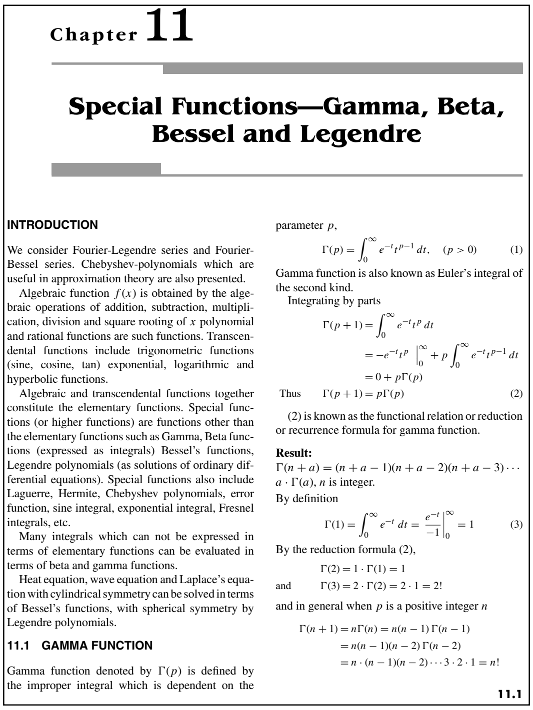
Visual Walkthrough

INTRODUCTION



Chapter 11

Special Functions—Gamma, Beta, Bessel and Legendre

INTRODUCTION

We consider Fourier-Legendre series and Fourier-Bessel series. Chebyshev-polynomials which are useful in approximation theory are also presented.

Algebraic function $f(x)$ is obtained by the algebraic operations of addition, subtraction, multiplication, division and square rooting of x polynomial and rational functions are such functions. Transcendental functions include trigonometric functions (sine, cosine, tan) exponential, logarithmic and hyperbolic functions.

Algebraic and transcendental functions together constitute the elementary functions. Special functions (or higher functions) are functions other than the elementary functions such as Gamma, Beta functions (expressed as integrals) Bessel's functions, Legendre polynomials (as solutions of ordinary differential equations). Special functions also include Laguerre, Hermite, Chebyshev polynomials, error function, sine integral, exponential integral, Fresnel integrals, etc.

Many integrals which can not be expressed in terms of elementary functions can be evaluated in terms of beta and gamma functions.

Heat equation, wave equation and Laplace's equation with cylindrical symmetry can be solved in terms of Bessel's functions, with spherical symmetry by Legendre polynomials.

11.1 GAMMA FUNCTION

Gamma function denoted by $\Gamma(p)$ is defined by the improper integral which is dependent on the parameter p,

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt, \quad (p > 0) \tag{1}$$

Gamma function is also known as Euler's integral of the second kind.

Integrating by parts

$$\Gamma(p+1) = \int_0^\infty e^{-t} t^p\, dt = -e^{-t} t^p \Big|_0^\infty + p\int_0^\infty e^{-t} t^{p-1}\, dt = 0 + p\Gamma(p)$$

Thus $\quad \Gamma(p+1) = p\Gamma(p) \qquad (2)$

(2) is known as the functional relation or reduction or recurrence formula for gamma function.

Result:
$\Gamma(n+a) = (n+a-1)(n+a-2)(n+a-3)\cdots a\cdot\Gamma(a)$, n is integer.

By definition

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \frac{e^{-t}}{-1}\Big|_0^\infty = 1 \tag{3}$$

By the reduction formula (2),

$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$

and $\quad \Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$

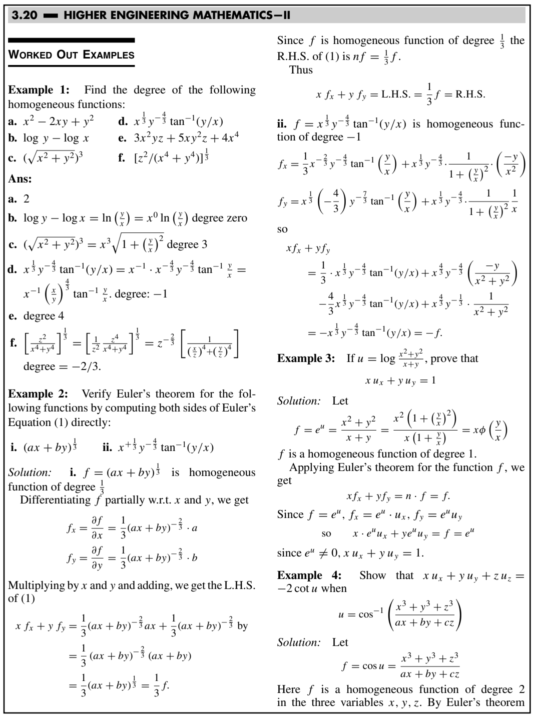
and in general when p is a positive integer n

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\,\Gamma(n-1) = n(n-1)(n-2)\,\Gamma(n-2) = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

11.1

Chapter Introduction provide a quick look into the concepts that will be discussed in the chapter

Examples



3.20 — HIGHER ENGINEERING MATHEMATICS—II

WORKED OUT EXAMPLES

Example 1: Find the degree of the following homogeneous functions:

a. $x^2 - 2xy + y^2$ **d.** $x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x)$
b. $\log y - \log x$ **e.** $3x^2yz + 5xy^2z + 4x^4$
c. $(\sqrt{x^2+y^2})^3$ **f.** $[z^2/(x^4+y^4)]^{\frac{1}{3}}$

Ans:

a. 2

b. $\log y - \log x = \ln\left(\frac{y}{x}\right) = x^0 \ln\left(\frac{y}{x}\right)$ degree zero

c. $(\sqrt{x^2+y^2})^3 = x^3\sqrt{1+\left(\frac{y}{x}\right)^2}$ degree 3

d. $x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x) = x^{-1}\cdot x^{-\frac{4}{3}} y^{-\frac{4}{3}} \tan^{-1}\frac{y}{x} = x^{-1}\left(\frac{x}{y}\right)^{\frac{4}{3}} \tan^{-1}\frac{y}{x}$. degree: -1

e. degree 4

f. $\left[\frac{z^2}{x^4+y^4}\right]^{\frac{1}{3}} = \left[\frac{1}{z^2}\frac{z^4}{x^4+y^4}\right]^{\frac{1}{3}} = z^{-\frac{2}{3}}\left[\frac{1}{\left(\frac{x}{z}\right)^4+\left(\frac{y}{z}\right)^4}\right]$
degree $= -2/3$.

Example 2: Verify Euler's theorem for the following functions by computing both sides of Euler's Equation (1) directly:

i. $(ax+by)^{\frac{1}{3}}$ **ii.** $x^{+\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x)$

Solution: **i.** $f = (ax+by)^{\frac{1}{3}}$ is homogeneous function of degree $\frac{1}{3}$

Differentiating f partially w.r.t. x and y, we get

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{3}(ax+by)^{-\frac{2}{3}}\cdot a$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{3}(ax+by)^{-\frac{2}{3}}\cdot b$$

Multiplying by x and y and adding, we get the L.H.S. of (1)

$$x\,f_x + y\,f_y = \frac{1}{3}(ax+by)^{-\frac{2}{3}}ax + \frac{1}{3}(ax+by)^{-\frac{2}{3}}by = \frac{1}{3}(ax+by)^{-\frac{2}{3}}(ax+by) = \frac{1}{3}(ax+by)^{\frac{1}{3}} = \frac{1}{3}f.$$

Since f is homogeneous function of degree $\frac{1}{3}$ the R.H.S. of (1) is $nf = \frac{1}{3}f$.

Thus

$$x\,f_x + y\,f_y = \text{L.H.S.} = \frac{1}{3}f = \text{R.H.S.}$$

ii. $f = x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x)$ is homogeneous function of degree -1

$$f_x = \frac{1}{3}x^{-\frac{2}{3}} y^{-\frac{4}{3}} \tan^{-1}\left(\frac{y}{x}\right) + x^{\frac{1}{3}} y^{-\frac{4}{3}}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-y}{x^2}\right)$$

$$f_y = x^{\frac{1}{3}}\left(-\frac{4}{3}\right) y^{-\frac{7}{3}} \tan^{-1}\left(\frac{y}{x}\right) + x^{\frac{1}{3}} y^{-\frac{4}{3}}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}$$

so

$$x f_x + y f_y = \frac{1}{3}\cdot x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x) + x^{\frac{4}{3}} y^{-\frac{4}{3}}\left(\frac{-y}{x^2+y^2}\right) - \frac{4}{3}x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x) + x^{\frac{4}{3}} y^{-\frac{1}{3}}\cdot\frac{1}{x^2+y^2} = -x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(y/x) = -f.$$

Example 3: If $u = \log\frac{x^2+y^2}{x+y}$, prove that

$$x\,u_x + y\,u_y = 1$$

Solution: Let

$$f = e^u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}{x\left(1+\frac{y}{x}\right)} = x\phi\left(\frac{y}{x}\right)$$

f is a homogeneous function of degree 1.

Applying Euler's theorem for the function f, we get

$$xf_x + yf_y = n\cdot f = f.$$

Since $f = e^u$, $f_x = e^u\cdot u_x$, $f_y = e^u u_y$

so $\quad x\cdot e^u u_x + ye^u u_y = f = e^u$

since $e^u \neq 0$, $x\,u_x + y\,u_y = 1$.

Example 4: Show that $x\,u_x + y\,u_y + z\,u_z = -2\cot u$ when

$$u = \cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$$

Solution: Let

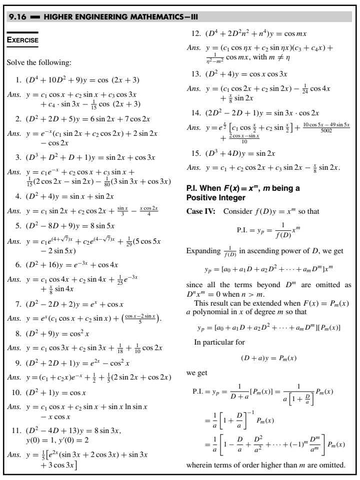
$$f = \cos u = \frac{x^3+y^3+z^3}{ax+by+cz}$$

Here f is a homogeneous function of degree 2 in the three variables x, y, z. By Euler's theorem

Every chapter contains worked out example problems which will guide the student while understanding the concepts and working out the exercise problems

Exercises

In all the chapters there are exercise problems within the text for the students to solve. This will hone their problem-solving skills like nothing else can. The answers to these exercises are provided alongside for the students to verify



9.16 — HIGHER ENGINEERING MATHEMATICS—III

EXERCISE

Solve the following:

1. $(D^4 + 10D^2 + 9)y = \cos(2x+3)$

Ans. $y = c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\cdot\sin 3x - \frac{1}{15}\cos(2x+3)$

2. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x$

Ans. $y = e^{-x}(c_1\sin 2x + c_2\cos 2x) + 2\sin 2x - \cos 2x$

3. $(D^3 + D^2 + D + 1)y = \sin 2x + \cos 3x$

Ans. $y = c_1e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{15}(2\cos 2x - \sin 2x) - \frac{1}{80}(3\sin 3x + \cos 3x)$

4. $(D^2 + 4)y = \sin x + \sin 2x$

Ans. $y = c_1\sin 2x + c_2\cos 2x + \frac{\sin x}{3} - \frac{x\cos 2x}{4}$

5. $(D^2 - 8D + 9)y = 8\sin 5x$

Ans. $y = c_1e^{(4+\sqrt{7})x} + c_2e^{(4-\sqrt{7})x} + \frac{1}{29}(5\cos 5x - 2\sin 5x)$

6. $(D^2 + 16)y = e^{-3x} + \cos 4x$

Ans. $y = c_1\cos 4x + c_2\sin 4x + \frac{1}{25}e^{-3x} + \frac{x}{8}\sin 4x$

7. $(D^2 - 2D + 2)y = e^x + \cos x$

Ans. $y = e^x(c_1\cos x + c_2\sin x) + \left(\frac{\cos x - 2\sin x}{5}\right)$.

8. $(D^2 + 9)y = \cos^2 x$

Ans. $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{18} + \frac{1}{10}\cos 2x$

9. $(D^2 + 2D + 1)y = e^{2x} - \cos^2 x$

Ans. $y = (c_1 + c_2x)e^{-x} + \frac{1}{2} + \frac{1}{5}(2\sin 2x + \cos 2x)$

10. $(D^2 + 1)y = \cos x$

Ans. $y = c_1\cos x + c_2\sin x + \sin x\ln\sin x - x\cos x$

11. $(D^2 - 4D + 13)y = 8\sin 3x$, $y(0) = 1$, $y'(0) = 2$

Ans. $y = \frac{1}{5}\left[e^{2x}(\sin 3x + 2\cos 3x) + \sin 3x + 3\cos 3x\right]$

12. $(D^4 + 2D^2n^2 + n^4)y = \cos mx$

Ans. $y = (c_1\cos\eta x + c_2\sin\eta x)(c_3 + c_4x) + \frac{1}{\eta^2 - m^2}\cos mx$, with $m \neq \eta$

13. $(D^2 + 4)y = \cos x\cos 3x$

Ans. $y = (c_1\cos 2x + c_2\sin 2x) - \frac{1}{24}\cos 4x + \frac{x}{8}\sin 2x$

14. $(2D^2 - 2D + 1)y = \sin 3x \cdot \cos 2x$

Ans. $y = e^{\frac{x}{2}}\left[c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}\right] + \frac{10\cos 5x - 49\sin 5x}{5002} + \frac{2\cos x - \sin x}{10}$

15. $(D^3 + 4D)y = \sin 2x$

Ans. $y = c_1 + c_2\cos 2x + c_3\sin 2x - \frac{x}{8}\sin 2x$.

P.I. When $F(x) = x^m$, m being a Positive Integer

Case IV: Consider $f(D)y = x^m$ so that

$$\text{P.I.} = y_p = \frac{1}{f(D)}x^m$$

Expanding $\frac{1}{f(D)}$ in ascending power of D, we get

$$y_p = [a_0 + a_1D + a_2D^2 + \cdots + a_mD^m]x^m$$

since all the terms beyond D^m are omitted as $D^nx^m = 0$ when $n > m$.

This result can be extended when $F(x) = P_m(x)$ a polynomial in x of degree m so that

$$y_p = [a_0 + a_1D + a_2D^2 + \cdots + a_mD^m][P_m(x)]$$

In particular for

$$(D + a)y = P_m(x)$$

we get

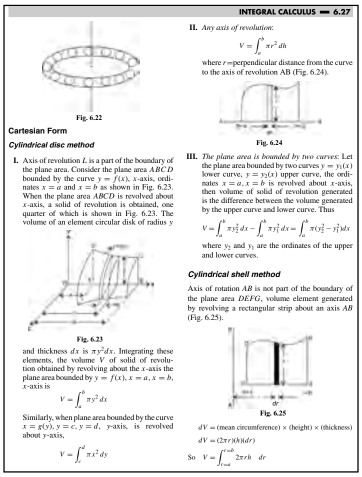
$$\text{P.I.} = y_p = \frac{1}{D+a}[P_m(x)] = \frac{1}{a\left[1 + \frac{D}{a}\right]}P_m(x)$$

$$= \frac{1}{a}\left[1 + \frac{D}{a}\right]^{-1}P_m(x)$$

$$= \frac{1}{a}\left[1 - \frac{D}{a} + \frac{D^2}{a^2} + \cdots + (-1)^m\frac{D^m}{a^m}\right]P_m(x)$$

wherein terms of order higher than m are omitted.

Figures



INTEGRAL CALCULUS — 6.27

Fig. 6.22

Cartesian Form

Cylindrical disc method

I. Axis of revolution L is a part of the boundary of the plane area. Consider the plane area $ABCD$ bounded by the curve $y = f(x)$, x-axis, ordinates $x = a$ and $x = b$ as shown in Fig. 6.23. When the plane area $ABCD$ is revolved about x-axis, a solid of revolution is obtained, one quarter of which is shown in Fig. 6.23. The volume of an element circular disk of radius y

Fig. 6.23

and thickness dx is πy^2dx. Integrating these elements, the volume V of solid of revolution obtained by revolving about the x-axis the plane area bounded by $y = f(x)$, $x = a$, $x = b$, x-axis is

$$V = \int_a^b \pi y^2\,dx$$

Similarly, when plane area bounded by the curve $x = g(y)$, $y = c$, $y = d$, y-axis, is revolved about y-axis,

$$V = \int_c^d \pi x^2\,dy$$

II. *Any axis of revolution*:

$$V = \int_a^b \pi r^2\,dh$$

where r=perpendicular distance from the curve to the axis of revolution AB (Fig. 6.24).

Fig. 6.24

III. *The plane area is bounded by two curves*: Let the plane area bounded by two curves $y = y_1(x)$ lower curve, $y = y_2(x)$ upper curve, the ordinates $x = a, x = b$ is revolved about x-axis, then volume of solid of revolution generated is the difference between the volume generated by the upper curve and lower curve. Thus

$$V = \int_a^b \pi y_2^2\,dx - \int_a^b \pi y_1^2\,dx = \int_a^b \pi(y_2^2 - y_1^2)dx$$

where y_2 and y_1 are the ordinates of the upper and lower curves.

Cylindrical shell method

Axis of rotation AB is not part of the boundary of the plane area $DEFG$, volume element generated by revolving a rectangular strip about an axis AB (Fig. 6.25).

dr

Fig. 6.25

$$dV = (\text{mean circumference}) \times (\text{height}) \times (\text{thickness})$$

$$dV = (2\pi r)(h)(dr)$$

$$\text{So}\quad V = \int_{r=a}^{r=b} 2\pi rh\quad dr$$

Figures are used exhaustively in the text to illustrate the concepts and methods described.

Web supplement

The book is accompanied by a dedicated website at
http://www.mhhe.com/ramanahem
containing additional chapters on the following topics for the students

- Matrices & Determinants
- Sequence and Series
- Analytical Solid Geometry
- Calculus of Variations
- Linear Programming

It also has chapter-wise summaries.

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