simplify as much as possible.

e) Select a value for x, and show that your expression in part d) holds true for that value.

f) **Use Technology** Use graphing technology to illustrate your identity.

C Extend and Challenge

20. a) **Use Technology** Use graphing technology to determine whether it is reasonable to conjecture that
$\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$
is an identity.

b) If it appears to be an identity, prove the identity. If not, determine a counterexample.

21. Prove that $\cos^4 x - \sin^4 x = \cos 2x$.

22. Refer to question 20 in Section 4.4. Use the addition formula for tangents to derive the double angle formula for tangents.

23. **Math Contest** Prove.

a) $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

b) $\cos^{-1} a + \cos^{-1} b$
$= \cos^{-1}\left(ab - \sqrt{1 - a^2}\sqrt{1 - b^2}\right)$

Extension

Use *The Geometer's Sketchpad*® to Sketch and Manipulate Three-Dimensional Structures in a Two-Dimensional Representation

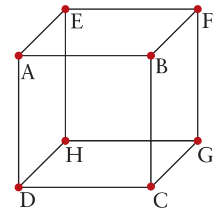
Tools

- computer with *The Geometer's Sketchpad*®

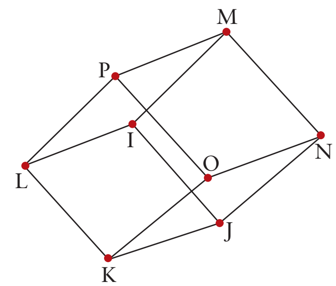
You can use *The Geometer's Sketchpad*® to get a feel for the operation of the kind of software used for engineering design or computer graphics imaging. This software makes heavy use of trigonometry to render three-dimensional objects on a two-dimensional computer screen, as you have seen in some parts of the Chapter Problem.

Although *The Geometer's Sketchpad*® is a two-dimensional program, it can be used to represent and animate three-dimensional representations in a limited way.

Part 1: *Make an Object Fly and Turn*



1. Open *The Geometer's Sketchpad*®. Use the **Line Tool** to construct a representation of a cube, as shown, near the upper left of the work area.
2. Construct a quadrilateral interior for the left (ADHE) and right (BCGF) faces and assign each a different colour.
3. You will move the structure to the lower right corner of the work space, causing it to rotate and dilate on the way. Construct a representation of the desired final form of the structure, as shown.



CONNECTIONS

If the final object is different from the initial object, the initial object will appear to transform into the final object. In computer graphics imaging, this is known as *morphing*.

4. Point A will move to point I, B to J, and so on. To achieve this, select the points in the order A, I, B, J, C, K, D, L, E, M, F, N, G, O, H, and P. This process maps point A to point I, point B to point J, and so on.
5. From the **Edit** menu, choose **Action Buttons**. Choose **Movement…**. Click on **OK**. Select the button marked **Move Points**. From the **Edit** menu, choose **Properties**, and then choose the **Label** tab. Change the label to **Fly**. Click on **OK**.
6. Hide the destination points I to P, as well as the line segments that join them.
7. Click on the button labelled **Fly**. Observe the movement. Notice that the original structure appears to move toward you as it rotates.
8. To return the structure to the starting location, hold the Ctrl key, and press **Z**.

Part 2: *Fold Up a Box*

1. Start with the representation of the cube from Part 1. Map A to F, B to E, D to G, and C to H.

2. From the **Edit** menu, choose **Action Buttons**. Choose **Movement….** Click on **OK**. Right-click on the button marked **Move Points**, and change the label to **Fold**.

3. Click on the button labelled **Fold**. Observe the movement.

Part 3: *Combine Actions*

You can combine the actions in Parts 1 and 2. You can make them occur simultaneously or sequentially.

1. Select the **Fly** button and the **Fold** button.

2. From the **Edit** menu, choose **Action Buttons** and then **Presentation….** Ensure that the **Simultaneously** radio button is selected. Click on **OK**. Rename the button **Simultaneous**.

3. Click on the **Simultaneous** button. Observe what happens.

4. Repeat steps 1 and 2, but this time select the **Sequentially** radio button. Rename the button **Sequential**.

5. Click on the **Sequential** button. Observe what happens.

Part 4: *Experiment*

1. Delete all points, segments, and buttons except the original cube.

2. Construct two different sets of destination points. Create an action button for each set.

3. What happens if you try to combine the action buttons simultaneously? sequentially?

4. Drag your sets of destination points to form patterns of interest to you. Experiment with the action buttons to produce interesting movements.

5. Create a third and fourth set of destination points, as well as action buttons. Experiment with different combinations of movements.

6. Use *The Geometer's Sketchpad*® to represent and manipulate a three-dimensional object of your own design. Features of your sketch can include, but are not limited to,
   - rotations
   - dilations
   - translations
   - colour
   - multiple movements
   - simultaneous and sequential movements
   - morphing to a different shape

7. **Reflect** What are some of the limitations of a two-dimensional program like *The Geometer's Sketchpad*® in representing three-dimensional objects? Describe some of the features that a true three-dimensional dynamic geometry program would need.

CONNECTIONS

One of the first intensive applications of computer graphics imaging was the science fiction television series Babylon 5, which first aired in the 1990s. The graphics designers used the engineering design program AutoCad® to render spaceships, space stations, and even aliens.

Chapter 4 REVIEW

4.1 Radian Measure

1. Determine the approximate radian measure, to the nearest hundredth, for each angle.

a) 33° **b)** 138° **c)** 252° **d)** 347°

2. Determine the approximate degree measure, to the nearest tenth, for each angle.

a) 1.24 **b)** 2.82 **c)** 4.78 **d)** 6.91

3. Determine the exact radian measure of each angle.

a) 75° **b)** 20° **c)** 12° **d)** 9°

4. Determine the exact degree measure of each angle.

a) $\frac{2\pi}{5}$ **b)** $\frac{4\pi}{9}$ **c)** $\frac{7\pi}{12}$ **d)** $\frac{11\pi}{18}$

5. The turntable in a microwave oven rotates 12 times per minute while the oven is operating. Determine the angular velocity of the turntable in

a) degrees per second

b) radians per second

6. Turntables for playing vinyl records have four speeds, in revolutions per minute (rpm): 16, $33\frac{1}{3}$, 45, and 78. Determine the angular velocity for each speed in

a) degrees per second

b) radians per second

Summarize your results in a table.

4.2 Trigonometric Ratios and Special Angles

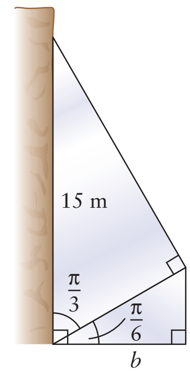
7. Use a calculator to determine the six trigonometric ratios of the angle $\frac{4\pi}{11}$. Round your answers to four decimal places.

8. Determine an exact value for each expression.

a) $\dfrac{\cot\frac{\pi}{4}}{\cos\frac{\pi}{3}\csc\frac{\pi}{2}}$

b) $\cos\frac{\pi}{6}\csc\frac{\pi}{3} + \sin\frac{\pi}{4}$

9. A ski lodge is constructed with one side along a vertical cliff such that it has a height of 15 m, as shown. Determine an exact measure for the base of the lodge, b.



4.3 Equivalent Trigonometric Expressions

10. Given that $\cot\frac{2\pi}{7} = \tan z$, first express $\frac{2\pi}{7}$ as a difference between $\frac{\pi}{2}$ and an angle, and then apply a cofunction identity to determine the measure of angle z.

11. Given that $\cos\frac{5\pi}{9} = \sin y$, first express $\frac{5\pi}{9}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle y.

12. Given that $\tan\frac{4\pi}{9} \doteq 5.6713$, determine the following, to four decimal places, without using a calculator. Justify your answers.

a) $\cot\frac{\pi}{18}$ **b)** $\tan\frac{13\pi}{9}$

13. Given that $\sin x = \cos\frac{3\pi}{11}$ and that x lies in the second quadrant, determine the measure of angle x.

4.4 Compound Angle Formulas

14. Use an appropriate compound angle formula to express as a single trigonometric function, and then determine an exact value of each.

a) $\sin\frac{5\pi}{12}\cos\frac{\pi}{4} + \cos\frac{5\pi}{12}\sin\frac{\pi}{4}$

b) $\sin\frac{5\pi}{12}\cos\frac{\pi}{4} - \cos\frac{5\pi}{12}\sin\frac{\pi}{4}$

c) $\cos\frac{5\pi}{12}\cos\frac{\pi}{4} - \sin\frac{5\pi}{12}\sin\frac{\pi}{4}$

d) $\cos\frac{5\pi}{12}\cos\frac{\pi}{4} + \sin\frac{5\pi}{12}\sin\frac{\pi}{4}$

15. Angles x and y are located in the first quadrant such that $\sin x = \frac{4}{5}$ and $\cos y = \frac{7}{25}$.
   a) Determine an exact value for $\cos x$.
   b) Determine an exact value for $\sin y$.
   c) Determine an exact value for $\sin(x + y)$.

16. Angle x lies in the third quadrant, and $\tan x = \frac{7}{24}$.
   a) Determine an exact value for $\cos 2x$.
   b) Determine an exact value for $\sin 2x$.

17. Determine an exact value for $\cos \frac{13\pi}{12}$.

4.5 Prove Trigonometric Identities

18. Prove that $\sin(2\pi - x) = -\sin x$.

19. a) Prove that $\sec x = \frac{2(\cos x \sin 2x - \sin x \cos 2x)}{\sin 2x}$.
   b) **Use Technology** Use graphing technology to illustrate the identity in part a).

20. Prove that $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$.

21. a) **Use Technology** Use graphing technology to determine whether it is reasonable to conjecture that $\cos 3x = \cos^3 x - 3 \cos x \sin x$.
   b) Outline a method that you could use to develop a formula for $\cos 3x$ in terms of $\sin x$ and $\cos x$.

22. Consider the equation $\cos 2x = 2 \sin x \sec x$. Either prove that it is an identity, or determine a counterexample to show that it is not an identity.

23. Prove that $(\sin 2x)(\tan x + \cot x) = 2$.

CHAPTER PROBLEM WRAP-UP

In Section 4.1, question 18; Section 4.2, question 19; Section 4.3, question 22; and Section 4.4, question 17, you solved problems involving trigonometry in the context of transportation and transportation engineering.

a) A highway enters a city and runs east for 2 km. Then, the highway turns southward through an angle of $\frac{\pi}{12}$ and runs for another 2 km before it leaves the city. To divert through traffic around the city, the highway engineers will design an arc-shaped bypass road with its centre at the bend in the highway and a radius of 2 km.
   i) Sketch a diagram of the old highway and the bypass. Mark all given information.
   ii) Determine the length of the bypass, to the nearest hundredth of a kilometre.

b) The bypass road in part a) runs over a ravine bridged by a cable suspension bridge with a central tower 50 m high. The top of the tower is connected to each end of the bridge by two cables. The angle of inclination of the first cable is x. The angle of inclination of the second cable is twice that of the first cable.
   i) Sketch the bridge, the tower, and the four cables. Mark all given information.
   ii) Show that the total length of the two cables on one side of the central tower is given by the expression $l = \frac{50(1 + 2\cos x)}{\sin 2x}$.

Chapter 4 PRACTICE TEST

For questions 1 to 7, select the best answer.

1. The exact radian measure of 105° is

A $\frac{5\pi}{12}$

B $\frac{7\pi}{12}$

C $\frac{15\pi}{24}$

D $\frac{13\pi}{12}$

2. The exact degree measure of $\frac{13\pi}{24}$ is

A 22.5°

B 75°

C 97.5°

D 142.5°

3. Use a calculator to determine $\sin\frac{15\pi}{17}$, rounded to four decimal places.

A −0.9325

B 0.0484

C 0.3612

D 0.9988

4. Which of these is an equivalent trigonometric expression for $\sin x$?

A $-\cos\left(\frac{\pi}{2} - x\right)$

B $\sin\left(\frac{\pi}{2} - x\right)$

C $\cos\left(x + \frac{\pi}{2}\right)$

D $-\cos\left(x + \frac{\pi}{2}\right)$

5. The exact value of $\cos\frac{5\pi}{3}$ is

A $\frac{\sqrt{3}}{2}$

B $\frac{1}{2}$

C $-\frac{\sqrt{3}}{2}$

D $-\frac{2}{\sqrt{3}}$

6. Given that $\cot\theta = -1$ and angle θ lies in the second quadrant, what is $\sec\theta$?

A $\frac{1}{\sqrt{2}}$

B $-\frac{1}{\sqrt{2}}$

C $-\sqrt{2}$

D $\sqrt{2}$

7. Simplify $\cos\frac{\pi}{5}\cos\frac{\pi}{6} - \sin\frac{\pi}{5}\sin\frac{\pi}{6}$.

A $\cos\frac{11\pi}{30}$

B $\sin\frac{11\pi}{30}$

C $\cos\frac{\pi}{30}$

D $\sin\frac{\pi}{30}$

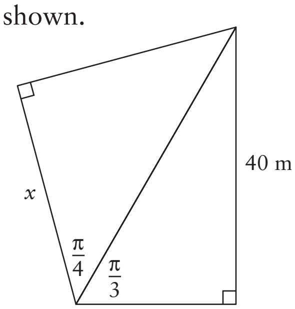
8. a) The Moon orbits Earth every 27.3 days. Determine the angular velocity of the Moon, in degrees per day and in radians per day.

b) The radius of the orbit of the Moon is about 384 400 km. How far does the Moon move along an arc of its orbit every day?

9. Determine an exact value for the expression

$$\frac{\sin\frac{\pi}{3} - \cos\frac{5\pi}{6}}{1 - \tan\frac{3\pi}{4}\cot\frac{\pi}{4}}.$$

10. An irregular building lot in a subdivision has the shape shown.



a) Determine an exact expression for the length of side x.

b) Determine an approximation for x, to the nearest tenth of a metre.

11. Given that $\cos \frac{7\pi}{18} \doteq 0.3420$, determine the following, to four decimal places, without using a calculator. Justify your answers.

a) $\sin \frac{\pi}{9}$

b) $\sin \frac{8\pi}{9}$

12. a) Determine an exact value for $\sin \frac{17\pi}{12}$.

b) Determine the answer to part a) using a different method.

13. Angle x is in the second quadrant and angle y is in the first quadrant such that $\sin x = \frac{7}{25}$ and $\cos y = \frac{5}{13}$.

a) Determine an exact value for $\cos x$.

b) Determine an exact value for $\sin y$.

c) Determine an exact value for $\cos(x - y)$.

14. The manufacturer of a propeller for a small aircraft mandates a maximum operating angular velocity of 300 rad/s. Determine whether it is safe to install this propeller on an aircraft whose engine is expected to run at a maximum of 2800 rpm.

15. A cruise ship sailed north for 50 km. Then, the captain turned the ship eastward through an angle of $\frac{\pi}{6}$ and sailed an additional 50 km. Without using a calculator, determine an exact expression for the direct distance from the start to the finish of the cruise.

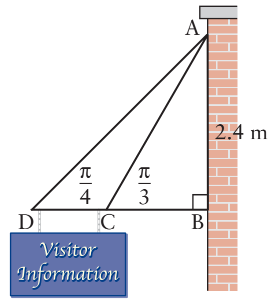
16. Prove that $(\cos x - \sin x)^2 = 1 - \sin 2x$.

17. Prove that $2\cos x \cos y = \cos(x + y) + \cos(x - y)$.

18. Use a counterexample to show that $\cos 2x + \sin 2y = 2\sin(x + y)\cos(x - y)$ is not an identity.

19. Prove that $(\csc x - \cot x)^2 = 1 - \frac{\cos x}{1 + \cos x}$.

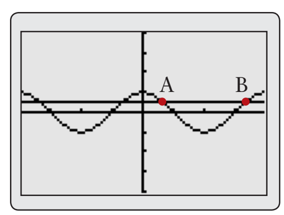
20. A truss is used to hold up a heavy sign on a building, as shown. The truss is attached to the building along side AB, which measures 2.4 m. Determine an exact expression for the length CD.



21. **Use Technology** A graphing calculator or graphing software can be used to deduce equivalent trigonometric expressions.

a) Ensure that your graphing calculator is set to Radian mode. Graph $y = \cos x$. From the **ZOOM** menu, select **7:ZTrig**. Determine the angle that is represented by each tick on the horizontal axis.

b) Plot a horizontal line at $y = 0.5$. Determine the coordinates of two successive points A and B where the line intersects the cosine graph. Deduce an equivalent trigonometric expression for the cosine ratio from these coordinates.

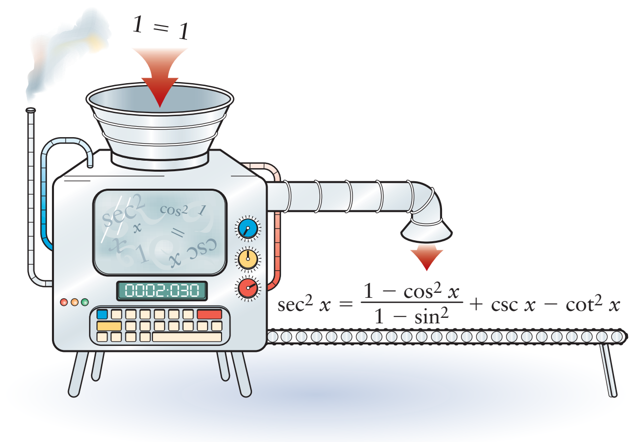


c) Clear the horizontal line graph. Graph $y = \sin x$. Inspect the intersection of the sine and cosine graphs. Use the intersection points to deduce one of the trigonometric identities for sine and cosine.

d) Does this method of deducing the identities constitute a proof of each identity? Justify your answer.

TASK

Make Your Own Identity



Make your own trigonometric identity and then challenge your friends to prove that it is true.

- Start with a statement you know is true for all values of x: for example $\sin^2 x = \sin^2 x$.
- Progressively change each side into an equivalent, but more complex form: for example $1 - \cos^2 x = \cos 2x + \cos^2 x$.
- Continue replacing terms with equivalent expressions until you decide that your identity is sufficiently complex.
- **Use Technology** Graph the left side of your identity with a thin line, and the right side with a thick line. Do the two overlap? How is this evidence, but not proof, that you have an identity?
- Check your work by proving your identity in your notebook.
- Trade identities with a partner, and prove your partner's identity.
- Work with your partner to develop a "Hints For Proving Identities" list.