CHAPTER 12

Kinetics of Particles: Newton's Second Law

As a car travels on the curved portion of a racetrack, it is subjected to a component of acceleration directed toward the center of curvature of its path. The force of gravity and the other external forces which act on the car must be considered as well as the components of acceleration. The relation existing among force, mass, and acceleration will be studied in this chapter.

KINETICS OF PARTICLES: NEWTON'S SECOND LAW

12.1. INTRODUCTION

Newton's first and third laws of motion were used extensively in statics to study bodies at rest and the forces acting upon them. These two laws are also used in dynamics; in fact, they are sufficient for the study of the motion of bodies which have no acceleration. However, when bodies are accelerated, that is, when the magnitude or the direction of their velocity changes, it is necessary to use Newton's second law of motion to relate the motion of the body with the forces acting on it.

In this chapter we will discuss Newton's second law and apply it to the analysis of the motion of particles. As we state in Sec. 12.2, if the resultant of the forces acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force. Moreover, the ratio of the magnitudes of the resultant force and of the acceleration can be used to define the *mass* of the particle.

In Sec. 12.3, the *linear momentum* of a particle is defined as the product $\mathbf{L} = m\mathbf{v}$ of the mass m and velocity $\mathbf{v}$ of the particle, and it is demonstrated that Newton's second law can be expressed in an alternative form relating the rate of change of the linear momentum with the resultant of the forces acting on that particle.

Section 12.4 stresses the need for consistent units in the solution of dynamics problems and provides a review of the International System of Units (SI units) and the system of U.S. customary units.

In Secs. 12.5 and 12.6 and in the Sample Problems which follow, Newton's second law is applied to the solution of engineering problems, using either rectangular components or tangential and normal components of the forces and accelerations involved. We recall that an actual body—including bodies as large as a car, rocket, or airplane—can be considered as a particle for the purpose of analyzing its motion as long as the effect of a rotation of the body about its mass center can be ignored.

The second part of the chapter is devoted to the solution of problems in terms of radial and transverse components, with particular emphasis on the motion of a particle under a central force. In Sec. 12.7, the *angular momentum* $\mathbf{H}_O$ of a particle about a point O is defined as the moment about O of the linear momentum of the particle: $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$. It then follows from Newton's second law that the rate of change of the angular momentum $\mathbf{H}_O$ of a particle is equal to the sum of the moments about O of the forces acting on that particle.

Section 12.9 deals with the motion of a particle under a *central force,* that is, under a force directed toward or away from a fixed point O. Since such a force has zero moment about O, it follows that the angular momentum of the particle about O is conserved. This property greatly simplifies the analysis of the motion of a particle under a central force; in Sec. 12.10 it is applied to the solution of problems involving the orbital motion of bodies under gravitational attraction.

Sections 12.11 through 12.13 are optional. They present a more extensive discussion of orbital motion and contain a number of problems related to space mechanics.

12.2. NEWTON'S SECOND LAW OF MOTION

Newton's second law can be stated as follows:

If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

Newton's second law of motion is best understood by imagining the following experiment: A particle is subjected to a force $\mathbf{F}_1$ of constant direction and constant magnitude F_1. Under the action of that force, the particle is observed to move in a straight line and *in the direction of the force* (Fig. 12.1*a*). By determining the position of the particle at various instants, we find that its acceleration has a constant magnitude a_1. If the experiment is repeated with forces $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , of different magnitude or direction (Fig. 12.1*b* and *c*), we find each time that the particle moves in the direction of the force acting on it and that the magnitudes a_1, a_2, a_3, . . . , of the accelerations are proportional to the magnitudes F_1, F_2, F_3, . . . , of the corresponding forces:

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = \cdots = \text{constant}$$

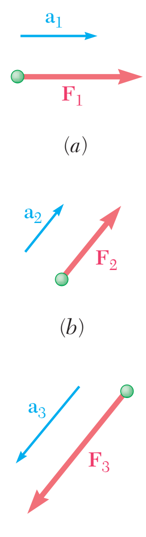


Fig. 12.1

The constant value obtained for the ratio of the magnitudes of the forces and accelerations is a characteristic of the particle under consideration; it is called the *mass* of the particle and is denoted by m. When a particle of mass m is acted upon by a force $\mathbf{F}$, the force $\mathbf{F}$ and the acceleration $\mathbf{a}$ of the particle must therefore satisfy the relation

$$\mathbf{F} = m\mathbf{a} \tag{12.1}$$

This relation provides a complete formulation of Newton's second law; it expresses not only that the magnitudes of $\mathbf{F}$ and $\mathbf{a}$ are proportional but also (since m is a positive scalar) that the vectors $\mathbf{F}$ and $\mathbf{a}$ have the same direction (Fig. 12.2). We should note that Eq. (12.1) still holds when $\mathbf{F}$ is not constant but varies with time in magnitude or direction. The magnitudes of $\mathbf{F}$ and $\mathbf{a}$ remain proportional, and the two vectors have the same direction at any given instant. However, they will not, in general, be tangent to the path of the particle.

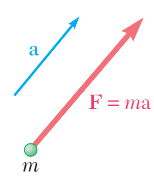


Fig. 12.2

When a particle is subjected simultaneously to several forces, Eq. (12.1) should be replaced by

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{12.2}$$

where $\Sigma\mathbf{F}$ represents the sum, or resultant, of all the forces acting on the particle.

It should be noted that the system of axes with respect to which the acceleration $\mathbf{a}$ is determined is not arbitrary. These axes must have a constant orientation with respect to the stars, and their origin must either be attached to the sun† or move with a constant velocity with

†More accurately, to the mass center of the solar system.

respect to the sun. Such a system of axes is called a *newtonian frame of reference*.† A system of axes attached to the earth does *not* constitute a newtonian frame of reference, since the earth rotates with respect to the stars and is accelerated with respect to the sun. However, in most engineering applications, the acceleration **a** can be determined with respect to axes attached to the earth and Eqs. (12.1) and (12.2) used without any appreciable error. On the other hand, these equations do not hold if **a** represents a relative acceleration measured with respect to moving axes, such as axes attached to an accelerated car or to a rotating piece of machinery.

We observe that if the resultant $\Sigma\mathbf{F}$ of the forces acting on the particle is zero, it follows from Eq. (12.2) that the acceleration **a** of the particle is also zero. If the particle is initially at rest ($\mathbf{v}_0 = 0$) with respect to the newtonian frame of reference used, it will thus remain at rest ($\mathbf{v} = 0$). If originally moving with a velocity $\mathbf{v}_0$, the particle will maintain a constant velocity $\mathbf{v} = \mathbf{v}_0$; that is, it will move with the constant speed v_0 in a straight line. This, we recall, is the statement of Newton's first law (Sec. 2.10). Thus, Newton's first law is a particular case of Newton's second law and can be omitted from the fundamental principles of mechanics.

12.3. LINEAR MOMENTUM OF A PARTICLE. RATE OF CHANGE OF LINEAR MOMENTUM

Replacing the acceleration **a** by the derivative $d\mathbf{v}/dt$ in Eq. (12.2), we write

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

or, since the mass m of the particle is constant,

$$\Sigma\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \qquad (12.3)$$

The vector $m\mathbf{v}$ is called the *linear momentum,* or simply the *momentum,* of the particle. It has the same direction as the velocity of the particle, and its magnitude is equal to the product of the mass m and the speed v of the particle (Fig. 12.3). Equation (12.3) expresses that *the resultant of the forces acting on the particle is equal to the rate of change of the linear momentum of the particle*. It is in this form that the second law of motion was originally stated by Newton. Denoting by **L** the linear momentum of the particle,

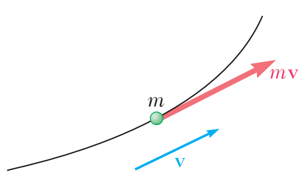


Fig. 12.3

$$\mathbf{L} = m\mathbf{v} \qquad (12.4)$$

and by $\dot{\mathbf{L}}$ its derivative with respect to t, we can write Eq. (12.3) in the alternative form

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \qquad (12.5)$$

†Since stars are not actually fixed, a more rigorous definition of a newtonian frame of reference (also called an *inertial system*) is *one with respect to which Eq. (12.2) holds*.

It should be noted that the mass m of the particle is assumed to be constant in Eqs. (12.3) to (12.5). Equation (12.3) or (12.5) should therefore not be used to solve problems involving the motion of bodies, such as rockets, which gain or lose mass. Problems of that type will be considered in Sec. 14.12.†

It follows from Eq. (12.3) that the rate of change of the linear momentum $m\mathbf{v}$ is zero when $\Sigma\mathbf{F} = 0$. Thus, *if the resultant force acting on a particle is zero, the linear momentum of the particle remains constant, in both magnitude and direction*. This is the principle of *conservation of linear momentum* for a particle, which can be recognized as an alternative statement of Newton's first law (Sec. 2.10).

12.4. SYSTEMS OF UNITS

In using the fundamental equation $\mathbf{F} = m\mathbf{a}$, the units of force, mass, length, and time cannot be chosen arbitrarily. If they are, the magnitude of the force $\mathbf{F}$ required to give an acceleration $\mathbf{a}$ to the mass m will *not* be numerically equal to the product ma; it will be only proportional to this product. Thus, we can choose three of the four units arbitrarily but must choose the fourth unit so that the equation $\mathbf{F} = m\mathbf{a}$ is satisfied. The units are then said to form a system of consistent kinetic units.

Two systems of consistent kinetic units are currently used by American engineers, the International System of Units (SI units‡) and the system of U.S. customary units. Both systems were discussed in detail in Sec. 1.3 and are described only briefly in this section.

International System of Units (SI Units). In this system, the base units are the units of length, mass, and time, and are called, respectively, the *meter* (m), the *kilogram* (kg), and the *second* (s). All three are arbitrarily defined (Sec. 1.3). The unit of force is a derived unit. It is called the *newton* (N) and is defined as the force which gives an acceleration of 1 m/s^2 to a mass of 1 kg (Fig. 12.4). From Eq. (12.1) we write

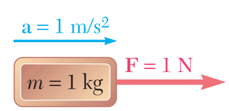


Fig 12.4

$$1 \text{ N} = (1 \text{ kg})(1 \text{ m/s}^2) = 1 \text{ kg} \cdot \text{m/s}^2$$

The SI units are said to form an *absolute* system of units. This means that the three base units chosen are independent of the location where measurements are made. The meter, the kilogram, and the second may be used anywhere on the earth; they may even be used on another planet. They will always have the same significance.

The *weight* $\mathbf{W}$ of a body, or *force of gravity* exerted on that body, should, like any other force, be expressed in newtons. Since a body subjected to its own weight acquires an acceleration equal to the acceleration of gravity g, it follows from Newton's second law that the magnitude W of the weight of a body of mass m is

$$W = mg \tag{12.6}$$

†On the other hand, Eqs. (12.3) and (12.5) do hold in *relativistic mechanics*, where the mass m of the particle is assumed to vary with the speed of the particle.

‡SI stands for *Système International d'Unités* (French).

Recalling that $g = 9.81 \text{ m/s}^2$, we find that the weight of a body of mass 1 kg (Fig. 12.5) is

$$W = (1 \text{ kg})(9.81 \text{ m/s}^2) = 9.81 \text{ N}$$

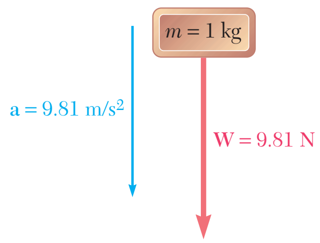


Fig 12.5

Multiples and submultiples of the units of length, mass, and force are frequently used in engineering practice. They are, respectively, the *kilometer* (km) and the *millimeter* (mm); the *megagram*† (Mg) and the *gram* (g); and the *kilonewton* (kN). By definition,

$$1 \text{ km} = 1000 \text{ m} \qquad 1 \text{ mm} = 0.001 \text{ m}$$
$$1 \text{ Mg} = 1000 \text{ kg} \qquad 1 \text{ g} = 0.001 \text{ kg}$$
$$1 \text{ kN} = 1000 \text{ N}$$

The conversion of these units to meters, kilograms, and newtons, respectively, can be effected simply by moving the decimal point three places to the right or to the left.

Units other than the units of mass, length, and time can all be expressed in terms of these three base units. For example, the unit of linear momentum can be obtained by recalling the definition of linear momentum and writing

$$mv = (\text{kg})(\text{m/s}) = \text{kg} \cdot \text{m/s}$$

U.S. Customary Units. Most practicing American engineers still commonly use a system in which the base units are the units of length, force, and time. These units are, respectively, the *foot* (ft), the *pound* (lb), and the *second* (s). The second is the same as the corresponding SI unit. The foot is defined as 0.3048 m. The pound is defined as the *weight* of a platinum standard, called the *standard pound*, which is kept at the National Institute of Standards and Technology outside Washington and the mass of which is 0.453 592 43 kg. Since the weight of a body depends upon the gravitational attraction of the earth, which varies with location, it is specified that the standard pound should be placed at sea level and at a latitude of 45° to properly define a force of 1 lb. Clearly, the U.S. customary units do not form an absolute system of units. Because of their dependence upon the gravitational attraction of the earth, they are said to form a *gravitational* system of units.

While the standard pound also serves as the unit of mass in commercial transactions in the United States, it cannot be so used in engineering computations since such a unit would not be consistent with the base units defined in the preceding paragraph. Indeed, when acted upon by a force of 1 lb, that is, when subjected to its own weight, the standard pound receives the acceleration of gravity, $g = 32.2 \text{ ft/s}^2$ (Fig. 12.6), and not the unit acceleration required by Eq. (12.1). The unit of mass consistent with the foot, the pound, and the second is the mass which receives an acceleration of 1 ft/s^2 when a force of 1 lb is applied to it (Fig. 12.7). This unit, sometimes called a *slug*, can be derived from the equation $F = ma$ after substituting 1 lb and 1 ft/s^2 for F and a, respectively. We write

$$F = ma \qquad 1 \text{ lb} = (1 \text{ slug})(1 \text{ ft/s}^2)$$

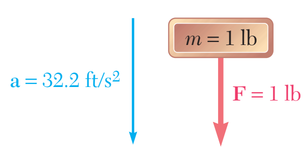


Fig 12.6

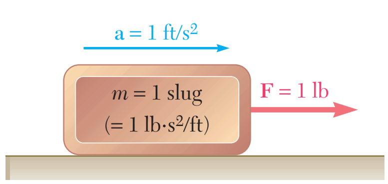


Fig 12.7

†Also known as a *metric ton*.

and obtain

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = 1 \text{ lb} \cdot \text{s}^2\text{/ft}$$

Comparing Figs. 12.6 and 12.7, we conclude that the slug is a mass 32.2 times larger than the mass of the standard pound.

The fact that bodies are characterized in the U.S. customary system of units by their weight in pounds rather than by their mass in slugs was a convenience in the study of statics, where we were dealing for the most part with weights and other forces and only seldomly with masses. However, in the study of kinetics, which involves forces, masses, and accelerations, it will be repeatedly necessary to express in slugs the mass m of a body, the weight W of which has been given in pounds. Recalling Eq. (12.6), we will write

$$m = \frac{W}{g} \tag{12.7}$$

where g is the acceleration of gravity ($g = 32.2 \text{ ft/s}^2$).

Units other than the units of force, length, and time can all be expressed in terms of these three base units. For example, the unit of linear momentum can be obtained by using the definition of linear momentum to write

$$mv = (\text{lb} \cdot \text{s}^2\text{/ft})(\text{ft/s}) = \text{lb} \cdot \text{s}$$

Conversion from One System of Units to Another. The conversion from U.S. customary units to SI units, and vice versa, was discussed in Sec. 1.4. You will recall that the conversion factors obtained for the units of length, force, and mass are, respectively,

Length:	$1 \text{ ft} = 0.3048 \text{ m}$
Force:	$1 \text{ lb} = 4.448 \text{ N}$
Mass:	$1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2\text{/ft} = 14.59 \text{ kg}$

Although it cannot be used as a consistent unit of mass, the mass of the standard pound is, by definition,

$$1 \text{ pound-mass} = 0.4536 \text{ kg}$$

This constant can be used to determine the *mass* in SI units (kilograms) of a body which has been characterized by its *weight* in U.S. customary units (pounds).

12.5. EQUATIONS OF MOTION

Consider a particle of mass m acted upon by several forces. We recall from Sec. 12.2 that Newton's second law can be expressed by the equation

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{12.2}$$

which relates the forces acting on the particle and the vector $m\mathbf{a}$ (Fig. 12.8). In order to solve problems involving the motion of a particle, however, it will be found more convenient to replace Eq. (12.2) by equivalent equations involving scalar quantities.

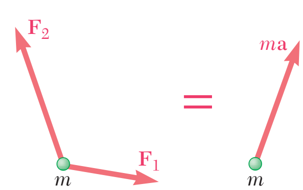


Fig. 12.8

Rectangular Components. Resolving each force **F** and the acceleration **a** into rectangular components, we write

$$\Sigma(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) = m(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

from which it follows that

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \tag{12.8}$$

Recalling from Sec. 11.11 that the components of the acceleration are equal to the second derivatives of the coordinates of the particle, we have

$$\Sigma F_x = m\ddot{x} \qquad \Sigma F_y = m\ddot{y} \qquad \Sigma F_z = m\ddot{z} \tag{12.8$'$}$$

Consider, as an example, the motion of a projectile. If the resistance of the air is neglected, the only force acting on the projectile after it has been fired is its weight $\mathbf{W} = -W\mathbf{j}$. The equations defining the motion of the projectile are therefore

$$m\ddot{x} = 0 \qquad m\ddot{y} = -W \qquad m\ddot{z} = 0$$

and the components of the acceleration of the projectile are

$$\ddot{x} = 0 \qquad \ddot{y} = -\frac{W}{m} = -g \qquad \ddot{z} = 0$$

where g is 9.81 m/s^2 or 32.2 ft/s^2. The equations obtained can be integrated independently, as shown in Sec. 11.11, to obtain the velocity and displacement of the projectile at any instant.

When a problem involves two or more bodies, equations of motion should be written for each of the bodies (see Sample Probs. 12.3 and 12.4). You will recall from Sec. 12.2 that all accelerations should be measured with respect to a newtonian frame of reference. In most engineering applications, accelerations can be determined with respect to axes attached to the earth, but relative accelerations measured with respect to moving axes, such as axes attached to an accelerated body, cannot be substituted for **a** in the equations of motion.

Photo 12.1 As it travels on the curved portion of a track, the bobsled is subjected to a normal component of acceleration directed toward the center of curvature of its path.

Tangential and Normal Components. Resolving the forces and the acceleration of the particle into components along the tangent to the path (in the direction of motion) and the normal (toward

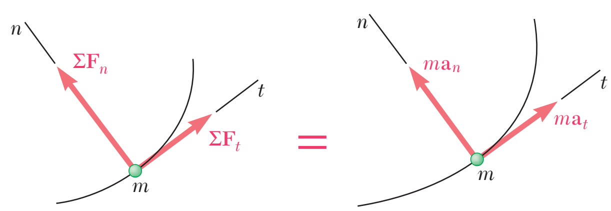


Fig. 12.9

the inside of the path) (Fig. 12.9), and substituting into Eq. (12.2), we obtain the two scalar equations

$$\Sigma F_t = ma_t \qquad \Sigma F_n = ma_n \tag{12.9}$$

Substituting for a_t and a_n from Eqs. (11.40), we have

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho} \tag{12.9$'$}$$

The equations obtained may be solved for two unknowns.

12.6. DYNAMIC EQUILIBRIUM

Returning to Eq. (12.2) and transposing the right-hand member, we write Newton's second law in the alternative form

$$\Sigma \mathbf{F} - m\mathbf{a} = 0 \tag{12.10}$$

which expresses that if we add the vector $-m\mathbf{a}$ to the forces acting on the particle, *we obtain a system of vectors equivalent to zero* (Fig. 12.10). The vector $-m\mathbf{a}$, of magnitude ma and of *direction opposite* to that of the acceleration, is called an *inertia vector*. The particle may thus be considered to be in equilibrium under the given forces and the inertia vector. The particle is said to be in *dynamic equilibrium,* and the problem under consideration can be solved by the methods developed earlier in statics.

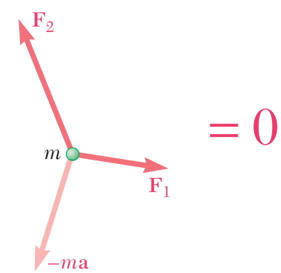


Fig. 12.10

In the case of coplanar forces, all the vectors shown in Fig. 12.10, *including the inertia vector,* can be drawn tip-to-tail to form a closed-vector polygon. Or the sums of the components of all the vectors in Fig. 12.10, again including the inertia vector, can be equated to zero. Using rectangular components, we therefore write

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \textbf{\textit{including inertia vector}} \tag{12.11}$$

When tangential and normal components are used, it is more convenient to represent the inertia vector by its two components $-m\mathbf{a}_t$ and $-m\mathbf{a}_n$ in the sketch itself (Fig. 12.11). The tangential component of the inertia vector provides a measure of the resistance the particle offers to a change in speed, while its normal component (also called *centrifugal force*) represents the tendency of the particle to leave its curved path. We should note that either of these two components may be zero under special conditions: (1) if the particle starts from rest, its initial velocity is zero and the normal component of the inertia vector is zero at $t = 0$; (2) if the particle moves at constant speed along its path, the tangential component of the inertia vector is zero and only its normal component needs to be considered.

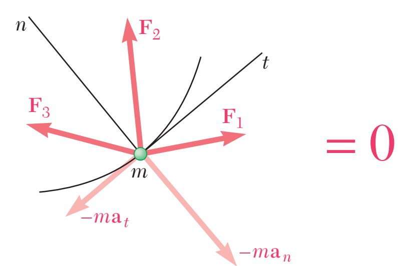
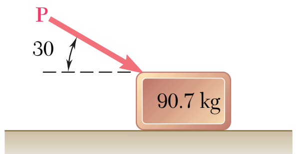


Fig. 12.11

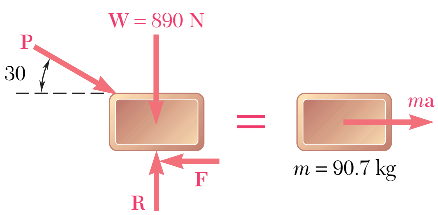
Because they measure the resistance that particles offer when we try to set them in motion or when we try to change the conditions of their motion, inertia vectors are often called *inertia forces*. The inertia forces, however, are not forces like the forces found in statics, which are either contact forces or gravitational forces (weights). Many people, therefore, object to the use of the word "force" when referring to the vector $-m\mathbf{a}$ or even avoid altogether the concept of dynamic equilibrium. Others point out that inertia forces and actual forces, such as gravitational forces, affect our senses in the same way and cannot be distinguished by physical measurements. A man riding in an elevator which is accelerated upward will have the feeling that his weight has suddenly increased; and no measurement made within the elevator could establish whether the elevator is truly accelerated or whether the force of attraction exerted by the earth has suddenly increased.

Sample problems have been solved in this text by the direct application of Newton's second law, as illustrated in Figs. 12.8 and 12.9, rather than by the method of dynamic equilibrium.



SAMPLE PROBLEM 12.1

A 90.7 kg block rests on a horizontal plane. Find the magnitude of the force **P** required to give the block an acceleration of 3 m/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$.



SOLUTION

The mass of the block is

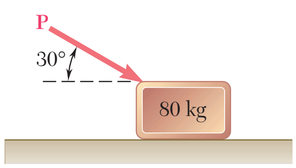
$$\text{Weight of block, } W = mg = 890 \text{ N.}$$
$$m = 90.7 \text{ kg}$$

We note that $F = \mu_k R = 0.25R$ and that $a = 3$ m/s^2. Expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\overset{+}{\rightarrow}\Sigma F_x = ma: \quad P\cos 30° - 0.25R = (90.7 \text{ kg } (3 \text{ m/s}^2)$$
$$P\cos 30° - 0.25R = 272 \text{ N} \qquad (1)$$
$$+\uparrow\Sigma F_y = 0: \quad R - P\sin 30° - 890 \text{ N} = 0 \qquad (2)$$

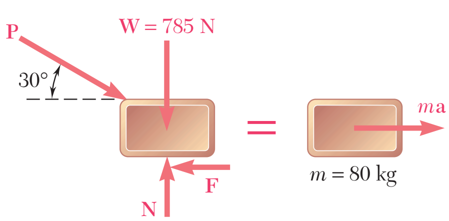
Solving (2) for N and substituting the result into (1), we obtain

$$R = P\sin 30° + 890 \text{ N}$$
$$P\cos 30° - 0.25(P\sin 30° + 890 \text{ N}) = 272 \text{ N} \qquad P = 667.3 \text{ N} \blacktriangleleft$$



SAMPLE PROBLEM 12.2

An 80-kg block rests on a horizontal plane. Find the magnitude of the force **P** required to give the block an acceleration of 2.5 m/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$.



SOLUTION

The weight of the block is

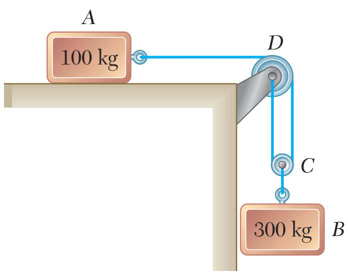
$$W = mg = (80 \text{ kg})(9.81 \text{ m/s}^2) = 785 \text{ N}$$

We note that $F = \mu_k N = 0.25N$ and that $a = 2.5$ m/s^2. Expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\overset{+}{\rightarrow}\Sigma F_x = ma: \quad P\cos 30° - 0.25N = (80 \text{ kg})(2.5 \text{ m/s}^2)$$
$$P\cos 30° - 0.25N = 200 \text{ N} \qquad (1)$$
$$+\uparrow\Sigma F_y = 0: \quad N - P\sin 30° - 785 \text{ N} = 0 \qquad (2)$$

Solving (2) for N and substituting the result into (1), we obtain

$$N = P\sin 30° + 785 \text{ N}$$
$$P\cos 30° - 0.25(P\sin 30° + 785 \text{ N}) = 200 \text{ N} \qquad P = 535 \text{ N} \blacktriangleleft$$



SAMPLE PROBLEM 12.3

The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.

SOLUTION

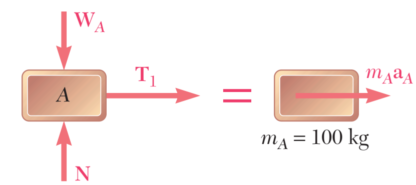
Kinematics. We note that if block A moves through x_A to the right, block B moves down through

$$x_B = \tfrac{1}{2}x_A$$

Differentiating twice with respect to t, we have

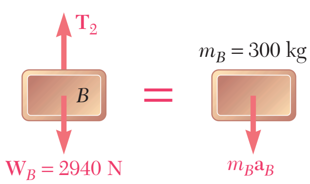
$$a_B = \tfrac{1}{2}a_A \qquad (1)$$

Kinetics. We apply Newton's second law successively to block A, block B, and pulley C.



Block A. Denoting by T_1 the tension in cord ACD, we write

$$\overset{+}{\rightarrow}\Sigma F_x = m_A a_A: \qquad T_1 = 100a_A \qquad (2)$$



Block B. Observing that the weight of block B is

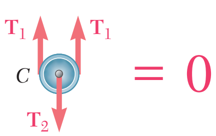
$$W_B = m_B g = (300 \text{ kg})(9.81 \text{ m/s}^2) = 2940 \text{ N}$$

and denoting by T_2 the tension in cord BC, we write

$$+\!\downarrow\Sigma F_y = m_B a_B: \qquad 2940 - T_2 = 300a_B$$

or, substituting for a_B from (1),

$$2940 - T_2 = 300(\tfrac{1}{2}a_A)$$
$$T_2 = 2940 - 150a_A \qquad (3)$$



Pulley C. Since m_C is assumed to be zero, we have

$$+\!\downarrow\Sigma F_y = m_C a_C = 0: \qquad T_2 - 2T_1 = 0 \qquad (4)$$

Substituting for T_1 and T_2 from (2) and (3), respectively, into (4) we write

$$2940 - 150a_A - 2(100a_A) = 0$$
$$2940 - 350a_A = 0 \qquad a_A = 8.40 \text{ m/s}^2 \blacktriangleleft$$

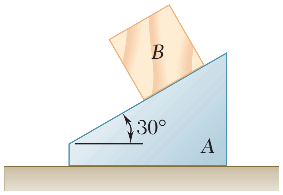
Substituting the value obtained for a_A into (1) and (2), we have

$$a_B = \tfrac{1}{2}a_A = \tfrac{1}{2}(8.40 \text{ m/s}^2) \qquad a_B = 4.20 \text{ m/s}^2 \blacktriangleleft$$
$$T_1 = 100a_A = (100 \text{ kg})(8.40 \text{ m/s}^2) \qquad T_1 = 840 \text{ N} \blacktriangleleft$$

Recalling (4), we write

$$T_2 = 2T_1 \qquad T_2 = 2(840 \text{ N}) \qquad T_2 = 1680 \text{ N} \blacktriangleleft$$

We note that the value obtained for T_2 is *not* equal to the weight of block B.

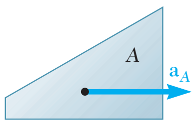


SAMPLE PROBLEM 12.4

The 5.4 kg block B starts from rest and slides on the 13.6 kg wedge A, which is supported by a horizontal surface. Neglecting friction, determine (a) the acceleration of the wedge, (b) the acceleration of the block relative to the wedge.

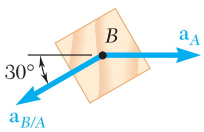
SOLUTION

Kinematics. We first examine the acceleration of the wedge and the acceleration of the block.



Wedge A. Since the wedge is constrained to move on the horizontal surface, its acceleration $\mathbf{a}_A$ is horizontal. We will assume that it is directed to the right.

Block B. The acceleration $\mathbf{a}_B$ of block B can be expressed as the sum of the acceleration of A and the acceleration of B relative to A. We have

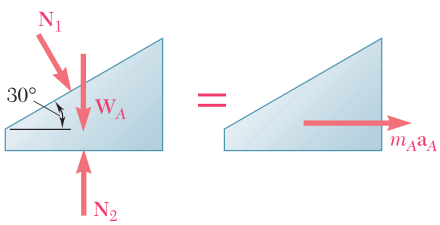


$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

where $\mathbf{a}_{B/A}$ is directed along the inclined surface of the wedge.

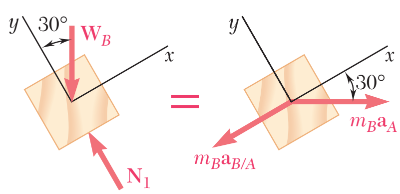
Kinetics. We draw the free-body diagrams of the wedge and of the block and apply Newton's second law.

Wedge A. We denote the forces exerted by the block and the horizontal surface on wedge A by $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively.



$$\xrightarrow{+}\Sigma F_x = m_A a_A: \qquad N_1 \sin 30° = m_A a_A$$
$$0.5N_1 = (W_A/g)a_A \qquad (1)$$

Block B. Using the coordinate axes shown and resolving $\mathbf{a}_B$ into its components $\mathbf{a}_A$ and $\mathbf{a}_{B/A}$, we write



$$+\nearrow\Sigma F_x = m_B a_x: \qquad -W_B \sin 30° = m_B a_A \cos 30° - m_B a_{B/A}$$
$$-W_B \sin 30° = (W_B/g)(a_A \cos 30° - a_{B/A})$$
$$a_{B/A} = a_A \cos 30° + g \sin 30° \qquad (2)$$
$$+\nwarrow\Sigma F_y = m_B a_y: \qquad N_1 - W_B \cos 30° = -m_B a_A \sin 30°$$
$$N_1 - W_B \cos 30° = -(W_B/g)a_A \sin 30° \qquad (3)$$

a. Acceleration of Wedge A. Substituting for N_1 from Eq. (1) into Eq. (3), we have

$$2(W_A/g)a_A - W_B \cos 30° = -(W_B/g)a_A \sin 30°$$

mass of block B m_B = 5.4 kg
Weight of block B $W_B = m_B g$ = 53 N
mass of Wedge A m_A = 13.6 kg
Weight of Wedge A $W_A = m_A g$ = 133.4 N

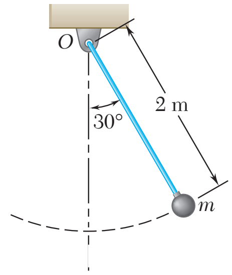
Solving for a_A and substituting the numerical data, we write

$$a_A = \frac{W_B \cos 30°}{2W_A + W_B \sin 30°} g = \frac{(53\,\text{N}) \cos 30°}{2(133.4\,\text{N}) + (53\,\text{N}) \sin 30°}(9.81\ \text{m/s}^2)$$

$$a_A = +1.54\ \text{m/s}^2 \qquad \mathbf{a}_A = 1.54\ \text{m/s}^2 \rightarrow \blacktriangleleft$$

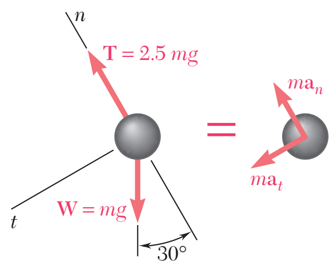
b. Acceleration of Block B Relative to A. Substituting the value obtained for a_A into Eq. (2), we have

$$a_{B/A} = (1.54\ \text{m/s}^2) \cos 30° + (9.81\ \text{m/s}^2) \sin 30°$$
$$a_{B/A} = +6.24\ \text{m/s}^2 \qquad \mathbf{a}_{B/A} = 6.24\ \text{m/s}^2 \measuredangle 30° \blacktriangleleft$$



SAMPLE PROBLEM 12.5

The bob of a 2-m pendulum describes an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and the acceleration of the bob in that position.



SOLUTION

The weight of the bob is $W = mg$; the tension in the cord is thus 2.5 mg. Recalling that $\mathbf{a}_n$ is directed toward O and assuming $\mathbf{a}_t$ as shown, we apply Newton's second law and obtain

$+\swarrow \Sigma F_t = ma_t$: $\quad mg \sin 30° = ma_t$

$a_t = g \sin 30° = +4.90 \text{ m/s}^2 \qquad \mathbf{a}_t = 4.90 \text{ m/s}^2 \swarrow$ ◀

$+\nwarrow \Sigma F_n = ma_n$: $\quad 2.5\, mg - mg \cos 30° = ma_n$

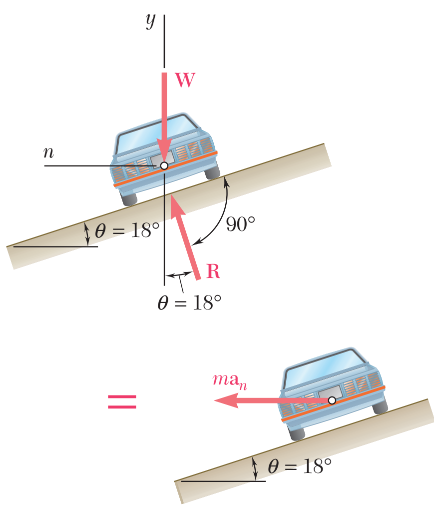
$a_n = 1.634\, g = +16.03 \text{ m/s}^2 \qquad \mathbf{a}_n = 16.03 \text{ m/s}^2 \nwarrow$ ◀

Since $a_n = v^2/\rho$, we have $v^2 = \rho a_n = (2 \text{ m})(16.03 \text{ m/s}^2)$

$v = \pm 5.66 \text{ m/s} \qquad \mathbf{v} = 5.66 \text{ m/s} \nearrow\!\!\swarrow$ (up or down) ◀

SAMPLE PROBLEM 12.6

Determine the rated speed of a highway curve of radius $\rho = 122$ m banked through an angle $\theta = 18°$. The *rated speed* of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted on its wheels.



SOLUTION

The car travels in a *horizontal* circular path of radius ρ. The normal component $\mathbf{a}_n$ of the acceleration is directed toward the center of the path; its magnitude is $a_n = v^2/\rho$, where v is the speed of the car in m/s. The mass m of the car is W/g, where W is the weight of the car. Since no lateral friction force is to be exerted on the car, the reaction $\mathbf{R}$ of the road is shown perpendicular to the roadway. Applying Newton's second law, we write

$$+\uparrow \Sigma F_y = 0: \qquad R \cos \theta - W = 0 \qquad R = \frac{W}{\cos \theta} \tag{1}$$

$$\overset{+}{\leftarrow} \Sigma F_n = ma_n: \qquad R \sin \theta = \frac{W}{g} a_n \tag{2}$$

Substituting for R from (1) into (2), and recalling that $a_n = v^2/\rho$,

$$\frac{W}{\cos \theta} \sin \theta = \frac{W}{g}\frac{v^2}{\rho} \qquad v^2 = g\rho \tan \theta$$

Substituting $\rho = 122$ m and $\theta = 18°$ into this equation, we obtain

$$v^2 = (9.81 \text{ m/s}^2)(122 \text{ m}) \tan 18°$$

$v = 19.7 \text{ m/s} \qquad v = 19.7 \text{ m/s}$ ◀

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will apply *Newton's second law of motion,* $\Sigma\mathbf{F} = m\mathbf{a}$, to relate the forces acting on a particle to the motion of the particle.

1. *Writing the equations of motion.* When applying Newton's second law to the types of motion discussed in this lesson, you will find it most convenient to express the vectors **F** and **a** in terms of either their rectangular components or their tangential and normal components.

a. When using rectangular components, and recalling from Sec. 11.11 the expressions found for a_x, a_y, and a_z, you will write

$$\Sigma F_x = m\ddot{x} \qquad \Sigma F_y = m\ddot{y} \qquad \Sigma F_z = m\ddot{z}$$

b. When using tangential and normal components, and recalling from Sec. 11.13 the expressions found for a_t and a_n, you will write

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho}$$

2. *Drawing a free-body diagram* showing the applied forces *and an equivalent diagram* showing the vector $m\mathbf{a}$ or its components will provide you with a pictorial representation of Newton's second law [Sample Probs. 12.1 through 12.6]. These diagrams will be of great help to you when writing the equations of motion. Note that when a problem involves two or more bodies, it is usually best to consider each body separately.

3. *Applying Newton's second law.* As we observed in Sec. 12.2, the acceleration used in the equation $\Sigma\mathbf{F} = m\mathbf{a}$ should always be the *absolute acceleration* of the particle (that is, it should be measured with respect to a newtonian frame of reference). Also, *if the sense of the acceleration* **a** *is unknown* or is not easily deduced, assume an arbitrary sense for **a** (usually the positive direction of a coordinate axis) and then let the solution provide the correct sense. Finally, note how the solutions of Sample Probs. 12.3 and 12.4 were divided into a *kinematics* portion and a *kinetics* portion, and how in Sample Prob. 12.4 we used two systems of coordinate axes to simplify the equations of motion.

4. *When a problem involves dry friction,* be sure to review the relevant sections of *Statics* [Secs. 8.1 to 8.3] before attempting to solve that problem. In particular, you should know when each of the equations $F = \mu_s N$ and $F = \mu_k N$ may be used. You should also recognize that if the motion of a system is not specified, it is necessary first to assume a possible motion and then to check the validity of that assumption.

5. *Solving problems involving relative motion.* When a body B moves with respect to a body A, as in Sample Prob. 12.4, it is often convenient to express the acceleration of B as

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

where $\mathbf{a}_{B/A}$ is the acceleration of B relative to A, that is, the acceleration of B as observed from a frame of reference attached to A and in translation. If B is observed to move in a straight line, $\mathbf{a}_{B/A}$ will be directed along that line. On the other hand, if B is observed to move along a circular path, the relative acceleration $\mathbf{a}_{B/A}$ should be resolved into components tangential and normal to that path.

6. *Finally, always consider the implications of any assumption you make.* Thus, in a problem involving two cords, if you assume that the tension in one of the cords is equal to its maximum allowable value, check whether any requirements set for the other cord will then be satisfied. For instance, will the tension T in that cord satisfy the relation $0 \leq T \leq T_{\max}$? That is, will the cord remain taut and will its tension be less than its maximum allowable value?

Problems

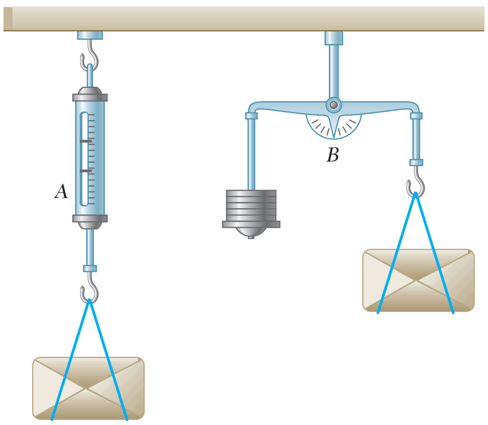


Fig. *P12.3*

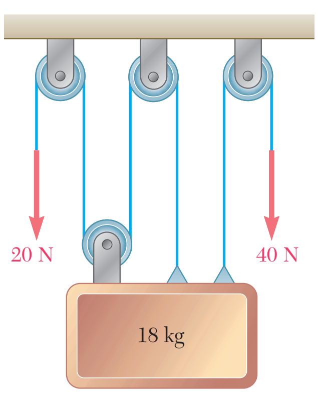


Fig. P12.5

12.1 The acceleration due to gravity on Mars is 3.75 m/s^2. Knowing that the mass of a silver bar has been officially designated as 20 kg, determine, on Mars, its weight in newtons.

12.2 The value of the acceleration of gravity at any latitude ϕ is given by $g = 9.7087(1 + 0.0053 \sin^2 \phi)$ m/s^2, where the effect of the rotation of the earth as well as the fact that the earth is not spherical have been taken into account. Knowing that the mass of a gold bar has been officially designated as 2 kg, determine to four significant figures its mass in kilograms and its weight in newtons at a latitude of (*a*) 0°, (*b*) 45°, (*c*) 60°.

12.3 A spring scale *A* and a lever scale *B* having equal lever arms are fastened to the roof on an elevator, and identical packages are attached to the scales as shown. Knowing that when the elevator moves downward with an acceleration of .6 m/s^2 the spring scale indicates a load of 3.2 kg, determine (*a*) the weight of the packages, (*b*) the load indicated by the spring scale and the mass needed to balance the lever scale when the elevator moves upward with an acceleration of .6 m/s^2.

12.4 A Global Positioning System (GPS) satellite is in a circular orbit 20157 km above the surface of the earth and completes one orbit every 12 h. Knowing that the magnitude of the linear momentum of the satellite is 340194 kg · s and the radius of the earth is 6370 km, determine (*a*) the mass of the satellite, (*b*) the weight of the satellite before it was launched from earth.

12.5 The 18 kg block starts from rest and moves upward when constant forces of 20 N and 40 N are applied to supporting ropes. Neglecting the masses of the pulleys and the effect of friction, determine the speed of the block after it has moved .457 m.

12.6 A motorist traveling at a speed of 108 km/h suddenly applies the brakes and comes to a stop after skidding 75 m. Determine (*a*) the time required for the car to stop, (*b*) the coefficient of friction between the tires and the pavement.

12.7 A 1400-kg automobile is driven down a 4º incline at a speed of 88 km/h when the brakes are applied, causing a total braking force of 7500 N to be applied to the automobile. Determine the distance traveled by the automobile before it comes to a stop.

12.8 In the braking test of a sports car its velocity is reduced from 112.16 km/h to zero in a distance of 51.8 m with slipping impending. Knowing that the coefficient of kinetic friction is 80 percent of the coefficient of static friction, determine (*a*) the coefficient of static friction, (*b*) the stopping distance for the same initial velocity if the car skids. Ignore air resistance and rolling resistance.

12.9 A .09 kg model rocket is launched vertically from rest at time $t = 0$ with a constant thrust of 4 N for one second and no thrust for $t > 1$ s. Neglecting air resistance and the decrease in mass of the rocket, determine (*a*) the maximum height h reached by the rocket, (*b*) the time required to reach this maximum height.

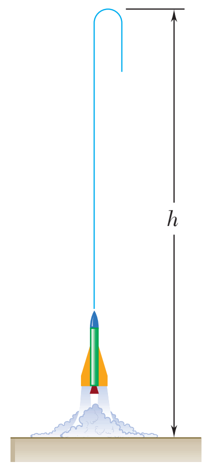


Fig. P12.9

12.10 A 40-kg package is at rest on an incline when a force **P** is applied to it. Determine the magnitude of **P** if 4 s is required for the package to travel 10 m up the incline. The static and kinetic coefficients of friction between the package and the incline are 0.30 and 0.25, respectively.

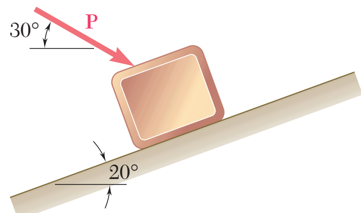


Fig. P12.10

12.11 If an automobile's braking distance from 100 km/h is 60 m on level pavement, determine the automobile's braking distance from 100 km/h when it is (*a*) going up a 6° incline, (*b*) going down a 2-percent incline.

12.12 The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between the blocks and the incline, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

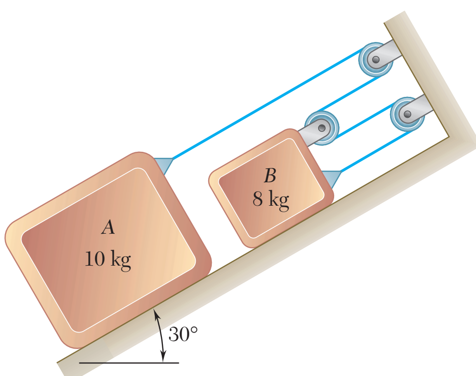


Fig. *P12.12* and P12.13

12.13 The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and assuming that the coefficients of friction between both blocks and the incline are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

12.14 A light train made up of two cars is traveling at 88 km/h when the brakes are applied to both cars. Knowing that car A has a weight of 24947.56 kg and car B has a weight of 19958 kg and that the braking force is 31137.5 N on each car, determine (*a*) the distance traveled by the train before it comes to a stop, (*b*) the force in the coupling between the cars while the train is slowing down.

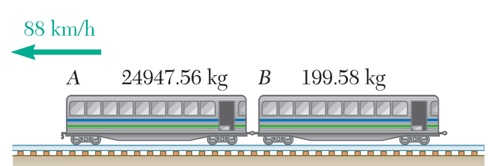


Fig. *P12.14*

12.15 Solve Prob. 12.14, assuming that the brakes of car B fail to operate.

12.16 Block A weighs 356 N, and block B weighs 71 N. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. Knowing that $P = 0$, determine (*a*) the acceleration of block B, (*b*) the tension in the cord.

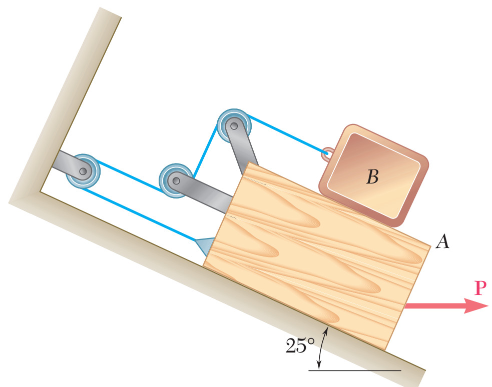


Fig. P12.16 and P12.17

12.17 Block A weighs 356 N, and block B weighs 71 N. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. Knowing that $\mathbf{P} = 44.5\text{ N} \rightarrow$, determine (*a*) the acceleration of block B, (*b*) the tension in the cord.

12.18 Boxes A and B are at rest on a conveyor belt that is initially at rest. The belt is suddenly started in an upward direction so that slipping occurs between the belt and the boxes. Knowing that the coefficients of kinetic friction between the belt and the boxes are $(\mu_k)_A = 0.30$ and $(\mu_k)_B = 0.32$, determine the initial acceleration of each box.

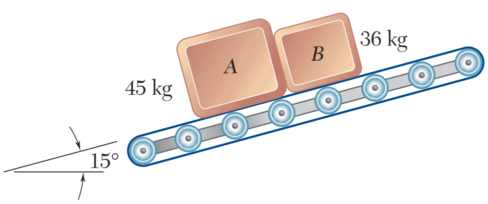


Fig. *P12.18*

12.19 The system shown is initially at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys, determine (*a*) the acceleration of each block, (*b*) the tension in each cable.

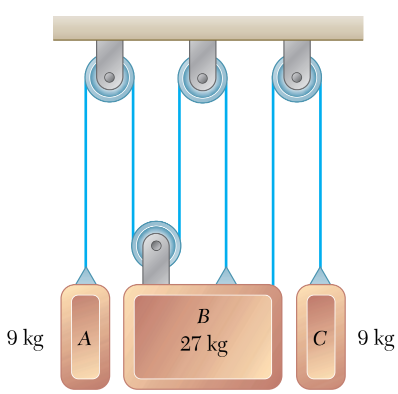


Fig. P12.19

12.20 Each of the systems shown is initially at rest. Neglecting axle friction and the masses of the pulleys, determine for each system (*a*) the acceleration of block A, (*b*) the velocity of block A after it has moved through 1.5 m, (*c*) the time required for block A to reach a velocity of 3 m/s.

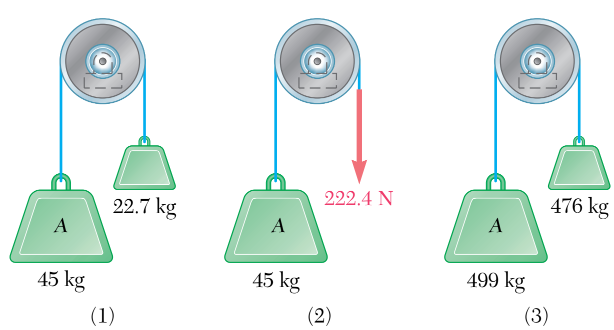


Fig. P12.20

12.21 The flatbed trailer carries two 1360.8 kg beams with the upper beam secured by a cable. The coefficients of static friction between the two beams and between the lower beam and the bed of the trailer are 0.25 and 0.30, respectively. Knowing that the load does not shift, determine (*a*) the maximum acceleration of the trailer and the corresponding tension in the cable, (*b*) the maximum deceleration of the trailer.

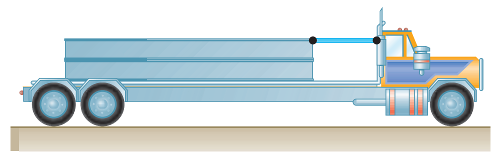


Fig. P12.21

12.22 The 10-kg block *B* is supported by the 40-kg block *A* which is pulled up an incline by a constant 500 N force. Neglecting friction between the block and the incline and knowing that block *B* does not slip on block *A*, determine the smallest allowable value of the coefficient of static friction between the blocks.

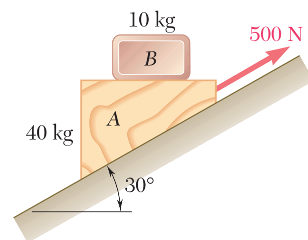


Fig. P12.22

12.23 A package is at rest on a conveyor belt which is initially at rest. The belt is started and moves to the right for 1.5 s with a constant acceleration of 3.2 m/s^2. The belt then moves with a constant deceleration $\mathbf{a}_2$ and comes to a stop after a total displacement of 4.6 m. Knowing that the coefficients of friction between the package and the belt are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (*a*) the deceleration $\mathbf{a}_2$ of the belt, (*b*) the displacement of the package relative to the belt as the belt comes to a stop.

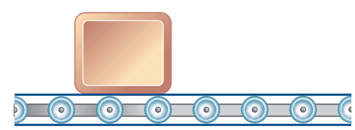


Fig. P12.23

12.24 To transport a series of bundles of shingles *A* to a roof, a contractor uses a motor-driven lift consisting of a horizontal platform *BC* which rides on rails attached to the sides of a ladder. The lift starts from rest and initially moves with a constant acceleration $\mathbf{a}_1$ as shown. The lift then decelerates at a constant rate $\mathbf{a}_2$ and comes to rest at *D*, near the top of the ladder. Knowing that the coefficient of static friction between the bundle of shingles and the horizontal platform is 0.30, determine the largest allowable acceleration $\mathbf{a}_1$ and the largest allowable deceleration $\mathbf{a}_2$ if the bundle is not to slide on the platform.

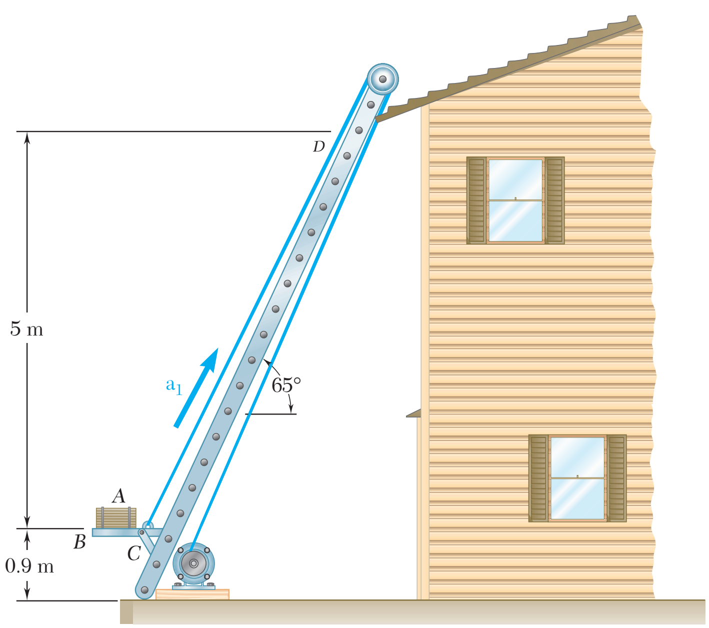


Fig. ***P12.24***

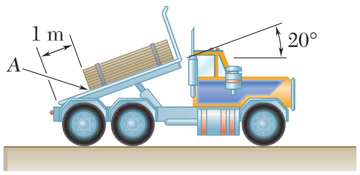


Fig. ***P12.25***

12.25 To unload a bound stack of plywood from a truck, the driver first tilts the bed of the truck and then accelerates from rest. Knowing that the coefficients of friction between the bottom sheet of plywood and the bed are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the smallest acceleration of the truck which will cause the stack of plywood to slide, (*b*) the acceleration of the truck which causes corner *A* of the stack of plywood to reach the end of the bed in 0.4 s.

12.26 The propellers of a ship of mass m can produce a propulsive force $\mathbf{F}_0$; they produce a force of the same magnitude but opposite direction when the engines are reversed. Knowing that the ship was proceeding forward at its maximum speed v_0 when the engines were put into reverse, determine the distance the ship travels before coming to a stop. Assume that the frictional resistance of the water varies directly with the square of the velocity.

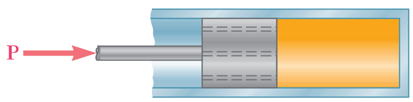


Fig. P12.27

12.27 A constant force **P** is applied to a piston and rod of total mass m to make them move in a cylinder filled with oil. As the piston moves, the oil is forced through orifices in the piston and exerts on the piston a force of magnitude kv in a direction opposite to the motion of the piston. Knowing that the piston starts from rest at $t = 0$ and $x = 0$, show that the equation relating x, v, and t, where x is the distance traveled by the piston and v is the speed of the piston, is linear in each of the variables.

12.28 A 4-kg projectile is fired vertically with an initial velocity of 90 m/s, reaches a maximum height, and falls to the ground. The aerodynamic drag **D** has a magnitude $D = 0.0024\, v^2$ where D and v are expressed in newtons and m/s, respectively. Knowing that the direction of the drag is always opposite to the direction of the velocity, determine (*a*) the maximum height of the trajectory, (*b*) the speed of the projectile when it reaches the ground.

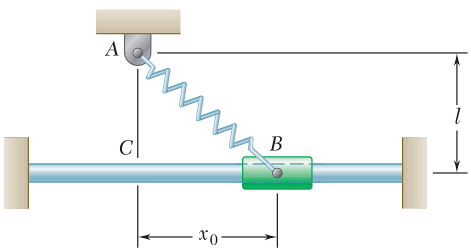


Fig. P12.29

12.29 A spring *AB* of constant k is attached to a support *A* and to a collar of mass m. The unstretched length of the spring is l. Knowing that the collar is released from rest at $x = x_0$ and neglecting friction between the collar and the horizontal rod, determine the magnitude of the velocity of the collar as it passes through point *C*.

12.30 The system of three 10-kg blocks is supported in a vertical plane and is initially at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys, determine (*a*) the change in position of block *A* after 0.5 s, (*b*) the tension in the cable.

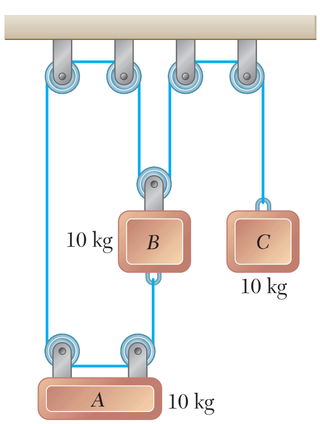


Fig. P12.30

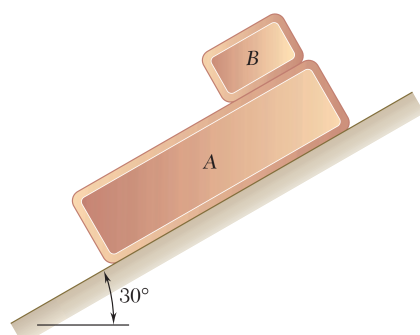


Fig. P12.31

12.31 The coefficients of friction between block *B* and block *A* are $\mu_s = 0.12$ and $\mu_k = 0.10$ and the coefficients of friction between block *A* and the incline are $\mu_s = 0.24$ and $\mu_k = 0.20$. The masses of block *A* and block *B* are 10 kg and 5 kg, respectively. Knowing that the system is released from rest in the position shown, determine (*a*) the acceleration of *A*, (*b*) the velocity of *B* relative to *A* at $t = 0.5$ s.

12.32 The weights of blocks A, B, and C are $w_A = w_C = 89$ N, and $w_B = 44.5$ N. Knowing that $P = 222.4$ N and neglecting the masses of the pulleys and the effect of friction, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

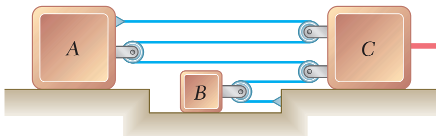


Fig. P12.32 and P12.33

12.33 The coefficients of friction between the three blocks and the horizontal surfaces are $\mu_s = 0.25$ and $\mu_k = 0.20$. The weights of the blocks are $w_A = w_C = 89$ N, and $w_B = 44.5$ N. Knowing that the blocks are initially at rest and that C moves to the right through .73 m in 0.4 s, determine (*a*) the acceleration of each block, (*b*) the tension in the cable, (*c*) the force **P**. Neglect axle friction and the masses of the pulleys.

12.34 A 25-kg block A rests on an inclined surface, and a 15-kg counterweight B is attached to a cable as shown. Neglecting friction, determine the acceleration of A and the tension in the cable immediately after the system is released from rest.

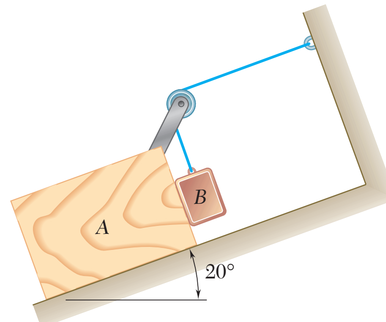


Fig. *P12.34*

12.35 A 250-kg crate B is suspended from a cable attached to a 20-kg trolley A which rides on an inclined I-beam as shown. Knowing that at the instant shown the trolley has an acceleration of 0.4 m/s^2 up to the right, determine (*a*) the acceleration of B relative to A, (*b*) the tension in cable CD.

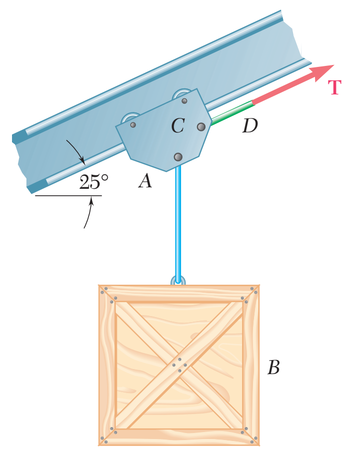


Fig. *P12.35*

12.36 A 2-kg ball revolves in a horizontal circle as shown at a constant speed of 1.5 m/s. Knowing that $L = 600$ mm, determine (*a*) the angle θ that the cord forms with the vertical, (*b*) the tension in the cord.

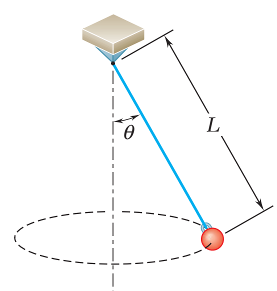


Fig. P12.36

12.37 A single wire ACB of length 2 m passes through a ring at C that is attached to a sphere which revolves at a constant speed v in the horizontal circle shown. Knowing that $\theta_1 = 60°$ and $\theta_2 = 30°$ and that the tension is the same in both portions of the wire, determine the speed v.

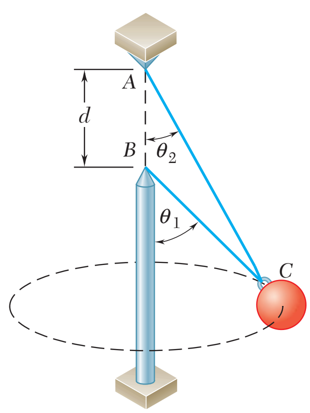


Fig. P12.37 and *P12.38*

12.38 Two wires AC and BC are tied to a 6.8 kg sphere which revolves at a constant speed v in the horizontal circle shown. Knowing that $\theta_1 = 50°$ and $\theta_2 = 25°$ and that $d = 1.2$ m, determine the range of values of v for which both wires are taut.

12.39 During a hammer thrower's practice swings, the 7.3 kg head A of the hammer revolves at a constant speed v in a horizontal circle as shown. If $\rho = 0.9$ m and $\theta = 60°$, determine (*a*) the tension in wire BC, (*b*) the speed of the hammer's head.

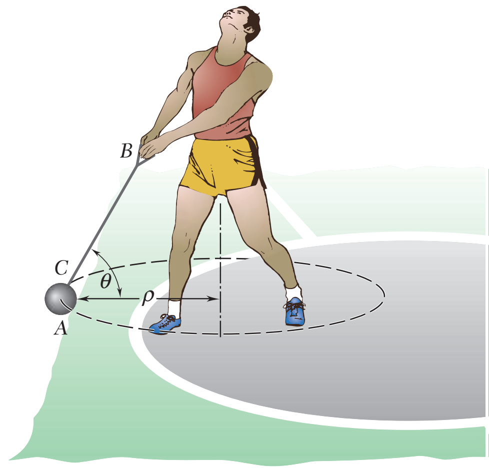


Fig. *P12.39*

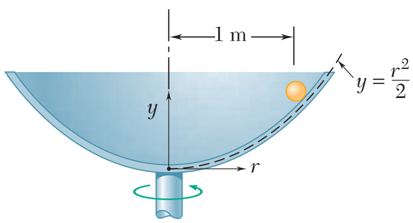


Fig. P12.40

12.40 A 1-kg sphere is at rest relative to a parabolic dish which rotates at a constant rate about a vertical axis. Neglecting friction and knowing that $r = 1$ m, determine (*a*) the speed v of the sphere, (*b*) the magnitude of the normal force exerted by the sphere on the inclined surface of the dish.

12.41 A 1-kg collar C slides without friction along the rod OA and is attached to rod BC by a frictionless pin. The rods rotate in the horizontal plane. At the instant shown, BC is rotating counterclockwise and the speed of C is 1 m/s, increasing at a rate of 1.3 m/s. Determine at this instant, (*a*) the tension in rod BC, (*b*) the force exerted by the collar on rod OA.

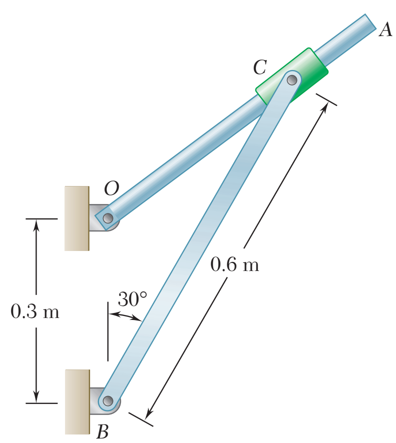


Fig. P12.41

***12.42** The 0.5-kg flyballs of a centrifugal governor revolve at a constant speed v in the horizontal circle of 150-mm radius shown. Neglecting the mass of links AB, BC, AD, and DE and requiring that the links support only tensile forces, determine the range of the allowable values of v so that the magnitudes of the forces in the links do not exceed 75 N.

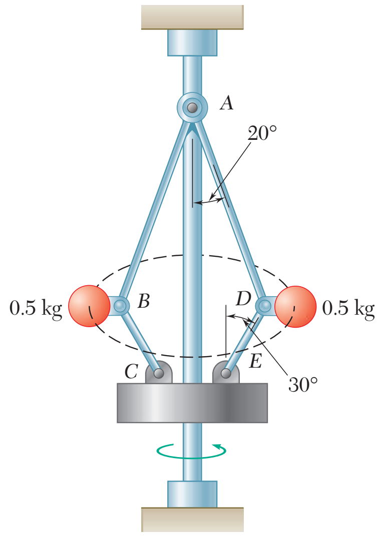


Fig. P12.42

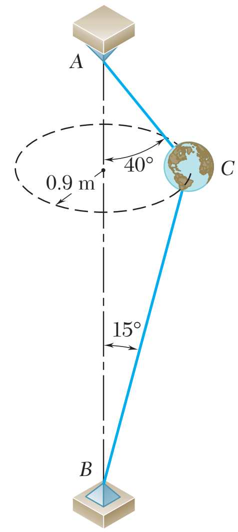


Fig. *P12.43*

***12.43** As part of an outdoor display, a 5-kg model C of the earth is attached to wires AC and BC and revolves at a constant speed v in the horizontal circle shown. Determine the range of the allowable values of v if both wires are to remain taut and if the tension in either of the wires is not to exceed 116 N.

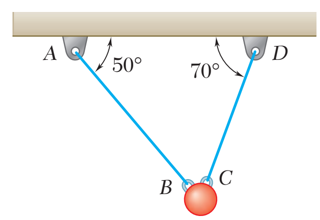


Fig. P12.44

12.44 A small sphere of weight W is held as shown by two wires AB and CD. If wire AB is cut, determine the tension in the other wire (a) before AB is cut, (b) immediately after AB has been cut.

12.45 A series of small packages are being moved by a thin conveyor belt that passes over a 300-mm-radius idler pulley. The belt starts from rest at time $t = 0$ and its speed increases at a constant rate of 150 mm/s^2. Knowing that the coefficient of static friction between the packages and the belt is 0.75, determine the time at which the first package slips.

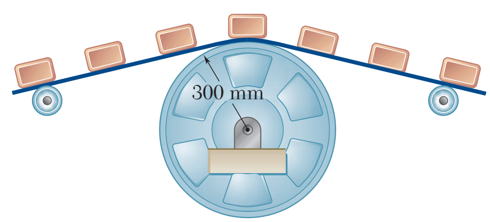


Fig. P12.45

12.46 An airline pilot climbs to a new flight level along the path shown. Knowing that the speed of the airplane decreases at a constant rate from 165 m/s at point A to 146 m/s at point C, determine the magnitude of the abrupt change in the force exerted on a 90.7 kg passenger as the airplane passes point B.

12.47 An airline pilot climbs to a new flight level along the path shown. The motion of the airplane between A and B is defined by the relation $s = 3t(180 - t)$, where s is the arc length in meter, t is the time in seconds, and $t = 0$ when the airplane is at point A. Determine the force exerted by his seat on a 74.8 kg passenger (a) just after the airplane passes point A, (b) just before the airplane reaches point B.

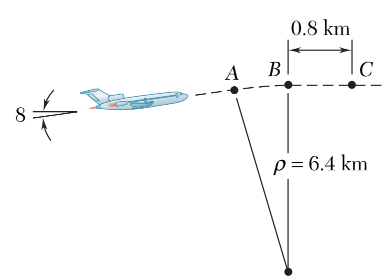


Fig. P12.46 and P12.47

12.48 During a high-speed chase, an 1100-kg sports car traveling at a speed of 160 km/h just loses contact with the road as it reaches the crest A of a hill. (*a*) Determine the radius of curvature ρ of the vertical profile of the road at A. (*b*) Using the value of ρ found in part *a*, determine the force exerted on a 70-kg driver by the seat of his 1400-kg car as the car, traveling at a constant speed of 80 km/h, passes through A.

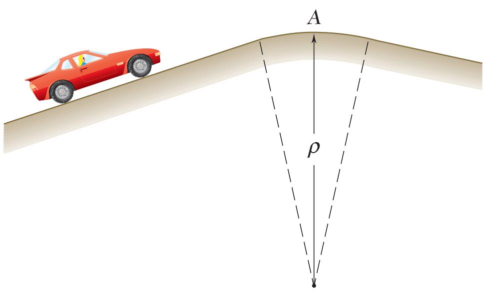


Fig. P12.48

12.49 A small 0.2-kg sphere B is given a downward velocity $\mathbf{v}_0$ and swings freely in the vertical plane, first about O and then about the peg A after the cord comes in contact with the peg. Determine the largest allowable velocity $\mathbf{v}_0$ if the tension in the cord is not to exceed 10 N.

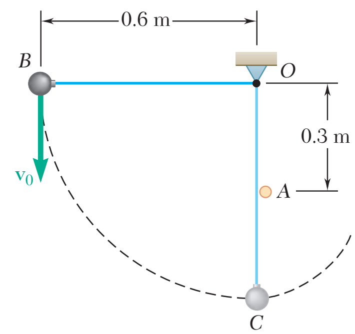


Fig. P12.49

12.50 A .23 kg block B fits inside a small cavity cut in arm OA, which rotates in the vertical plane at a constant rate such that $v = 2.74$ m/s. Knowing that the spring exerts on block B a force of magnitude $P = 1.33$ N and neglecting the effect of friction, determine the range of values of θ for which block B is in contact with the face of the cavity closest to the axis of rotation.

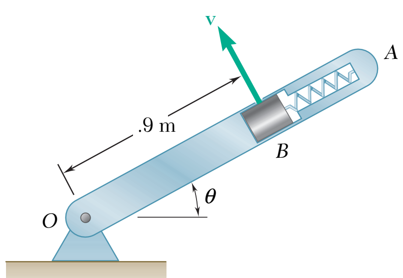


Fig. *P12.50*

12.51 A 54.4 kg pilot flies a jet trainer in a half vertical loop of 1097-m radius so that the speed of the trainer decreases at a constant rate. Knowing that the pilot's apparent weights at points A and C are 1690 N and 356 N, respectively, determine the force exerted on her by the seat of the trainer when the trainer is at point B.

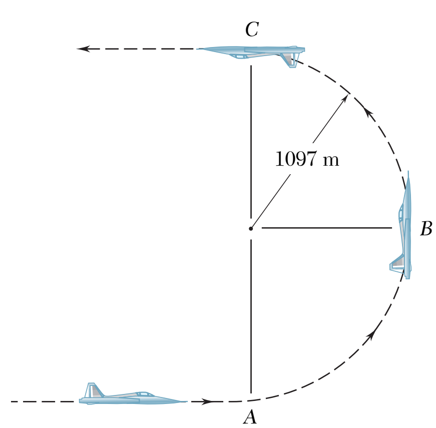


Fig. *P12.51*

12.52 A car is traveling on a banked road at a constant speed v. Determine the range of values of v for which the car does not skid. Express your answer in terms of the radius r of the curve, the banking angle θ, and the angle of static friction ϕ_s between the tires and the pavement.

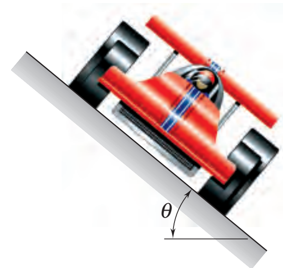


Fig. P12.52 and P12.53

12.53 A curve in a speed track has a radius of 200 m and a rated speed of 180 km/h. (See Sample Prob. 12.6 for the definition of rated speed.) Knowing that a racing car starts skidding on the curve when traveling at a speed of 320 km/h, determine (*a*) the banking angle θ, (*b*) the coefficient of static friction between the tires and the track under the prevailing conditions, (*c*) the minimum speed at which the same car could negotiate that curve.

12.54 Tilting trains such as the *Acela*, which runs from Washington to New York to Boston, are designed to travel safely at high speeds on curved sections of track which were built for slower, conventional trains. As it enters a curve, each car is tilted by hydraulic actuators mounted on its trucks. The tilting feature of the cars also increases passenger comfort by eliminating or greatly reducing the side force $\mathbf{F}_s$ (parallel to the floor of the car) to which passengers feel subjected. For a train traveling at 200 km/h on a curved section of track banked at an angle $\theta = 8°$ and with a rated speed of 120 km/h, determine (*a*) the magnitude of the side force felt by a passenger of weight W in a standard car with no tilt ($\phi = 0$), (*b*) the required angle of tilt ϕ if the passenger is to feel no side force. (See Sample Problem 12.6 for the definition of rated speed.)

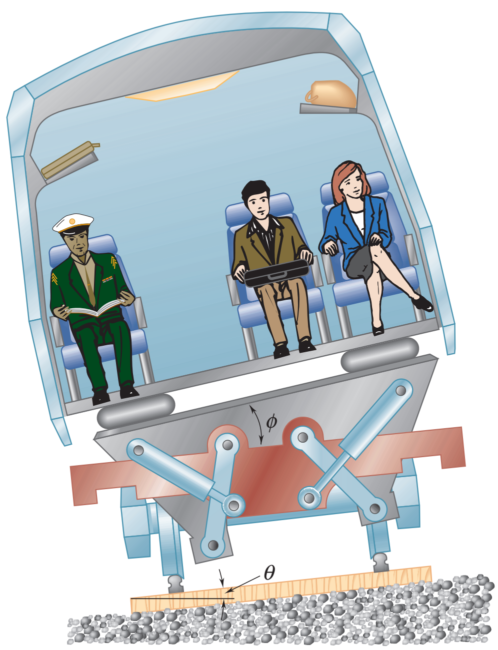


Fig. P12.54 and P12.55

12.55 Tests carried out with the tilting trains described in Prob. 12.54 revealed that passengers feel queasy when they see through the car windows that the train is rounding a curve at high speed, yet do not feel any side force. Designers, therefore, prefer to reduce, but not eliminate that force. For the train of Prob. 12.54, determine the required angle of tilt ϕ if passengers are to feel side forces equal to 12 percent of their weights.

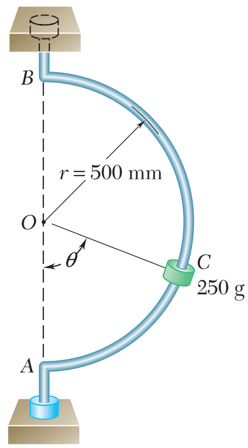


Fig. P12.56

12.56 A small 250-g collar C can slide on a semicircular rod which is made to rotate about the vertical AB at a constant rate of 7.5 rad/s. Determine the three values of θ for which the collar will not slide on the rod, assuming no friction between the collar and the rod.

12.57 For the collar and rod of Prob. 12.56, and assuming that the coefficient of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, indicate whether the collar will slide on the rod if it is released in the position corresponding to (*a*) $\theta = 75°$, (*b*) $\theta = 40°$. Also, determine the magnitude and direction of the friction force exerted on the collar immediately after release.

12.58 A small block B fits inside a slot cut in arm OA which rotates in a vertical plane at a constant rate. The block remains in contact with the end of the slot closest to A and its speed is 1.28 m/s for $0 \le \theta \le 150°$. Knowing that the block begins to slide when $\theta = 150°$, determine the coefficient of static friction between the block and the slot.

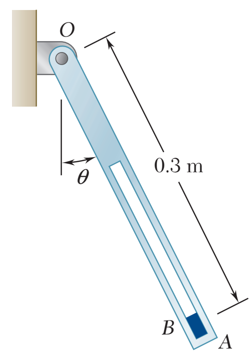


Fig. *P12.58*

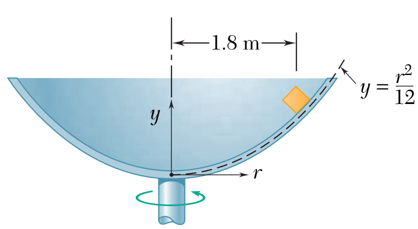


Fig. *P12.59*

12.59 A 2.7 kg block is at rest relative to a parabolic dish which rotates at a constant rate about a vertical axis. Knowing that the coefficient of static friction is 0.5 and that $r = 1.8$ m, determine the maximum allowable speed v of the block.

12.60 Four seconds after a polisher is started from rest, small tufts of fleece from along the circumference of the 25.4 cm-diameter polishing pad are observed to fly free of the pad. If the polisher is started so that the fleece along the circumference undergoes a constant tangential acceleration of 3.7 m/s^2, determine (*a*) the speed v of a tuft as it leaves the pad, (*b*) the magnitude of the force required to free the tuft if the average weight of a tuft is 1.5×10^{-6} kg.

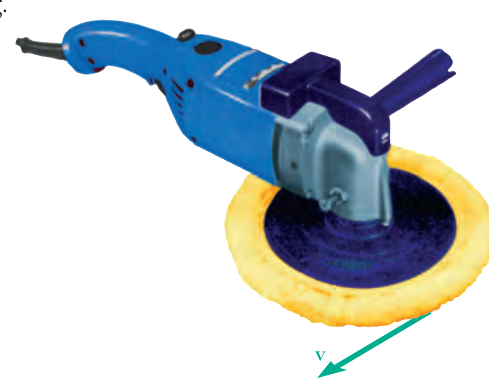


Fig. P12.60

12.61 A turntable A is built into a stage for use in a theatrical production. It is observed during a rehearsal that a trunk B starts to slide on the turntable 12 s after the turntable begins to rotate. Knowing that the trunk undergoes a constant tangential acceleration of 0.23 m/s^2, determine the coefficient of static friction between the trunk and the turntable.

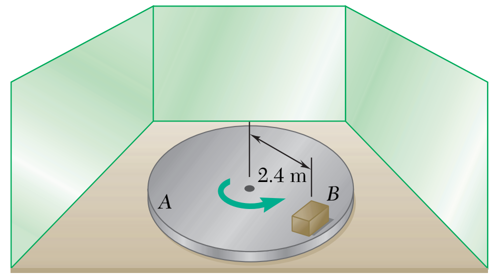


Fig. P12.61

12.62 The parallel-link mechanism $ABCD$ is used to transport a component I between manufacturing processes at stations E, F, and G by picking it up at a station when $\theta = 0$ and depositing it at the next station when $\theta = 180°$. Knowing that member BC remains horizontal throughout its motion and that links AB and CD rotate at a constant rate in a vertical plane in such a way that $v_B = 0.7$ m/s, determine (*a*) the minimum value of the coefficient of static friction between the component and BC if the component is not to slide on BC while being transferred, (*b*) the values of θ for which sliding is impending.

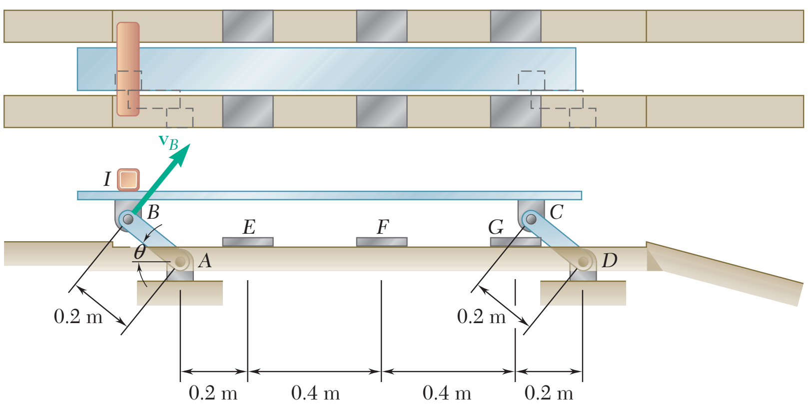


Fig. *P12.62*

12.63 Knowing that the coefficients of friction between the component I and member BC of the mechanism of Prob. 12.62 are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (*a*) the maximum allowable speed v_B if the component is not to slide on BC while being transferred, (*b*) the values of θ for which sliding is impending.

12.64 In the cathode-ray tube shown, electrons emitted by the cathode and attracted by the anode pass through a small hole in the anode and then travel in a straight line with a speed v_0 until they strike the screen at A. However, if a difference of potential V is established between the two parallel plates, the electrons will be subjected to a force $\mathbf{F}$ perpendicular to the plates while they travel between the plates and will strike the screen at point B, which is at a distance δ from A. The magnitude of the force $\mathbf{F}$ is $F = eV/d$, where $-e$ is the charge of an electron and d is the distance between the plates. Neglecting the effects of gravity, derive an expression for the deflection δ in terms of V, v_0, the charge $-e$ and the mass m of an electron, and the dimensions d, l, and L.

12.65 In Prob. 12.64, determine the smallest allowable value of the ratio d/l in terms of e, m, v_0, and V if at $x = l$ the minimum permissible distance between the path of the electrons and the positive plate is $0.075d$.

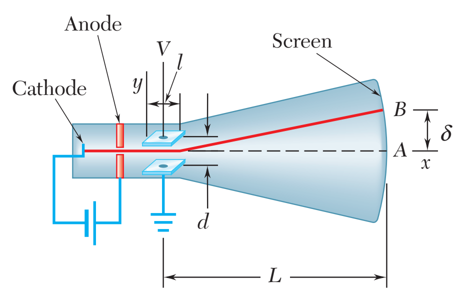


Fig. P12.64 and P12.65

12.7. ANGULAR MOMENTUM OF A PARTICLE. RATE OF CHANGE OF ANGULAR MOMENTUM

Consider a particle P of mass m moving with respect to a newtonian frame of reference $Oxyz$. As we saw in Sec. 12.3, the linear momentum of the particle at a given instant is defined as the vector $m\mathbf{v}$ obtained by multiplying the velocity $\mathbf{v}$ of the particle by its mass m. The moment about O of the vector $m\mathbf{v}$ is called the *moment of momentum*, or the *angular momentum*, of the particle about O at that instant and is denoted by $\mathbf{H}_O$. Recalling the definition of the moment of a vector (Sec. 3.6) and denoting by $\mathbf{r}$ the position vector of P, we write

$$\boxed{\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}} \tag{12.12}$$

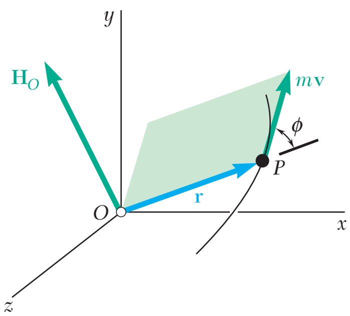


Fig. 12.12

and note that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$ and of magnitude

$$\boxed{H_O = rmv \sin \phi} \tag{12.13}$$

where ϕ is the angle between $\mathbf{r}$ and $m\mathbf{v}$ (Fig. 12.12). The sense of $\mathbf{H}_O$ can be determined from the sense of $m\mathbf{v}$ by applying the right-hand rule. The unit of angular momentum is obtained by multiplying the units of length and of linear momentum (Sec. 12.4). With SI units, we have

$$(\text{m})(\text{kg} \cdot \text{m/s}) = \text{kg} \cdot \text{m}^2/\text{s}$$

With U.S. customary units, we write

$$(\text{ft})(\text{lb} \cdot \text{s}) = \text{ft} \cdot \text{lb} \cdot \text{s}$$

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into components and applying formula (3.10), we write

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{12.14}$$


Photo 12.2 During a hammer thrower's practice swings, the hammer acquires angular momentum about a vertical axis at the center of its circular path.

The components of $\mathbf{H}_O$, which also represent the moments of the linear momentum $m\mathbf{v}$ about the coordinate axes, can be obtained by expanding the determinant in (12.14). We have

$$\begin{aligned} H_x &= m(yv_z - zv_y) \\ H_y &= m(zv_x - xv_z) \\ H_z &= m(xv_y - yv_x) \end{aligned} \tag{12.15}$$

In the case of a particle moving in the xy plane, we have $z = v_z = 0$ and the components H_x and H_y reduce to zero. The angular momentum is thus perpendicular to the xy plane; it is then completely defined by the scalar

$$H_O = H_z = m(xv_y - yv_x) \tag{12.16}$$

which will be positive or negative according to the sense in which the particle is observed to move from O. If polar coordinates are used, we resolve the linear momentum of the particle into radial and transverse components (Fig. 12.13) and write

$$H_O = rmv \sin \phi = rmv_\theta \tag{12.17}$$

or, recalling from (11.45) that $v_\theta = r\dot{\theta}$,

$$H_O = mr^2\dot{\theta} \tag{12.18}$$

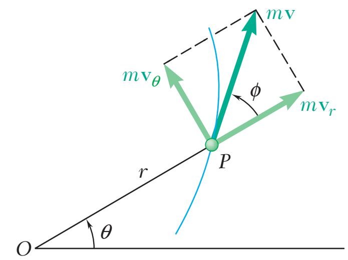


Fig. 12.13

Let us now compute the derivative with respect to t of the angular momentum $\mathbf{H}_O$ of a particle P moving in space. Differentiating both members of Eq. (12.12), and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a}$$

Since the vectors $\mathbf{v}$ and $m\mathbf{v}$ are collinear, the first term of the expression obtained is zero; and, by Newton's second law, $m\mathbf{a}$ is equal to the sum $\Sigma\mathbf{F}$ of the forces acting on P. Noting that $\mathbf{r} \times \Sigma\mathbf{F}$ represents the sum $\Sigma\mathbf{M}_O$ of the moments about O of these forces, we write

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \tag{12.19}$$

Equation (12.19), which results directly from Newton's second law, states that *the sum of the moments about O of the forces acting on the particle is equal to the rate of change of the moment of momentum, or angular momentum, of the particle about O.*

12.8. EQUATIONS OF MOTION IN TERMS OF RADIAL AND TRANSVERSE COMPONENTS

Consider a particle P, of polar coordinates r and θ, which moves in a plane under the action of several forces. Resolving the forces and the acceleration of the particle into radial and transverse components (Fig. 12.14) and substituting into Eq. (12.2), we obtain the two scalar equations

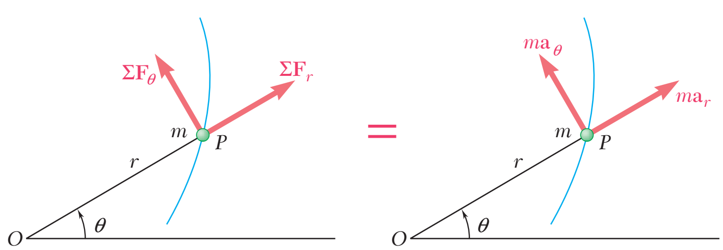


Fig 12.14

$$\Sigma F_r = ma_r \qquad \Sigma F_\theta = ma_\theta \tag{12.20}$$

Substituting for a_r and a_θ from Eqs. (11.46), we have

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2) \tag{12.21}$$

$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

The equations obtained can be solved for two unknowns.

Equation (12.22) could have been derived from Eq. (12.19). Recalling (12.18) and noting that $\Sigma M_O = r\Sigma F_\theta$, Eq. (12.19) yields

$$r\Sigma F_\theta = \frac{d}{dt}(mr^2\dot{\theta})$$
$$= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$$

and, after dividing both members by r,

$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

Photo 12.3 The path of specimen being tested in a centrifuge is a horizontal circle. The forces acting on the specimen and its acceleration can be resolved into radial and transverse components with r = constant.

12.9. MOTION UNDER A CENTRAL FORCE. CONSERVATION OF ANGULAR MOMENTUM

When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving *under a central force,* and the point O is referred to as the *center of force* (Fig. 12.15). Since the line of action of $\mathbf{F}$ passes through O, we must have $\Sigma\mathbf{M}_O = 0$ at any given instant. Substituting into Eq. (12.19), we therefore obtain

$$\dot{\mathbf{H}}_O = 0$$

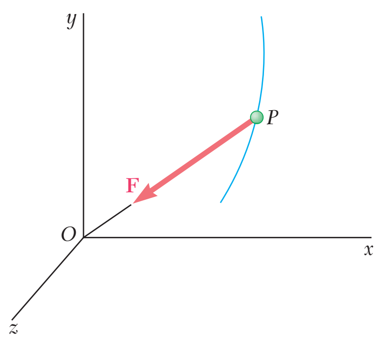


Fig. 12.15

for all values of t and, integrating in t,

$$\mathbf{H}_O = \text{constant} \tag{12.23}$$

We thus conclude that *the angular momentum of a particle moving under a central force is constant, in both magnitude and direction.*

Recalling the definition of the angular momentum of a particle (Sec. 12.7), we write

$$\mathbf{r} \times m\mathbf{v} = \mathbf{H}_O = \text{constant} \tag{12.24}$$

from which it follows that the position vector $\mathbf{r}$ of the particle P must be perpendicular to the constant vector $\mathbf{H}_O$. Thus, a particle under a central force moves in a fixed plane perpendicular to $\mathbf{H}_O$. The vector $\mathbf{H}_O$ and the fixed plane are defined by the initial position vector $\mathbf{r}_0$

and the initial velocity $\mathbf{v}_0$ of the particle. For convenience, let us assume that the plane of the figure coincides with the fixed plane of motion (Fig. 12.16).

Since the magnitude H_O of the angular momentum of the particle P is constant, the right-hand member in Eq. (12.13) must be constant. We therefore write

$$rmv \sin \phi = r_0 m v_0 \sin \phi_0 \tag{12.25}$$

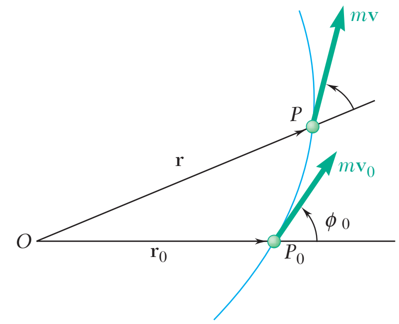


Fig. 12.16

This relation applies to the motion of any particle under a central force. Since the gravitational force exerted by the sun on a planet is a central force directed toward the center of the sun, Eq. (12.25) is fundamental to the study of planetary motion. For a similar reason, it is also fundamental to the study of the motion of space vehicles in orbit about the earth.

Alternatively, recalling Eq. (12.18), we can express the fact that the magnitude H_O of the angular momentum of the particle P is constant by writing

$$mr^2\dot{\theta} = H_O = \text{constant} \tag{12.26}$$

or, dividing by m and denoting by h the angular momentum per unit mass H_O/m,

$$\boxed{r^2\dot{\theta} = h} \tag{12.27}$$

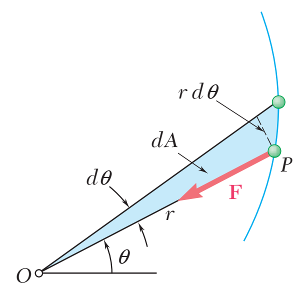


Fig. 12.17

Equation (12.27) can be given an interesting geometric interpretation. Observing from Fig. 12.17 that the radius vector OP sweeps an infinitesimal area $dA = \frac{1}{2}r^2\,d\theta$ as it rotates through an angle $d\theta$, and defining the *areal velocity* of the particle as the quotient dA/dt, we note that the left-hand member of Eq. (12.27) represents twice the areal velocity of the particle. We thus conclude that *when a particle moves under a central force, its areal velocity is constant.*

12.10. NEWTON'S LAW OF GRAVITATION

As you saw in the preceding section, the gravitational force exerted by the sun on a planet or by the earth on an orbiting satellite is an important example of a central force. In this section, you will learn how to determine the magnitude of a gravitational force.

In his *law of universal gravitation,* Newton states that two particles of masses M and m at a distance r from each other attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line joining the particles (Fig. 12.18). The common magnitude F of the two forces is

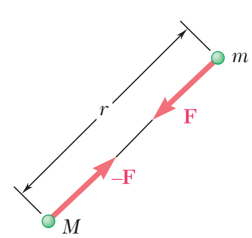


Fig. 12.18

$$\boxed{F = G\frac{Mm}{r^2}} \tag{12.28}$$

where G is a universal constant, called the *constant of gravitation.* Experiments show that the value of G is $(66.73 \pm 0.03) \times 10^{-12}$ $\mathrm{m}^3/\mathrm{kg} \cdot \mathrm{s}^2$ in SI units or approximately 34.4×10^{-9} $\mathrm{ft}^4/\mathrm{lb} \cdot \mathrm{s}^4$ in U.S. customary units. Gravitational forces exist between any pair of bodies, but their effect is appreciable only when one of the bodies has a very large mass. The effect of gravitational forces is apparent in the cases of the motion of a planet about the sun, of satellites orbiting about the earth, or of bodies falling on the surface of the earth.

Since the force exerted by the earth on a body of mass m located on or near its surface is defined as the weight $\mathbf{W}$ of the body, we can substitute the magnitude $W = mg$ of the weight for F, and the radius R of the earth for r, in Eq. (12.28). We obtain

$$W = mg = \frac{GM}{R^2} m \qquad \text{or} \qquad g = \frac{GM}{R^2} \tag{12.29}$$

where M is the mass of the earth. Since the earth is not truly spherical, the distance R from the center of the earth depends upon the point selected on its surface, and the values of W and g will thus vary with the altitude and latitude of the point considered. Another reason for the variation of W and g with latitude is that a system of axes attached to the earth does not constitute a newtonian frame of reference (see Sec. 12.2). A more accurate definition of the weight of a body should therefore include a component representing the centrifugal force due to the rotation of the earth. Values of g at sea level vary from 9.781 $\mathrm{m/s^2}$, or 32.09 $\mathrm{ft/s^2}$, at the equator to 9.833 $\mathrm{m/s^2}$, or 32.26 $\mathrm{ft/s^2}$, at the poles.†

The force exerted by the earth on a body of mass m located in space at a distance r from its center can be found from Eq. (12.28). The computations will be somewhat simplified if we note that according to Eq. (12.29), the product of the constant of gravitation G and the mass M of the earth can be expressed as

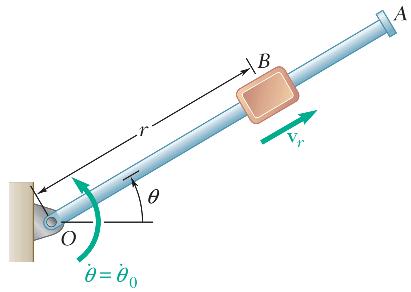
$$GM = gR^2 \tag{12.30}$$

where g and the radius R of the earth will be given their average values $g = 9.81\ \mathrm{m/s^2}$ and $R = 6.37 \times 10^6$ m in SI units‡ and $g = 32.2\ \mathrm{ft/s^2}$ and $R = (3960\ \mathrm{mi})(5280\ \mathrm{ft/mi})$ in U.S. customary units.

The discovery of the law of universal gravitation has often been attributed to the belief that, after observing an apple falling from a tree, Newton had reflected that the earth must attract an apple and the moon in much the same way. While it is doubtful that this incident actually took place, it may be said that Newton would not have formulated his law if he had not first perceived that the acceleration of a falling body must have the same cause as the acceleration which keeps the moon in its orbit. This basic concept of the continuity of gravitational attraction is more easily understood today, when the gap between the apple and the moon is being filled with artificial earth satellites.

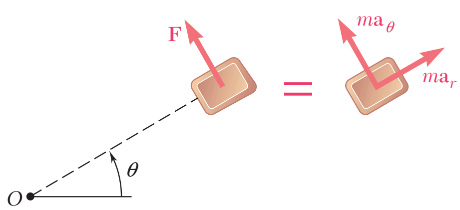
†A formula expressing g in terms of the latitude ϕ was given in Prob. 12.1.

‡The value of R is easily found if one recalls that the circumference of the earth is $2\pi R = 40 \times 10^6$ m.



SAMPLE PROBLEM 12.7

A block B of mass m can slide freely on a frictionless arm OA which rotates in a horizontal plane at a constant rate $\dot{\theta}_0$. Knowing that B is released at a distance r_0 from O, express as a function of r, (*a*) the component v_r of the velocity of B along OA, (*b*) the magnitude of the horizontal force $\mathbf{F}$ exerted on B by the arm OA.



SOLUTION

Since all other forces are perpendicular to the plane of the figure, the only force shown acting on B is the force $\mathbf{F}$ perpendicular to OA.

Equations of Motion. Using radial and transverse components.

$$+\nearrow\Sigma F_r = ma_r: \qquad 0 = m(\ddot{r} - r\dot{\theta}^2) \tag{1}$$

$$+\nwarrow\Sigma F_\theta = ma_\theta: \qquad F = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{2}$$

***a.* Component v_r of Velocity.** Since $v_r = \dot{r}$, we have

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

Substituting for $\ddot{r}$ in (1), recalling that $\dot{\theta} = \dot{\theta}_0$, and separating the variables,

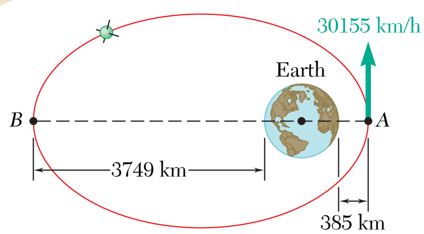
$$v_r\, dv_r = \dot{\theta}_0^2 r\, dr$$

Multiplying by 2, and integrating from 0 to v_r and from r_0 to r,

$$v_r^2 = \dot{\theta}_0^2(r^2 - r_0^2) \qquad v_r = \dot{\theta}_0(r^2 - r_0^2)^{1/2} \quad ◀$$

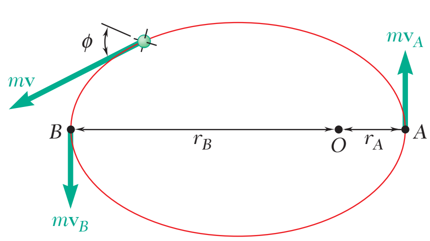
***b.* Horizontal Force F.** Setting $\dot{\theta} = \dot{\theta}_0$, $\ddot{\theta} = 0$, $\dot{r} = v_r$ in Eq. (2), and substituting for v_r the expression obtained in part *a*,

$$F = 2m\dot{\theta}_0(r^2 - r_0^2)^{1/2}\dot{\theta}_0 \qquad F = 2m\dot{\theta}_0^2(r^2 - r_0^2)^{1/2} \quad ◀$$



SAMPLE PROBLEM 12.8

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 30155 km/h from an altitude of 385 km. Determine the velocity of the satellite as it reaches its maximum altitude of 3749 km. It is recalled that the radius of the earth is 6345 km.



SOLUTION

Since the satellite is moving under a central force directed toward the center O of the earth, its angular momentum $\mathbf{H}_O$ is constant. From Eq. (12.13) we have

$$rmv \sin\phi = H_O = \text{constant}$$

which shows that v is minimum at B, where both r and $\sin\phi$ are maximum. Expressing conservation of angular momentum between A and B.

$$r_A m v_A = r_B m v_B$$

$$v_B = v_A\frac{r_A}{r_B} = (30{,}155 \text{ km/h})\cdot\frac{6345 \text{ km} + 385 \text{ km}}{6345 \text{ km} + 3749 \text{ km}}\cdot$$

$$v_B = 20{,}105 \text{ km/h} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we continued our study of Newton's second law by expressing the force and the acceleration in terms of their *radial and transverse components,* where the corresponding equations of motion are

$$\Sigma F_r = ma_r: \qquad \Sigma F_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\Sigma F_\theta = ma_\theta: \qquad \Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

We introduced the *moment of the momentum*, or the *angular momentum*, $\mathbf{H}_O$ of a particle about O:

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{12.12}$$

and found that $\mathbf{H}_O$ is constant when the particle moves under a *central force* with its center located at O.

1. *Using radial and transverse components.* Radial and transverse components were introduced in the last lesson of Chap. 11 [Sec. 11.14]; you should review that material before attempting to solve the following problems. Also, our comments in the preceding lesson regarding the application of Newton's second law (drawing a free-body diagram and a $m\mathbf{a}$ diagram, etc.) still apply [Sample Prob. 12.7]. Finally, note that the solution of that sample problem depends on the application of techniques developed in Chap. 11—you will need to use similar techniques to solve some of the problems of this lesson.

2. *Solving problems involving the motion of a particle under a central force.* In problems of this type, the angular momentum $\mathbf{H}_O$ of the particle about the center of force O is conserved. You will find it convenient to introduce the constant $h = H_O/m$ representing the angular momentum per unit mass. Conservation of the angular momentum of the particle P about O can then be expressed by either of the following equations

$$rv \sin \phi = h \qquad \text{or} \qquad r^2\dot{\theta} = h$$

where r and θ are the polar coordinates of P, and ϕ is the angle that the velocity $\mathbf{v}$ of the particle forms with the line OP (Fig. 12.16). The constant h can be determined from the initial conditions and either of the above equations can be solved for one unknown.

3. *In space-mechanics problems* involving the orbital motion of a planet about the sun, or a satellite about the earth, the moon, or some other planet, the central force **F** is the force of gravitational attraction; it is directed *toward* the center of force O and has the magnitude

$$F = G\frac{Mm}{r^2} \qquad (12.28)$$

Note that in the particular case of the gravitational force exerted by the earth, the product GM can be replaced by gR^2, where R is the radius of the earth [Eq. 12.30].

The following two cases of orbital motion are frequently encountered:

a. For a satellite in a circular orbit, the force **F** is normal to the orbit and you can write $F = ma_n$; substituting for F from Eq. (12.28) and observing that $a_n = v^2/\rho = v^2/r$, you will obtain

$$G\frac{Mm}{r^2} = m\frac{v^2}{r} \qquad \text{or} \qquad v^2 = \frac{GM}{r}$$

b. For a satellite in an elliptic orbit, the radius vector **r** and the velocity **v** of the satellite are perpendicular to each other at the points A and B which are, respectively, farthest and closest to the center of force O [Sample Prob. 12.8]. Thus, conservation of angular momentum of the satellite between these two points can be expressed as

$$r_A m v_A = r_B m v_B$$

Problems

12.66 A 0.5-kg block B slides without friction inside a slot cut in arm OA which rotates in a vertical plane at a constant rate, $\dot{\theta} = 2$ rad/s. At the instant when $\theta = 30°$, $r = 0.6$ m and the force exerted on the block by the arm is zero. Determine, at this instant, (*a*) the relative velocity of the block with respect to the arm, (*b*) the relative acceleration of the block with respect to the arm.

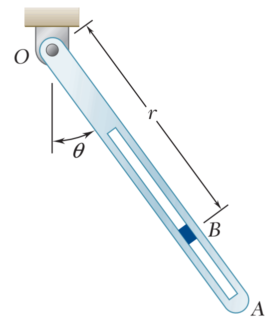


Fig. P12.66 and P12.67

12.67 A 0.5-kg block B slides without friction inside a slot cut in arm OA which rotates in a vertical plane. The motion of the rod is defined by the relation $\ddot{\theta} = 10$ rad/s^2, constant. At the instant when $\theta = 45°$, $r = 0.8$ m and the velocity of the block is zero. Determine, at this instant, (*a*) the force exerted on the block by the arm, (*b*) the relative acceleration of the block with respect to the arm.

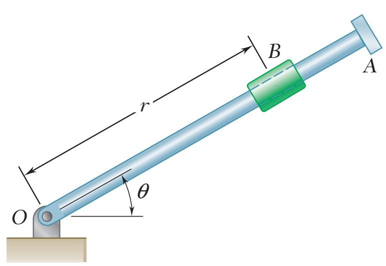


Fig. P12.68 and P12.69

12.68 The motion of a 1.8 kg block B in a horizontal plane is defined by the relations $r = 3t^2 - t^3$ and $\theta = 2t^2$, where r is expressed in meter, t in seconds, and θ in radians. Determine the radial and transverse components of the force exerted on the block when (*a*) $t = 0$, (*b*) $t = 1$ s.

12.69 The motion of a .45 kg block B in a horizontal plane is defined by the relations $r = 6(1 + \cos 2\pi t)$ and $\theta = 2\pi t$, where r is expressed in meter, t in seconds, and θ in radians. Determine the radial and transverse components of the force exerted on the block when (*a*) $t = 0$, (*b*) $t = 0.75$ s.

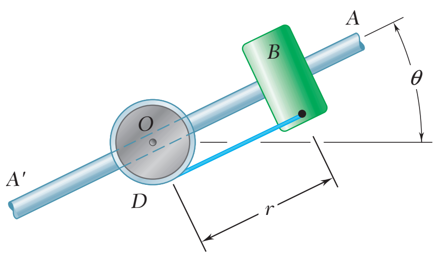


Fig. *P12.70*

12.70 The 2.7 kg collar B slides on the frictionless arm AA'. The arm is attached to drum D and rotates about O in a horizontal plane at the rate $\dot{\theta} = 0.8t$, where $\dot{\theta}$ and t are expressed in rad/s and seconds, respectively. As the arm-drum assembly rotates, a mechanism within the drum releases cord so that the collar moves outward from O with a constant speed of .457 m/s. Knowing that at $t = 0$, $r = 0$, determine the time at which the tension in the cord is equal to the magnitude of the horizontal force exerted on B by arm AA'.

12.71 The horizontal rod OA rotates about a vertical shaft according to the relation $\dot{\theta} = 10t$, where $\dot{\theta}$ and t are expressed in rad/s and seconds, respectively. A .23 kg collar B is held by a cord with a breaking strength of 1.8 kg. Neglecting friction, determine, immediately after the cord breaks, (*a*) the relative acceleration of the collar with respect to the rod, (*b*) the magnitude of the horizontal force exerted on the collar by the rod.

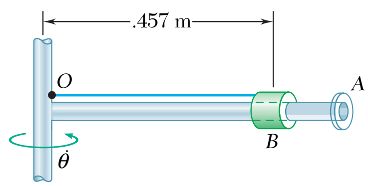


Fig. *P12.71*

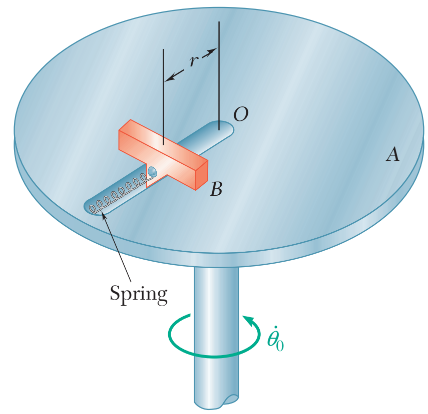


Fig. P12.72

12.72 Disk A rotates in a horizontal plane about a vertical axis at the constant rate of $\dot{\theta}_0 = 15$ rad/s. Slider B has a mass of 230 g and moves in a frictionless slot cut in the disk. The slider is attached to a spring of constant $k = 60$ N/m, which is undeformed when $r = 0$. Knowing that at a given instant the acceleration of the slider relative to the disk is $\ddot{r} = -12$ m/s^2 and that the horizontal force exerted on the slider by the disk is 9 N, determine at that instant (*a*) the distance r, (*b*) the radial component of the velocity of the slider.

12.73 A 1.5-kg collar is attached to a spring and slides without friction along a circular rod in a vertical plane. Knowing that the tension in the spring is 70 N and the speed of the collar is 3.8 m/s as it passes through point A, determine, at that instant, the radial and transverse components of acceleration of the collar.

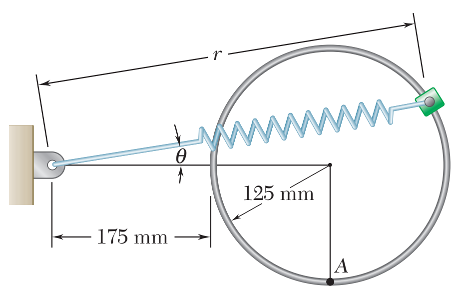


Fig. P12.73

12.74 The two blocks are released from rest when $r = .73$ m and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine (*a*) the initial tension in the cable, (*b*) the initial acceleration of block A, (*c*) the initial acceleration of block B.

12.75 The velocity of block A is 1.8 m/s to the right at the instant when $r = 2.4$ ft and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine, at this instant, (*a*) the tension in the cable, (*b*) the acceleration of block A, (*c*) the acceleration of block B.

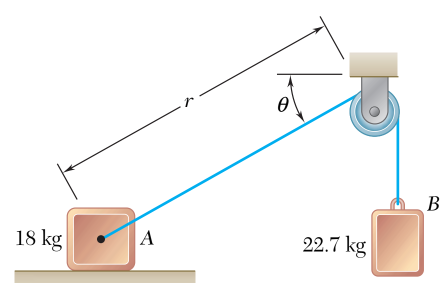


Fig. P12.74 and P12.75

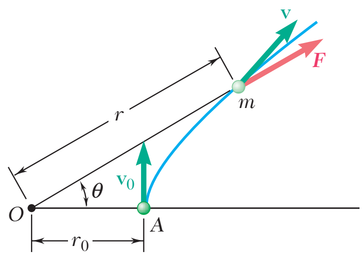


Fig. P12.76 and P12.77

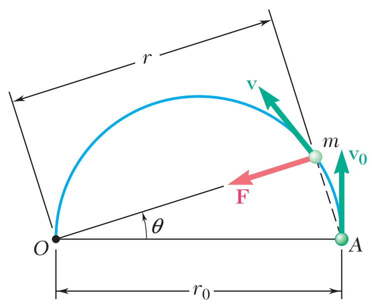


Fig. P12.78 and *P12.79*

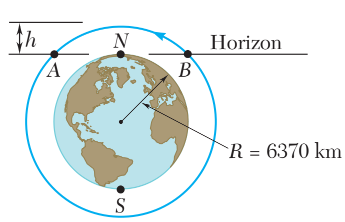


Fig. P12.85

12.76 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to line OA and moves under a central force $\mathbf{F}$ directed away from the center of force O. Knowing that the particle follows a path defined by the equation $r = r_0/\sqrt{\cos 2\theta}$ and using Eq. (12.27), express the radial and transverse components of the velocity $\mathbf{v}$ of the particle as functions of θ.

12.77 For the particle of Prob. 12.76, show (a) that the velocity of the particle and the central force $\mathbf{F}$ are proportional to the distance r from the particle to the center of force O, (b) that the radius of curvature of the path is proportional to r^3.

12.78 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to line OA and moves under a central force $\mathbf{F}$ along a semicircular path of diameter OA. Observing that $r = r_0 \cos\theta$ and using Eq. (12.27), show that the speed of the particle is $v = v_0/\cos^2\theta$.

12.79 For the particle of Prob. 12.78, determine the tangential component F_t of the central force $\mathbf{F}$ along the tangent to the path of the particle for (a) $\theta = 0$, (b) $\theta = 45°$.

12.80 The radius of the orbit of a moon of a given planet is three times as large as the radius of that planet. Denoting by ρ the mean density of the planet, show that the time required by the moon to complete one full revolution about the planet is $9(\pi/G\rho)^{1/2}$, where G is the constant of gravitation.

12.81 Communication satellites are placed in a geosynchronous orbit, that is, in a circular orbit such that they complete one full revolution about the earth in one sidereal day (23.934 h), and thus appear stationary with respect to the ground. Determine (a) the altitude of these satellites above the surface of the earth, (b) the velocity with which they describe their orbit. Give the answers in both SI and U.S. customary units.

12.82 Show that the radius r of the orbit of a moon of a given planet can be determined from the radius R of the planet, the acceleration of gravity at the surface of the planet, and the time τ required by the moon to complete one full revolution about the planet. Determine the acceleration of gravity at the surface of the planet Jupiter knowing that $R = 71142$ km, $\tau = 3.551$ days, and $r = 668162$ km for its moon Europa.

12.83 The orbit of the planet Venus is nearly circular with an orbital velocity of 125460.7 km/h. Knowing that the mean distance from the center of the sun to the center of Venus is 107.7×10^6 km and that the radius of the sun is 692×10^3 km, determine (a) the mass of the sun, (b) the acceleration of gravity at the surface of the sun.

12.84 The periodic times of the planet Jupiter's satellites, Ganymede and Callisto, have been observed to be 7.15 days and 16.69 days, respectively. Knowing that the mass of Jupiter is 319 times that of the earth and that the orbits of the two satellites are circular, determine (a) the radius of the orbit of Ganymede, (b) the velocity with which Callisto describes its orbit. Give the answers in SI units. (The *periodic time* of a satellite is the time it requires to complete one full revolution about the planet.)

12.85 The periodic time (see Prob. 12.84) of an earth satellite in a circular polar orbit is 120 min. Determine (a) the altitude h of the satellite, (b) the time during which the satellite is above the horizon for an observer located at the north pole.

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12.86 A space vehicle is in a circular orbit 320.5 km above the surface of the moon. Knowing that the radius and mass of the moon are 1730.5 km and 2.3×10^{21} kg respectively, determine (*a*) the acceleration of gravity at the surface of the moon, (*b*) the periodic time (see Prob. 12.84) of the space vehicle.

12.87 The periodic times (see Prob. 12.84) of the planet Saturn's satellites Tethys and Rhea have been observed to be 1.888 days and 4.52 days, respectively. Assuming the orbits are circular and knowing that the radius of the orbit of Tethys is 293.7×10^3 km, determine (*a*) the radius of the orbit of Rhea, (*b*) the mass of Saturn.

12.88 During a flyby of the earth, the velocity of a spacecraft is 10.4×10^3 m/s as it reaches its minimum altitude of 960 km above the surface at point *A*. At point *B* the spacecraft is observed to have an altitude of 8300 km. Assuming that the trajectory of the spacecraft is parabolic, determine its velocity at *B*.

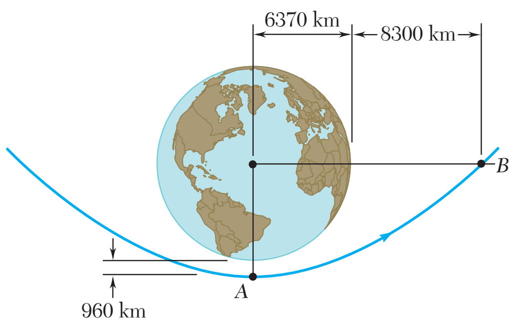


Fig. P12.88

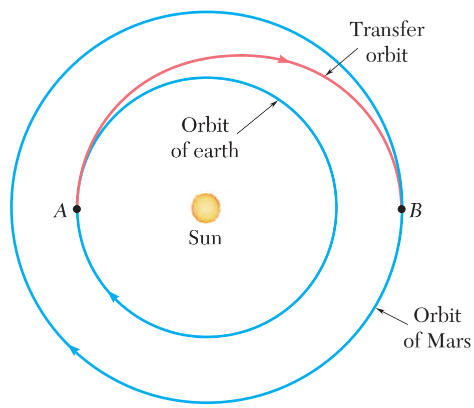


Fig. P12.89

12.89 As a first approximation to the analysis of a space flight from the earth to Mars, assume the orbits of the earth and Mars are circular and coplanar. The mean distances from the sun to the earth and to Mars are 149×10^6 km and 226.7×10^6 km respectively. To place the spacecraft into an elliptical transfer orbit at point *A*, its speed is increased over a short interval of time to v_A which is 2.9 km/s faster than the earth's orbital speed. When the spacecraft reaches point *B* on the elliptical transfer orbit, its speed v_B is increased to the orbital speed of Mars. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine the increase in speed required at *B*.

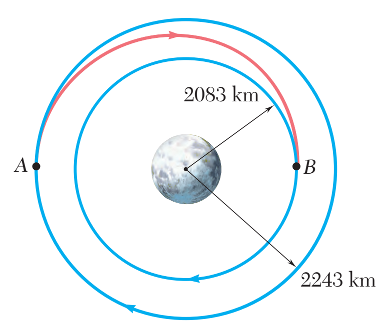


Fig. *P12.90*

12.90 A space vehicle is in a circular orbit of 2243 km radius around the moon. To transfer to a smaller orbit of 2083 km radius, the vehicle is first placed in an elliptic path *AB* by reducing its speed by 26.2 m/s as it passes through *A*. Knowing that the mass of the moon is 2.28×10^{21} kg determine (*a*) the speed of the vehicle as it approaches *B* on the elliptic path, (*b*) the amount by which its speed should be reduced as it approaches *B* to insert it into the smaller circular orbit.

12.91 A space shuttle S and a satellite A are in the circular orbits shown. In order for the shuttle to recover the satellite, the shuttle is first placed in an elliptic path BC by increasing its speed by $\Delta v_B = 85$ m/s as it passes through B. As the shuttle approaches C, its speed is increased by $\Delta v_C = 79$ m/s to insert it into a second elliptic transfer orbit CD. Knowing that the distance from O to C is 6900 km, determine the amount by which the speed of the shuttle should be increased as it approaches D to insert it into the circular orbit of the satellite.

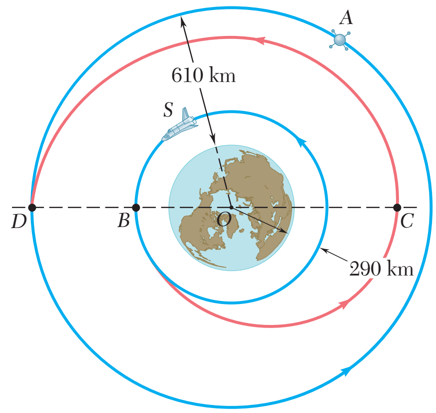


Fig. P12.91

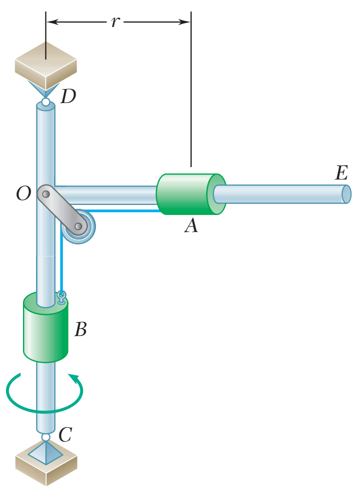


Fig. P12.92 and P12.93

12.92 Two 1.18 kg collars A and B can slide without friction on a frame, consisting of the horizontal rod OE and the vertical rod CD, which is free to rotate about CD. The two collars are connected by a cord running over a pulley that is attached to the frame at O and a stop prevents collar B from moving. The frame is rotating at the rate $\dot{\theta} = 10$ rad/s and $r = .18$ m when the stop is removed allowing collar A to move out along rod OE. Neglecting friction and the mass of the frame, determine (a) the tension in the cord and the acceleration of collar A relative to rod OE immediately after the stop is removed, (b) the transverse component of the velocity of collar A when $r = .27$ m.

12.93 Two 1.18 kg collars A and B can slide without friction on a frame, consisting of the horizontal rod OE and the vertical rod CD, which is free to rotate about CD. The two collars are connected by a cord running over a pulley that is attached to the frame at O and a stop prevents collar B from moving. The frame is rotating at the rate $\dot{\theta} = 12$ rad/s and $r = .18$ m when the stop is removed allowing collar A to move out along rod OE. Neglecting friction and the mass of the frame, determine, for the position $r = .37$ m, (a) the transverse component of the velocity of collar A, (b) the tension in the cord and the acceleration of collar A relative to the rod OE.

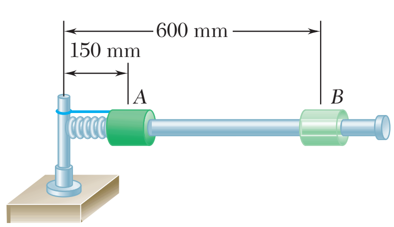


Fig. *P12.94*

12.94 A 300-g collar can slide on a horizontal rod which is free to rotate about a vertical shaft. The collar is initially held at A by a cord attached to the shaft and compresses a spring of constant 5 N/m, which is undeformed when the collar is located 750 mm from the shaft. As the rod rotates at the rate $\dot{\theta} = 12$ rad/s, the cord is cut and the collar moves out along the rod. Neglecting friction and the mass of the rod, determine for position B of the collar (a) the transverse component of the velocity of the collar, (b) the radial and transverse components of its acceleration, (c) the acceleration of the collar relative to the rod.

12.95 In Prob. 12.94, determine for position B of the collar, (a) the radial component of the velocity of the collar, (b) the value of $\ddot{\theta}$.

*12.11. TRAJECTORY OF A PARTICLE UNDER A CENTRAL FORCE

Consider a particle P moving under a central force $\mathbf{F}$. We propose to obtain the differential equation which defines its trajectory.

Assuming that the force $\mathbf{F}$ is directed toward the center of force O, we note that ΣF_r and ΣF_θ reduce, respectively, to $-F$ and zero in Eqs. (12.21) and (12.22). We therefore write

$$m(\ddot{r} - r\dot{\theta}^2) = -F \tag{12.31}$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{12.32}$$

These equations define the motion of P. We will, however, replace Eq. (12.32) by Eq. (12.27), which is equivalent to Eq. (12.32), as can easily be checked by differentiating it with respect to t, but which is more convenient to use. We write

$$r^2\dot{\theta} = h \qquad \text{or} \qquad r^2\frac{d\theta}{dt} = h \tag{12.33}$$

Equation (12.33) can be used to eliminate the independent variable t from Eq. (12.31). Solving Eq. (12.33) for $\dot{\theta}$ or $d\theta/dt$, we have

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} \tag{12.34}$$

from which it follows that

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{12.35}$$
$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta}$$

or, substituting for $\dot{r}$ from (12.35),

$$\ddot{r} = \frac{h}{r^2}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]$$
$$\ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{12.36}$$

Substituting for θ and $\ddot{r}$ from (12.34) and (12.36), respectively, in Eq. (12.31) and introducing the function $u = 1/r$, we obtain after reductions

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

In deriving Eq. (12.37), the force $\mathbf{F}$ was assumed directed toward O. The magnitude F should therefore be positive if $\mathbf{F}$ is actually directed toward O (attractive force) and negative if $\mathbf{F}$ is directed away from O (repulsive force). If F is a known function of r and thus of u, Eq. (12.37) is a differential equation in u and θ. This differential equation defines the trajectory followed by the particle under the central force $\mathbf{F}$. The equation of the trajectory can be obtained by solving the differential equation (12.37) for u as a function of θ and determining the constants of integration from the initial conditions.

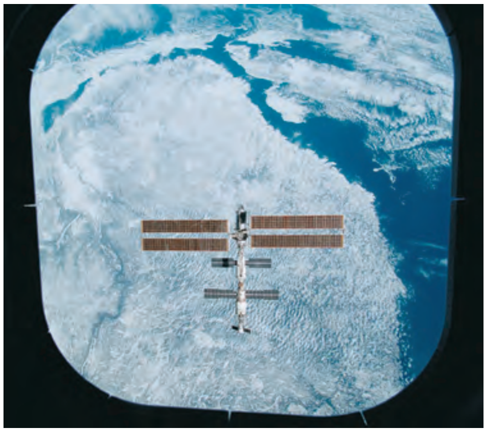


Photo 12.4 The International Space Station and the space vehicle are subjected to the gravitational pull of the earth and if all other forces are neglected, the trajectory of each will be a circle, an ellipse, a parabola, or a hyperbola.

*12.12. APPLICATION TO SPACE MECHANICS

After the last stages of their launching rockets have burned out, earth satellites and other space vehicles are subjected to only the gravitational pull of the earth. Their motion can therefore be determined from Eqs. (12.33) and (12.37), which govern the motion of a particle under a central force, after F has been replaced by the expression obtained for the force of gravitational attraction.† Setting in Eq. (12.37)

$$F = \frac{GMm}{r^2} = GMmu^2$$

where M = mass of earth
m = mass of space vehicle
r = distance from center of earth to vehicle
$u = 1/r$

we obtain the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \tag{12.38}$$

where the right-hand member is observed to be a constant.

The solution of the differential equation (12.38) is obtained by adding the particular solution $u = GM/h^2$ to the general solution $u = C \cos (\theta - \theta_0)$ of the corresponding homogeneous equation (that is, the equation obtained by setting the right-hand member equal to zero). Choosing the polar axis so that $\theta_0 = 0$, we write

$$\boxed{\frac{1}{r} = u = \frac{GM}{h^2} + C \cos \theta} \tag{12.39}$$

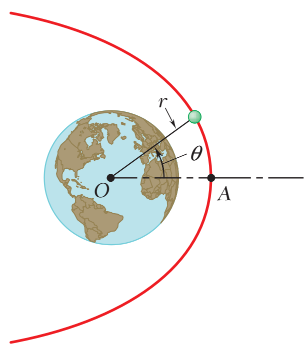


Fig. 12.19

Equation (12.39) is the equation of a *conic section* (ellipse, parabola, or hyperbola) in the polar coordinates r and θ. The origin O of the coordinates, which is located at the center of the earth, is a *focus* of this conic section, and the polar axis is one of its axes of symmetry (Fig. 12.19).

The ratio of the constants C and GM/h^2 defines the *eccentricity* ε of the conic section; letting

$$\varepsilon = \frac{C}{GM/h^2} = \frac{Ch^2}{GM} \tag{12.40}$$

we can write Eq. (12.39) in the form

$$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon \cos \theta) \tag{12.39$'$}$$

This equation represents three possible trajectories.

1. $\varepsilon > 1$, or $C > GM/h^2$: There are two values θ_1 and $-\theta_1$ of the polar angle, defined by $\cos \theta_1 = -GM/Ch^2$, for which

†It is assumed that the space vehicles considered here are attracted by the earth only and that their mass is negligible compared with the mass of the earth. If a vehicle moves very far from the earth, its path may be affected by the attraction of the sun, the moon, or another planet.

the right-hand member of Eq. (12.39) becomes zero. For both these values, the radius vector r becomes infinite; the conic section is a *hyperbola* (Fig. 12.20).

2. $\varepsilon = 1$, or $C = GM/h^2$: The radius vector becomes infinite for $\theta = 180°$; the conic section is a *parabola*.
3. $\varepsilon < 1$, or $C < GM/h^2$: The radius vector remains finite for every value of θ; the conic section is an *ellipse*. In the particular case when $\varepsilon = C = 0$, the length of the radius vector is constant; the conic section is a circle.

Let us now see how the constants C and GM/h^2, which characterize the trajectory of a space vehicle, can be determined from the vehicle's position and velocity at the beginning of its free flight. We will assume that, as is generally the case, the powered phase of its flight has been programmed in such a way that as the last stage of the launching rocket burns out, the vehicle has a velocity parallel to the surface of the earth (Fig. 12.21). In other words, we will assume that the space vehicle begins its free flight at the vertex A of its trajectory.†

Denoting the radius vector and speed of the vehicle at the beginning of its free flight by r_0 and v_0, respectively, we observe that the velocity reduces to its transverse component and, thus, that $v_0 = r_0\dot{\theta}_0$. Recalling Eq. (12.27), we express the angular momentum per unit mass h as

$$h = r_0^2\dot{\theta}_0 = r_0 v_0 \tag{12.41}$$

The value obtained for h can be used to determine the constant GM/h^2. We also note that the computation of this constant will be simplified if we use the relation obtained in Sec. 12.10:

$$GM = gR^2 \tag{12.30}$$

where R is the radius of the earth ($R = 6.37 \times 10^6$ m or 3960 mi) and g is the acceleration of gravity at the surface of the earth.

The constant C is obtained by setting $\theta = 0$, $r = r_0$ in (12.39):

$$C = \frac{1}{r_0} - \frac{GM}{h^2} \tag{12.42}$$

Substituting for h from (12.41), we can then easily express C in terms of r_0 and v_0.

Let us now determine the initial conditions corresponding to each of the three fundamental trajectories indicated above. Considering first the parabolic trajectory, we set C equal to GM/h^2 in Eq. (12.42) and eliminate h between Eqs. (12.41) and (12.42). Solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2GM}{r_0}}$$

We can easily check that a larger value of the initial velocity corresponds to a hyperbolic trajectory and a smaller value corresponds to an elliptic orbit. Since the value of v_0 obtained for the parabolic

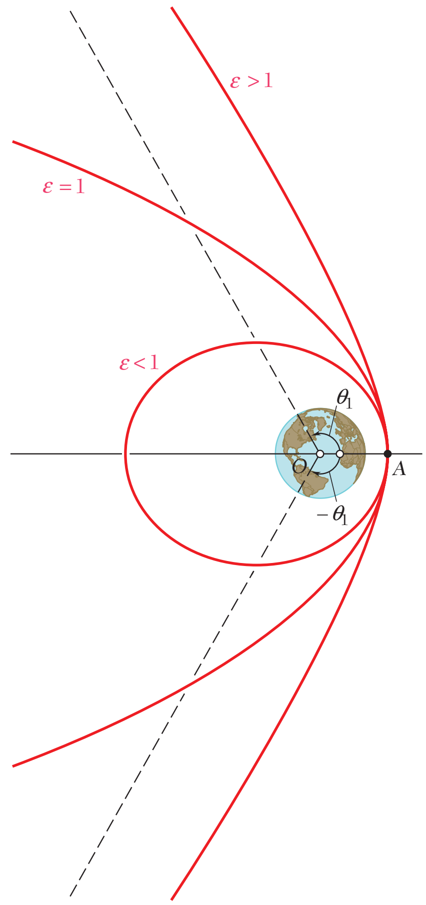


Fig. 12.20

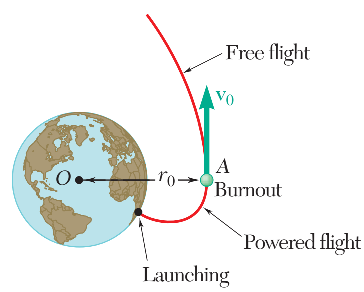


Fig. 12.21

†Problems involving oblique launchings will be considered in Sec. 13.9.

trajectory is the smallest value for which the space vehicle does not return to its starting point, it is called the *escape velocity.* We write therefore

$$v_{esc} = \sqrt{\frac{2GM}{r_0}} \qquad \text{or} \qquad v_{esc} = \sqrt{\frac{2gR^2}{r_0}} \tag{12.43}$$

if we make use of Eq. (12.30). We note that the trajectory will be (1) hyperbolic if $v_0 > v_{esc}$, (2) parabolic if $v_0 = v_{esc}$, and (3) elliptic if $v_0 < v_{esc}$.

Among the various possible elliptic orbits, the one obtained when $C = 0$, the *circular orbit,* is of special interest. The value of the initial velocity corresponding to a circular orbit is easily found to be

$$v_{circ} = \sqrt{\frac{GM}{r_0}} \qquad \text{or} \qquad v_{circ} = \sqrt{\frac{gR^2}{r_0}} \tag{12.44}$$

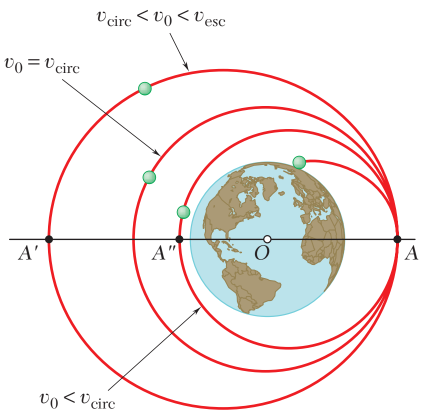


Fig. 12.22

if Eq. (12.30) is taken into account. We note from Fig. 12.22 that for values of v_0 larger than v_{circ} but smaller than v_{esc}, point A where free flight begins is the point of the orbit closest to the earth; this point is called the *perigee,* while point A', which is farthest away from the earth, is known as the *apogee.* For values of v_0 smaller than v_{circ}, point A is the apogee, while point A'', on the other side of the orbit, is the perigee. For values of v_0 much smaller than v_{circ}, the trajectory of the space vehicle intersects the surface of the earth; in such a case, the vehicle does not go into orbit.

Ballistic missiles, which were designed to hit the surface of the earth, also travel along elliptic trajectories. In fact, we should now realize that any object projected in vacuum with an initial velocity v_0 smaller than v_{esc} will move along an elliptic path. It is only when the distances involved are small that the gravitational field of the earth can be assumed uniform and that the elliptic path can be approximated by a parabolic path, as was done earlier (Sec. 11.11) in the case of conventional projectiles.

Periodic Time. An important characteristic of the motion of an earth satellite is the time required by the satellite to describe its orbit. This time, known as the *periodic time* of the satellite, is denoted by τ. We first observe, in view of the definition of areal velocity (Sec. 12.9), that τ can be obtained by dividing the area inside the orbit by the areal velocity. Noting that the area of an ellipse is equal to πab, where a and b denote the semimajor and semiminor axes, respectively, and that the areal velocity is equal to $h/2$, we write

$$\tau = \frac{2\pi ab}{h} \tag{12.45}$$

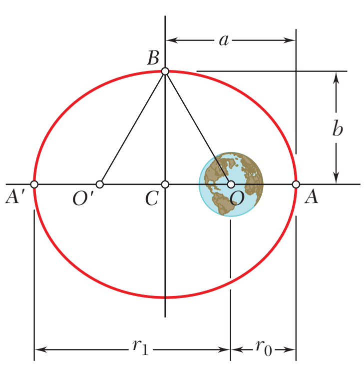


Fig. 12.23

While h can be readily determined from r_0 and v_0 in the case of a satellite launched in a direction parallel to the surface of the earth, the semiaxes a and b are not directly related to the initial conditions. Since, on the other hand, the values r_0 and r_1 of r corresponding to the perigee and apogee of the orbit can easily be determined from Eq. (12.39), we will express the semiaxes a and b in terms of r_0 and r_1.

Consider the elliptic orbit shown in Fig. 12.23. The earth's center is located at O and coincides with one of the two foci of the ellipse, while the points A and A' represent, respectively, the perigee and apogee of the orbit. We easily check that

$$r_0 + r_1 = 2a$$

and thus

$$a = \tfrac{1}{2}(r_0 + r_1) \tag{12.46}$$

Recalling that the sum of the distances from each of the foci to any point of the ellipse is constant, we write

$$O'B + BO = O'A + OA = 2a \qquad \text{or} \qquad BO = a$$

On the other hand, we have $CO = a - r_0$. We can therefore write

$$b^2 = (BC)^2 = (BO)^2 - (CO)^2 = a^2 - (a - r_0)^2$$
$$b^2 = r_0(2a - r_0) = r_0 r_1$$

and thus

$$b = \sqrt{r_0 r_1} \tag{12.47}$$

Formulas (12.46) and (12.47) indicate that the semimajor and semiminor axes of the orbit are equal, respectively, to the arithmetic and geometric means of the maximum and minimum values of the radius vector. Once r_0 and r_1 have been determined, the lengths of the semiaxes can be easily computed and substituted for a and b in formula (12.45).

*12.13. KEPLER'S LAWS OF PLANETARY MOTION

The equations governing the motion of an earth satellite can be used to describe the motion of the moon around the earth. In that case, however, the mass of the moon is not negligible compared with the mass of the earth, and the results obtained are not entirely accurate.

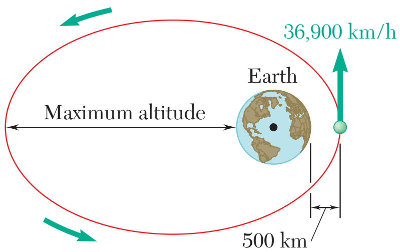
The theory developed in the preceding sections can also be applied to the study of the motion of the planets around the sun. Although another error is introduced by neglecting the forces exerted by the planets on one another, the approximation obtained is excellent. Indeed, even before Newton had formulated his fundamental theory, the properties expressed by Eq. (12.39), where M now represents the mass of the sun, and by Eq. (12.33) had been discovered by the German astronomer Johann Kepler (1571–1630) from astronomical observations of the motion of the planets.

Kepler's three *laws of planetary motion* can be stated as follows:

1. Each planet describes an ellipse, with the sun located at one of its foci.
2. The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
3. The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits.

The first law states a particular case of the result established in Sec. 12.12, and the second law expresses that the areal velocity of each planet is constant (see Sec. 12.9). Kepler's third law can also be derived from the results obtained in Sec. 12.12.†

†See Prob. 12.120.



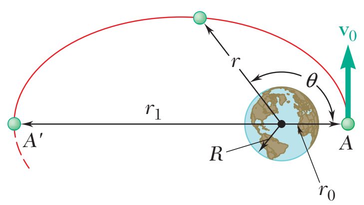
SAMPLE PROBLEM 12.9

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36 900 km/h from an altitude of 500 km. Determine (*a*) the maximum altitude reached by the satellite, (*b*) the periodic time of the satellite.

SOLUTION

***a.* Maximum Altitude.** After the satellite is launched, it is subjected only to the gravitational attraction of the earth; its motion is thus governed by Eq. (12.39),

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta \qquad (1)$$



Since the radial component of the velocity is zero at the point of launching A, we have $h = r_0 v_0$. Recalling that for the earth $R = 6370$ km, we compute

$$r_0 = 6370 \text{ km} + 500 \text{ km} = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$$
$$v_0 = 36\,900 \text{ km/h} = \frac{36.9 \times 10^6 \text{ m}}{3.6 \times 10^3 \text{ s}} = 10.25 \times 10^3 \text{ m/s}$$
$$h = r_0 v_0 = (6.87 \times 10^6 \text{ m})(10.25 \times 10^3 \text{ m/s}) = 70.4 \times 10^9 \text{ m}^2\text{/s}$$
$$h^2 = 4.96 \times 10^{21} \text{ m}^4\text{/s}^2$$

Since $GM = gR^2$, where R is the radius of the earth, we have

$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2 = 398 \times 10^{12} \text{ m}^3\text{/s}^2$$
$$\frac{GM}{h^2} = \frac{398 \times 10^{12} \text{ m}^3\text{/s}^2}{4.96 \times 10^{21} \text{ m}^4\text{/s}^2} = 80.3 \times 10^{-9} \text{ m}^{-1}$$

Substituting this value into (1), we obtain

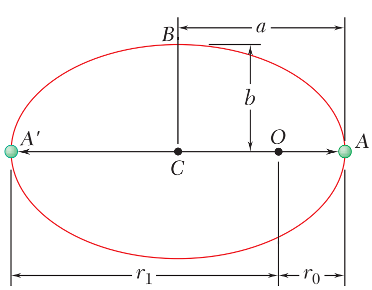
$$\frac{1}{r} = 80.3 \times 10^{-9} \text{ m}^{-1} + C\cos\theta \qquad (2)$$

Noting that at point A we have $\theta = 0$ and $r = r_0 = 6.87 \times 10^6$ m, we compute the constant C:

$$\frac{1}{6.87 \times 10^6 \text{ m}} = 80.3 \times 10^{-9} \text{ m}^{-1} + C\cos 0° \qquad C = 65.3 \times 10^{-9} \text{ m}^{-1}$$

At A', the point on the orbit farthest from the earth, we have $\theta = 180°$. Using (2), we compute the corresponding distance r_1:

$$\frac{1}{r_1} = 80.3 \times 10^{-9} \text{ m}^{-1} + (65.3 \times 10^{-9} \text{ m}^{-1})\cos 180°$$
$$r_1 = 66.7 \times 10^6 \text{ m} = 66\,700 \text{ km}$$
$$\textit{Maximum altitude} = 66\,700 \text{ km} - 6370 \text{ km} = 60\,300 \text{ km} \quad \blacktriangleleft$$



***b.* Periodic Time.** Since A and A' are the perigee and apogee, respectively, of the elliptic orbit, we use Eqs. (12.46) and (12.47) and compute the semimajor and semiminor axes of the orbit:

$$a = \tfrac{1}{2}(r_0 + r_1) = \tfrac{1}{2}(6.87 + 66.7)(10^6) \text{ m} = 36.8 \times 10^6 \text{ m}$$
$$b = \sqrt{r_0 r_1} = \sqrt{(6.87)(66.7)} \times 10^6 \text{ m} = 21.4 \times 10^6 \text{ m}$$
$$\tau = \frac{2\pi ab}{h} = \frac{2\pi(36.8 \times 10^6 \text{ m})(21.4 \times 10^6 \text{ m})}{70.4 \times 10^9 \text{ m}^2\text{/s}}$$
$$\tau = 70.3 \times 10^3 \text{ s} = 1171 \text{ min} = 19 \text{ h } 31 \text{ min} \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson, we continued our study of the motion of a particle under a central force and applied the results to problems in space mechanics. We found that the trajectory of a particle under a central force is defined by the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \qquad (12.37)$$

where u is the reciprocal of the distance r of the particle to the center of force ($u = 1/r$), F is the magnitude of the central force $\mathbf{F}$, and h is a constant equal to the angular momentum per unit mass of the particle. In space-mechanics problems, $\mathbf{F}$ is the force of gravitational attraction exerted on the satellite or spacecraft by the sun, earth, or other planet about which it travels. Substituting $F = GMm/r^2 = GMmu^2$ into Eq. (12.37), we obtain for that case

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \qquad (12.38)$$

where the right-hand member is a constant.

1. Analyzing the motion of satellites and spacecraft. The solution of the differential equation (12.38) defines the trajectory of a satellite or spacecraft. It was obtained in Sec. 12.12 and was given in the alternative forms

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta \qquad \text{or} \qquad \frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon\cos\theta) \qquad (12.39,\ 12.39')$$

Remember when applying these equations that $\theta = 0$ always corresponds to the perigee (the point of closest approach) of the trajectory (Fig. 12.19) and that h is a constant for a given trajectory. Depending on the value of the eccentricity ε, the trajectory will be a hyperbola, a parabola, or an ellipse.

a. ***$\varepsilon > 1$: The trajectory is a hyperbola,*** so that for this case the spacecraft never returns to its starting point.

b. ***$\varepsilon = 1$: The trajectory is a parabola.*** This is the limiting case between open (hyperbolic) and closed (elliptic) trajectories. We had observed for this case that the velocity v_0 at the perigee is equal to the escape velocity v_{esc},

$$v_0 = v_{esc} = \sqrt{\frac{2GM}{r_0}} \qquad (12.43)$$

Note that the escape velocity is the smallest velocity for which the spacecraft does not return to its starting point.

c. $\varepsilon < 1$: The trajectory is an elliptic orbit. For problems involving elliptic orbits, you may find that the relation derived in Prob. 12.102,

$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2GM}{h^2}$$

will be useful in the solution of subsequent problems. When you apply this equation, remember that r_0 and r_1 are the distances from the center of force to the perigee ($\theta = 0$) and apogee ($\theta = 180°$), respectively; that $h = r_0 v_0 = r_1 v_1$; and that, for a satellite orbiting the earth, $GM_{\text{earth}} = gR^2$, where R is the radius of the earth. Also recall that the trajectory is a circle when $\varepsilon = 0$.

2. Determining the point of impact of a descending spacecraft. For problems of this type, you may assume that the trajectory is elliptic and that the initial point of the descent trajectory is the apogee of the path (Fig. 12.22). Note that at the point of impact, the distance r in Eqs. (12.39) and (12.39′) is equal to the radius R of the body on which the spacecraft lands or crashes. In addition, we have $h = Rv_I \sin \phi_I$, where v_I is the speed of the spacecraft at impact and ϕ_I is the angle that its path forms with the vertical at the point of impact.

3. Calculating the time to travel between two points on a trajectory. For central force motion, the time t required for a particle to travel along a portion of its trajectory can be determined by recalling from Sec. 12.9 that the rate at which area is swept per unit time by the position vector $\mathbf{r}$ is equal to one-half of the angular momentum per unit mass h of the particle: $dA/dt = h/2$. It follows, since h is a constant for a given trajectory, that

$$t = \frac{2A}{h}$$

where A is the total area swept in the time t.

a. In the case of an elliptic trajectory, the time required to complete one orbit is called the *periodic time* and is expressed as

$$\tau = \frac{2(\pi ab)}{h} \tag{12.45}$$

where a and b are the semimajor and semiminor axes, respectively, of the ellipse and are related to the distances r_0 and r_1 by

$$a = \frac{1}{2}(r_0 + r_1) \qquad \text{and} \qquad b = \sqrt{r_0 r_1} \qquad (12.46,\ 12.47)$$

b. Kepler's third law provides a convenient relation between the periodic times of two satellites describing elliptic orbits about the same body [Sec. 12.13]. Denoting the semimajor axes of the two orbits by a_1 and a_2, respectively, and the corresponding periodic times by τ_1 and τ_2, we have

$$\frac{\tau_1^2}{\tau_2^2} = \frac{a_1^3}{a_2^3}$$

c. In the case of a parabolic trajectory, you may be able to use the expression given on the inside of the front cover of the book for a parabolic or a semiparabolic area to calculate the time required to travel between two points of the trajectory.

Problems

12.96 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to OA and moves under a central force $\mathbf{F}$ along an elliptic path defined by the equation $r = r_0/(2 - \cos\theta)$. Using Eq. (12.37), show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to the center of force O.

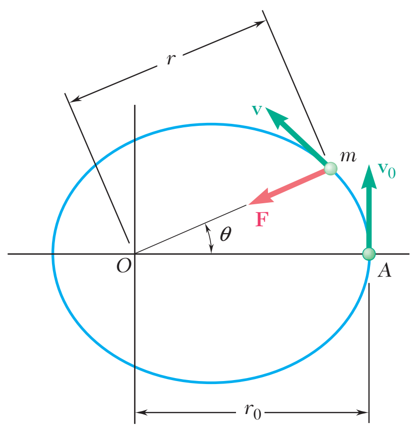


Fig. P12.96

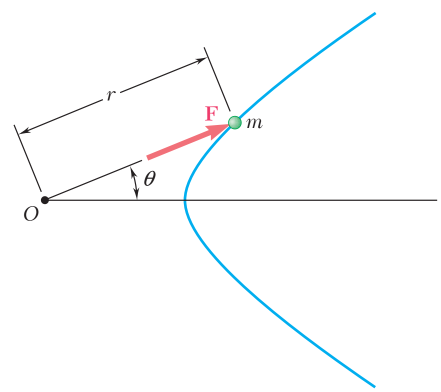


Fig. P12.97

12.97 A particle of mass m describes the path defined by the equation $r = r_0/(6\cos\theta - 5)$ under a central force $\mathbf{F}$ directed away from the center of force O. Using Eq. (12.37), show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to O.

12.98 A particle of mass m describes the parabola $y = x^2/4r_0$ under a central force $\mathbf{F}$ directed toward the center of force C. Using Eq. (12.37) and Eq. (12.39′) with $\varepsilon = 1$, show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to the center of force and that the angular momentum per unit mass $h = \sqrt{2GMr_0}$.

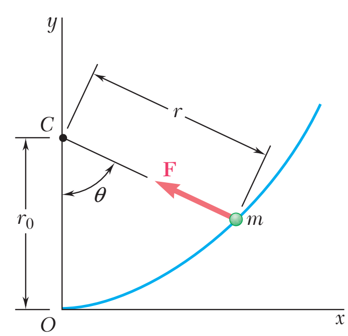


Fig. P12.98

12.99 A particle of mass m describes the logarithmic spiral $r = r_0 e^{b\theta}$ under a central force $\mathbf{F}$ directed toward the center of force O. Using Eq. (12.37) show that $\mathbf{F}$ is inversely proportional to the cube of the distance r from the particle to O.

12.100 It was observed that as the spacecraft Voyager 1 reached the point on its trajectory closest to the planet Jupiter, it was at a distance of 350×10^3 km from the center of the planet and had a velocity of 26.9 km/s. Determine the mass of Jupiter, assuming that the trajectory of the spacecraft was parabolic.

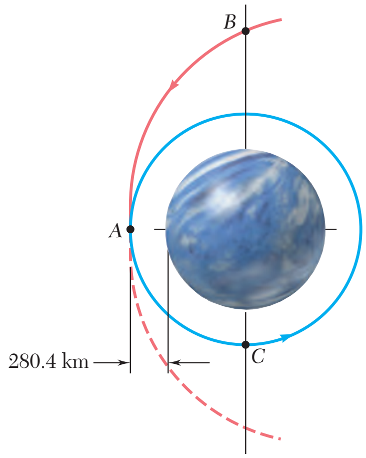


Fig. *P12.101*

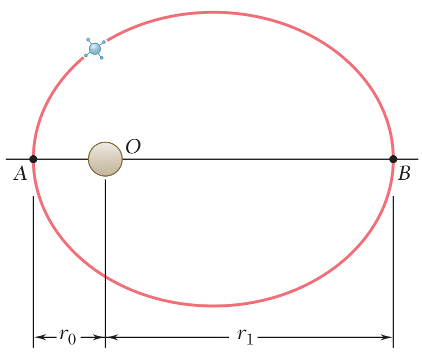


Fig. P12.104

12.101 As a space probe approaching the planet Venus on a parabolic trajectory reaches point A closest to the planet, its velocity is decreased to insert it into a circular orbit. Knowing that the mass and the radius of Venus are 151.5×10^{21} kg and 6026 km respectively, determine (*a*) the velocity of the probe as it approaches A, (*b*) the decrease in velocity required to insert it into a circular orbit.

12.102 It was observed that as the Galileo spacecraft reached the point of its trajectory closest to Io, a moon of the planet Jupiter, it was at a distance of 2820 km from the center of Io and had a velocity of 15 km/s. Knowing that the mass of Io is 0.01496 times the mass of the earth, determine the eccentricity of the trajectory of the spacecraft as it approached Io.

12.103 A small satellite of Jupiter, discovered in 2002, was reported to be an orbit of semimajor axis 23.6×10^6 km and eccentricity 0.45. Knowing that the mass of Jupiter is 318 times the mass of the earth, determine the maximum and minimum speeds of the satellite.

12.104 A satellite describes an elliptic orbit about a planet of mass M. Denoting by r_0 and r_1, respectively, the minimum and maximum values of the distance r from the satellite to the center of the planet, derive the relation

$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2GM}{h^2}$$

where h is the angular momentum per unit mass of the satellite.

12.105 A satellite describes a circular orbit at an altitude of 19019.4 km above the surface of the earth. Determine (*a*) the increase in speed required at point A for the satellite to achieve the escape velocity and enter a parabolic orbit, (*b*) the decrease in speed required at point A for the satellite to enter an elliptic orbit of minimum altitude 6345 km, (*c*) the eccentricity ε of the elliptic orbit.

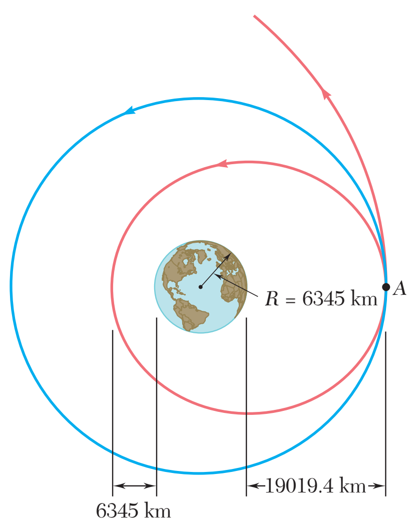


Fig. P12.105

12.106 The Chandra X-ray observatory, launched in 1999, achieved an elliptical orbit of minimum altitude 10 000 km and maximum altitude 140 000 km above the surface of the earth. Assuming that the observatory was transferred to this orbit from a circular orbit of altitude 10 000 km at point A, determine (a) the increase in speed required at A, (b) the speed of the observatory at B.

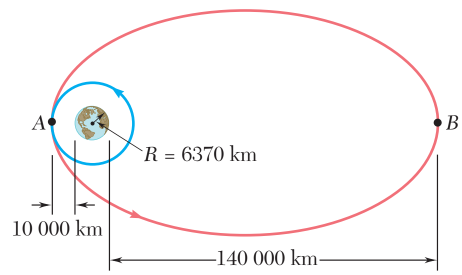


Fig. *P12.106*

12.107 As it describes an elliptic orbit about the sun, a spacecraft reaches a maximum distance of 325×10^6 km from the center of the sun at point A (called the *aphelion*) and a minimum distance of 148×10^6 km at point B (called the *perihelion*). To place the spacecraft in a smaller elliptic orbit with aphelion A' and perihelion B', where A' and B' are located 264.7×10^6 km and 137.6×10^6 km, respectively, from the center of the sun, the speed of the spacecraft is first reduced as it passes through A and then is further reduced as it passes through B'. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine (a) the speed of the spacecraft at A, (b) the amounts by which the speed of the spacecraft should be reduced at A and B' to insert it into the desired elliptic orbit.

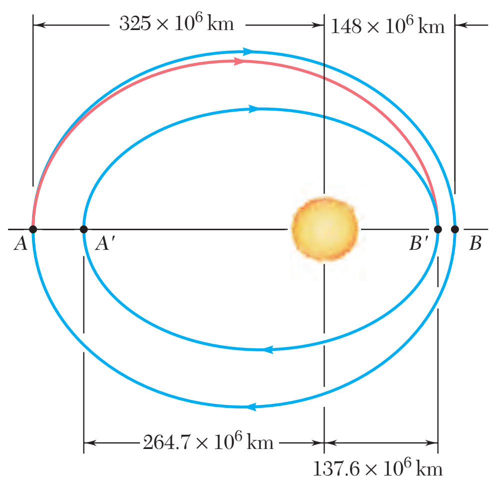


Fig. *P12.107*

12.108 A space probe is to be placed in a circular orbit of 9000-km radius about the planet Venus in a specified plane. As the probe reaches A, the point of its original trajectory closest to Venus, it is inserted in a first elliptic transfer orbit by reducing its speed of Δv_A. This orbit brings it to point B with a much reduced velocity. There the probe is inserted in a second transfer orbit located in the specified plane by changing the direction of its velocity and further reducing its speed by Δv_B. Finally, as the probe reaches point C, it is inserted in the desired circular orbit by reducing its speed by Δv_C. Knowing that the mass of Venus is 0.82 times the mass of the earth, that $r_A = 15 \times 10^3$ km and $r_B = 300 \times 10^3$ km, and that the probe approaches A on a parabolic trajectory, determine by how much the velocity of the probe should be reduced (a) at A, (b) at B, (c) at C.

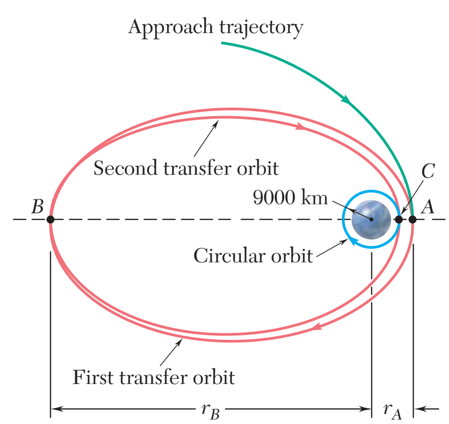


Fig. P12.108

12.109 For the space probe of Prob. 12.108, it is known that $r_A = 15 \times 10^3$ km and that the velocity of the probe is reduced to 6500 m/s as it passes through A. Determine (a) the distance from the center of Venus to point B, (b) the amounts by which the velocity of the probe should be reduced at B and C, respectively.

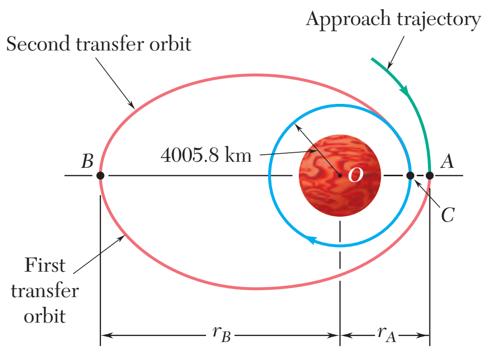


Fig. P12.110

12.110 A space probe is to be placed in a circular orbit of radius 4005.8 km about the planet Mars. As the probe reaches A, the point of its original trajectory closest to Mars, it is inserted into a first elliptic transfer orbit by reducing its speed. This orbit brings it to point B with a much reduced velocity. There the probe is inserted into a second transfer orbit by further reducing its speed. Knowing that the mass of Mars is 0.1074 times the mass of the earth, that $r_A = 9013$ km and $r_B = 180260$ km and that the probe approaches A on a parabolic trajectory, determine the time needed for the space probe to travel from A to B on its first transfer orbit.

12.111 Halley's comet travels in an elongated elliptic orbit for which the minimum distance from the sun in approximately $\frac{1}{2}r_E$, where $r_E = 148.9 \times 10^6$ km is the mean distance from the sun to the earth. Knowing that the periodic time of Halley's comet is about 76 years, determine the maximum distance from the sun reached by the comet.

12.112 A spacecraft and a satellite are at diametrically opposite positions in the same circular orbit of altitude 497 km above the earth. As it passes through point A, the spacecraft fires its engine for a short interval of time to increase its speed and enter an elliptic orbit. Knowing that the spacecraft returns to A at the same time the satellite reaches A after completing one and a half orbits, determine (a) the increase in speed required, (b) the periodic time for the elliptic orbit.

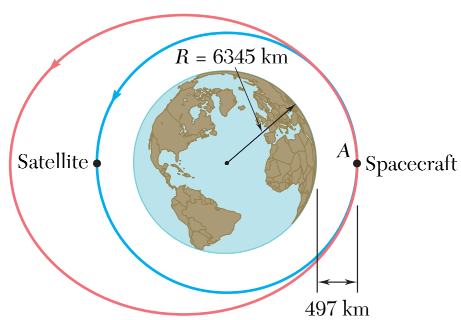


Fig. *P12.112*

12.113 Based on observations made during the 1996 sighting of the comet Hyakutake, it was concluded that the trajectory of the comet is a highly elongated ellipse for which the eccentricity is approximately $\varepsilon = 0.999887$. Knowing that for the 1996 sighting the minimum distance between the comet and the sun was $0.230R_E$, where R_E is the mean distance from the sun to the earth, determine the periodic time of the comet.

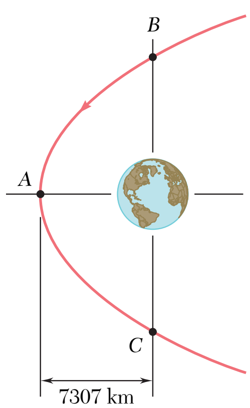


Fig. P12.114

12.114 It was observed that during its first flyby of the earth, the Galileo spacecraft had a velocity of 10.4 km/s as it reached its minimum distance of 7307 km from the center of the earth. Assuming that the trajectory of the spacecraft was parabolic, determine the time needed for the spacecraft to travel from B to C on its trajectory.

12.115 A space shuttle is in an elliptic orbit of eccentricity 0.0356 and a minimum altitude of 292 km above the surface of the earth. Knowing that the radius of the earth is 6345 km, determine the periodic time for the orbit.

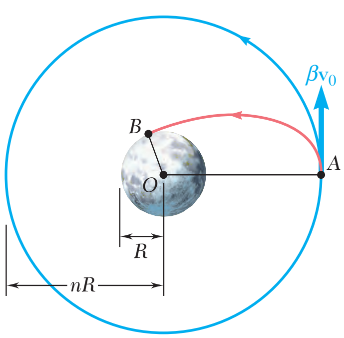


Fig. P12.116

12.116 A space probe is describing a circular orbit of radius nR with a speed v_0 about a planet of radius R and center O. As the probe passes through point A, its speed is reduced from v_0 to βv_0, where $\beta < 1$, to place the probe on a crash trajectory. Express in terms of n and β the angle AOB, where B denotes the point of impact of the probe on the planet.

12.117 Prior to the Apollo missions to the moon, several Lunar Orbiter spacecraft were used to photograph the lunar surface to obtain information regarding possible landing sites. At the conclusion of each mission, the trajectory of the spacecraft was adjusted so that the spacecraft would crash on the moon to further study the characteristics of the lunar surface. Shown below is the elliptic orbit of Lunar Orbiter 2. Knowing that the mass of the moon is 0.01230 times the mass of the earth, determine the amount by which the speed of the orbiter should be reduced at point B so that it impacts the lunar surface at point C. (*Hint*. Point B is the apogee of the elliptic impact trajectory.)

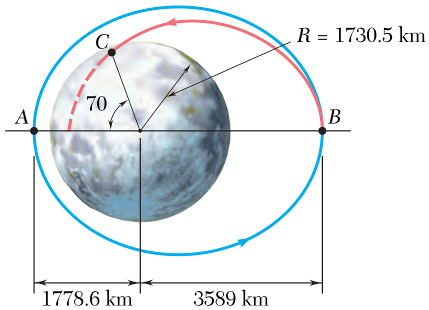


Fig. P12.117

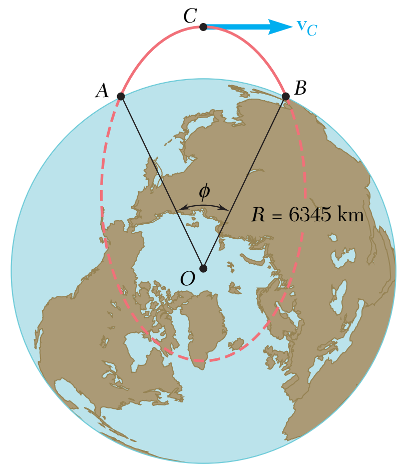


Fig. *P12.118*

12.118 A long-range ballistic trajectory between points A and B on the earth's surface consists of a portion of an ellipse with the apogee at point C. Knowing that point C is 1490 km above the surface of the earth and the range $R\phi$ of the trajectory is 5928.5 km, determine (a) the velocity of the projectile at C, (b) the eccentricity ε of the trajectory.

12.119 A space shuttle is describing a circular orbit at an altitude of 320.5 km above the surface of the earth. As it passes through A it fires its engine for a short interval of time to reduce its speed by 6 percent and begin its descent toward the earth. Determine the altitude of the shuttle at point B, knowing that the angle AOB is equal to 50°. (*Hint.* Point A is the apogee of the elliptic descent trajectory.)

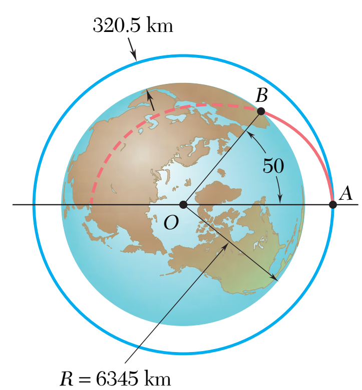


Fig. *P12.119*

12.120 A satellite describes an elliptic orbit about a planet. Denoting by r_0 and r_1 the distances corresponding, respectively, to the perigee and apogee of the orbit, show that the curvature of the orbit at each of these two points can be expressed as

$$\frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{r_0} + \frac{1}{r_1}\right)$$

12.121 (a) Express the eccentricity ε of the elliptic orbit described by a satellite about a planet in terms of the distances r_0 and r_1 corresponding, respectively, to the perigee and apogee of the orbit. (b) Use the result obtained in part a and the data given in Prob. 12.113, where $R_\varepsilon = 149 \times 10^6$ km, to determine the appropriate maximum distance from the sun reached by comet Hyakutake.

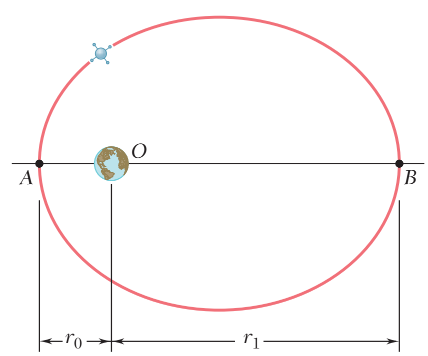


Fig. P12.120 and P12.121

12.122 Derive Kepler's third law of planetary motion from Eqs. (12.39) and (12.45).

12.123 Show that the angular momentum per unit mass h of a satellite describing an elliptic orbit of semimajor axis a and eccentricity ε about a planet of mass M can be expressed as

$$h = \sqrt{GMa(1 - \varepsilon^2)}$$

REVIEW AND SUMMARY FOR CHAPTER 12

This chapter was devoted to Newton's second law and its application to the analysis of the motion of particles.

Newton's second law

Denoting by m the mass of a particle, by $\Sigma\mathbf{F}$ the sum, or resultant, of the forces acting on the particle, and by $\mathbf{a}$ the acceleration of the particle relative to a *newtonian frame of reference* [Sec. 12.2], we wrote

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{12.2}$$

Linear momentum

Introducing the *linear momentum* of a particle, $\mathbf{L} = m\mathbf{v}$ [Sec. 12.3], we saw that Newton's second law can also be written in the form

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \tag{12.5}$$

which expresses that *the resultant of the forces acting on a particle is equal to the rate of change of the linear momentum of the particle.*

Consistent systems of units

Equation (12.2) holds only if a consistent system of units is used. With SI units, the forces should be expressed in newtons, the masses in kilograms, and the accelerations in m/s^2; with U.S. customary units, the forces should be expressed in pounds, the masses in lb · s^2/ft (also referred to as *slugs*), and the accelerations in ft/s^2 [Sec. 12.4].

Equations of motion for a particle

To solve a problem involving the motion of a particle, Eq. (12.2) should be replaced by equations containing scalar quantities [Sec. 12.5]. Using *rectangular components* of $\mathbf{F}$ and $\mathbf{a}$, we wrote

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \tag{12.8}$$

Using *tangential and normal components,* we had

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho} \tag{12.9$'$}$$

Dynamic equilibrium

We also noted [Sec. 12.6] that the equations of motion of a particle can be replaced by equations similar to the equilibrium equations used in statics if a vector $-m\mathbf{a}$ of magnitude ma but of sense opposite to that of the acceleration is added to the forces applied to the particle; the particle is then said to be in *dynamic equilibrium.* For the sake of uniformity, however, all the Sample Problems were solved by using the equations of motion, first with rectangular components [Sample Probs. 12.1 through 12.4], then with tangential and normal components [Sample Probs. 12.5 and 12.6].

Angular momentum

In the second part of the chapter, we defined the *angular momentum* $\mathbf{H}_O$ of a particle about a point O as the moment about O of the linear momentum $m\mathbf{v}$ of that particle [Sec. 12.7]. We wrote

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{12.12}$$

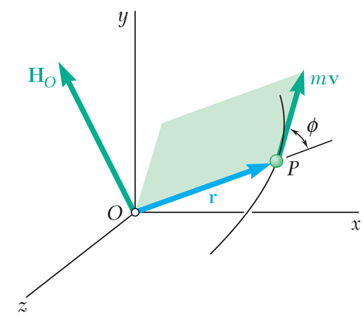


Fig. 12.24

and noted that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$ (Fig. 12.24) and of magnitude

$$H_O = rmv \sin \phi \tag{12.13}$$

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into rectangular components, we expressed the angular momentum $\mathbf{H}_O$ in the determinant form

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{12.14}$$

In the case of a particle moving in the xy plane, we have $z = v_z = 0$. The angular momentum is perpendicular to the xy plane and is completely defined by its magnitude. We wrote

$$H_O = H_z = m(xv_y - yv_x) \tag{12.16}$$

Rate of change of angular momentum

Computing the rate of change $\dot{\mathbf{H}}_O$ of the angular momentum $\mathbf{H}_O$, and applying Newton's second law, we wrote the equation

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \tag{12.19}$$

which states that *the sum of the moments about O of the forces acting on a particle is equal to the rate of change of the angular momentum of the particle about O.*

Radial and transverse components

In many problems involving the plane motion of a particle, it is found convenient to use *radial and transverse components* [Sec. 12.8, Sample Prob. 12.7] and to write the equations

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2) \tag{12.21}$$
$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

Motion under a central force

When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving *under a central force* [Sec. 12.9]. Since $\Sigma\mathbf{M}_O = 0$ at any given instant, it follows from Eq. (12.19) that $\dot{\mathbf{H}}_O = 0$ for all values of t and, thus, that

$$\mathbf{H}_O = \text{constant} \tag{12.23}$$

We concluded that *the angular momentum of a particle moving under a central force is constant, both in magnitude and direction,* and that the particle moves in a plane perpendicular to the vector $\mathbf{H}_O$.

Recalling Eq. (12.13), we wrote the relation

$$rmv \sin \phi = r_0 m v_0 \sin \phi_0 \tag{12.25}$$

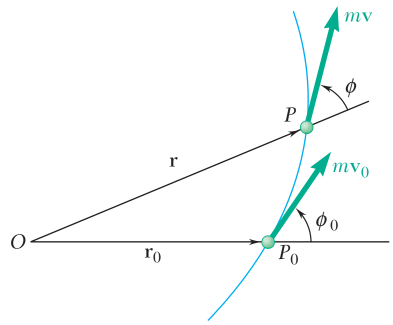


Fig. 12.25

for the motion of any particle under a central force (Fig. 12.25). Using polar coordinates and recalling Eq. (12.18), we also had

$$r^2\dot{\theta} = h \tag{12.27}$$

where h is a constant representing the angular momentum per unit mass, H_O/m, of the particle. We observed (Fig. 12.26) that the infinitesimal area dA swept by the radius vector OP as it rotates through $d\theta$ is equal to $\frac{1}{2}r^2\, d\theta$ and, thus, that the left-hand member of Eq. (12.27) represents twice the *areal velocity dA/dt* of the particle. Therefore, *the areal velocity of a particle moving under a central force is constant.*

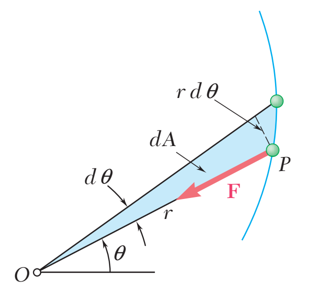


Fig. 12.26

Newton's law of universal gravitation

An important application of the motion under a central force is provided by the orbital motion of bodies under gravitational attraction [Sec. 12.10]. According to *Newton's law of universal gravitation,* two particles at a distance r from each other and of masses M and m, respectively, attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line joining the particles (Fig. 12.27). The common magnitude F of the two forces is

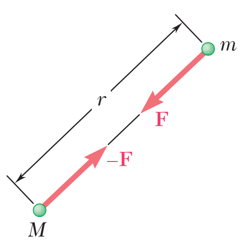


Fig. 12.27

$$F = G\frac{Mm}{r^2} \tag{12.28}$$

where G is the *constant of gravitation.* In the case of a body of mass m subjected to the gravitational attraction of the earth, the product GM, where M is the mass of the earth, can be expressed as

$$GM = gR^2 \tag{12.30}$$

where $g = 9.81\ \text{m/s}^2 = 32.2\ \text{ft/s}^2$ and R is the radius of the earth.

Orbital motion

It was shown in Sec. 12.11 that a particle moving under a central force describes a trajectory defined by the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

where $F > 0$ corresponds to an attractive force and $u = 1/r$. In the case of a particle moving under a force of gravitational attraction [Sec. 12.12], we substituted for F the expression given in Eq. (12.28). Measuring θ from the axis OA joining the focus O to the point A of the trajectory closest to O (Fig. 12.28), we found that the solution to Eq. (12.37) was

$$\frac{1}{r} = u = \frac{GM}{h^2} + C \cos \theta \qquad (12.39)$$

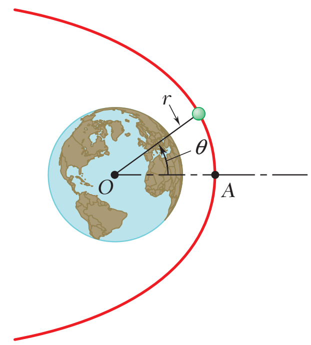


Fig. 12.28

This is the equation of a conic of eccentricity $\varepsilon = Ch^2/GM$. The conic is an *ellipse* if $\varepsilon < 1$, a *parabola* if $\varepsilon = 1$, and a hyperbola if $\varepsilon > 1$. The constants C and h can be determined from the initial conditions; if the particle is projected from point A ($\theta = 0$, $r = r_0$) with an initial velocity $\mathbf{v}_0$ perpendicular to OA, we have $h = r_0 v_0$ [Sample Prob. 12.9].

Escape velocity

It was also shown that the values of the initial velocity corresponding, respectively, to a parabolic and a circular trajectory were

$$v_{esc} = \sqrt{\frac{2GM}{r_0}} \qquad (12.43)$$

$$v_{circ} = \sqrt{\frac{GM}{r_0}} \qquad (12.44)$$

and that the first of these values, called the *escape velocity*, is the smallest value of v_0 for which the particle will not return to its starting point.

Periodic time

The *periodic time* τ of a planet or satellite was defined as the time required by that body to describe its orbit. It was shown that

$$\tau = \frac{2\pi ab}{h} \qquad (12.45)$$

where $h = r_0 v_0$ and where a and b represent the semimajor and semiminor axes of the orbit. It was further shown that these semiaxes are respectively equal to the arithmetic and geometric means of the maximum and minimum values of the radius vector r.

Kepler's laws

The last section of the chapter [Sec. 12.13] presented *Kepler's laws of planetary motion* and showed that these empirical laws, obtained from early astronomical observations, confirm Newton's laws of motion as well as his law of gravitation.

Review Problems

12.124 Determine the maximum theoretical speed that an automobile starting from rest can reach after traveling 402 km. Assume that the coefficient of static friction is 0.80 between the tires and the pavement and that (*a*) the automobile has front-wheel drive and the front wheels support 65 percent of the automobile's weight, (*b*) the automobile has rear-wheel drive and the rear wheels support 42 percent of the automobile's weight.

12.125 A tractor-trailer is traveling at 90 km/h when the driver applies his brakes. Knowing that the braking forces of the tractor and the trailer are 16 kN and 60 kN, respectively, determine (*a*) the distance traveled by the tractor-trailer before it comes to a stop, (*b*) the horizontal component of the force in the hitch between the tractor and the trailer while they are slowing down.

Fig. P12.125

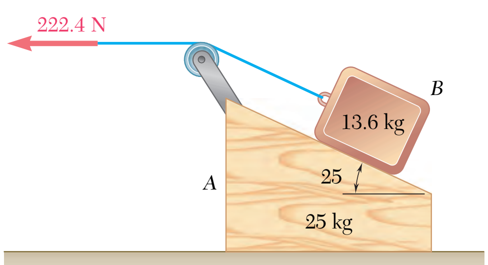


Fig. *P12.126*

12.126 The 13.6 kg block *B* is supported by the 25 kg block *A* and is attached to a cord to which a 222.4 N horizontal force is applied as shown. Neglecting friction, determine (*a*) the acceleration of block *A*, (*b*) the acceleration of block *B* relative to *A*.

12.127 A 81.6 kg wrecking ball *B* is attached to a 12.2 m long steel cable *AB* and swings in the vertical arc shown. Determine the tension in the cable (*a*) at the top *C* of the swing, where $\theta = 30°$, (*b*) at the bottom *D* of the swing, where the speed of *B* is 5.7 m/s.

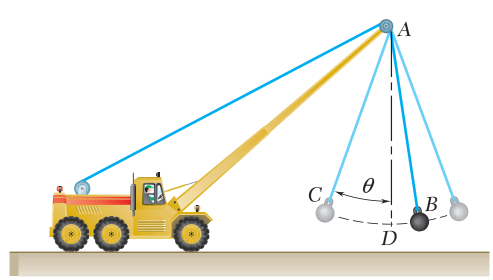


Fig. P12.127

12.128 A car traveling at a speed of 110 km/h approaches a curve of radius 50 m. Knowing that the coefficient of static friction between the tires and the road is 0.65, determine by how much the driver must reduce his speed to safely negotiate the curve if the banking angle is (*a*) $\theta = 10°$, (*b*) $\theta = -5°$, because of the presence of a sinkhole under the roadbed.

12.129 A small .24 kg collar *D* can slide on portion *AB* of a rod which is bent as shown. Knowing that the rod rotates about the vertical *AC* at a constant rate and that $\alpha = 40°$ and $r = 61$ cm, determine the range of values of the speed v for which the collar will not slide on the rod if the coefficient of static friction between the rod and the collar is 0.35.

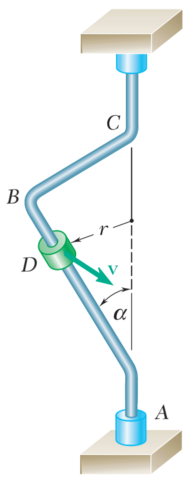


Fig. *P12.129*

12.130 Rod *OA* rotates about *O* in a horizontal plane. The motion of the 400-g collar *B* is defined by the relations $r = 500 + 300 \sin \pi t$ and $\theta = 2\pi(t^2 - 2t)$, where r is expressed in millimeters, t in seconds, and θ in radians. Determine the radial and transverse components of the force exerted on the collar when (*a*) $t = 0$, (*b*) $t = 0.8$ s.

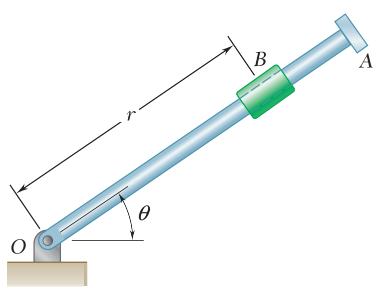


Fig. P12.130

12.131 The .12 kg pin *B* slides along the slot in the rotating arm *OC* and along the slot *DE* which is cut in a fixed horizontal plate. Neglecting friction and knowing that arm *OC* rotates at a constant rate $\dot{\theta}_0 = 10$ rad/s, determine for any given value of θ (*a*) the radial and transverse components of the resultant force **F** exerted on pin *B*, (*b*) the forces **P** and **Q** exerted on pin *B* by arm *OC* and the wall of the slot *DE*, respectively.

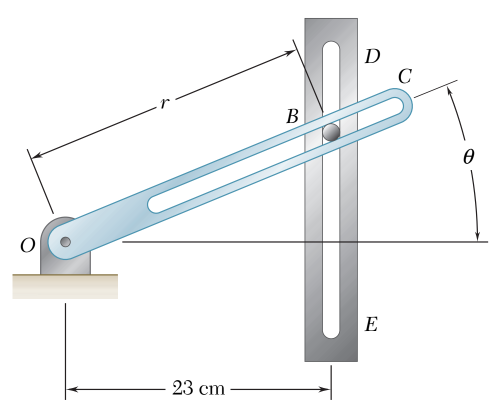


Fig. P12.131

12.132 Determine the mass of the earth knowing that the mean radius of the moon's orbit about the earth is 384.5 Mm and that the moon requires 27.32 days to complete one full revolution about the earth.

12.133 A 540-kg spacecraft first is placed into a circular orbit about the earth at an altitude of 4500 km and then is transferred to a circular orbit about the moon. Knowing that the mass of the moon is 0.01230 times the mass of the earth and that the radius of the moon is 1740 km, determine (*a*) the gravitational force exerted on the spacecraft as it was orbiting the earth, (*b*) the required radius of the orbit of the spacecraft about the moon if the periodic times of the two orbits are to be equal, (*c*) the acceleration of gravity at the surface of the moon. (The *periodic time* of a satellite is the time it requires to complete one full revolution about a planet.)

12.134 A .45 kg ball *A* and a .9 kg ball *B* are mounted on a horizontal rod which rotates freely about a vertical shaft. The balls are held in the positions shown by pins. The pin holding *B* is suddenly removed and the ball moves to position *C* as the rod rotates. Neglecting friction and the mass of the rod and knowing that the initial speed of *A* is $v_A = 2.4$ m/s, determine (*a*) the radial and transverse components of the acceleration of ball *B* immediately after the pin is removed, (*b*) the acceleration of ball *B* relative to the rod at that instant, (*c*) the speed of ball *A* after ball *B* has reached the stop at *C*.

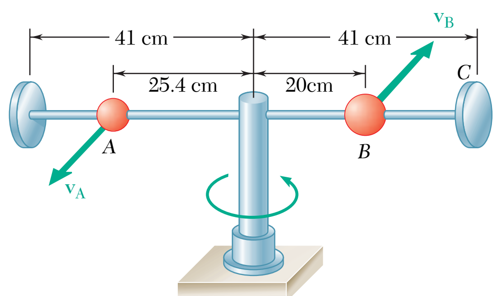


Fig. P12.134

12.135 A space shuttle is describing a circular orbit at an altitude of 563 km above the surface of the earth. As it passes through point *A*, it fires its engine for a short interval of time to reduce its speed by 152 m/s and begin its descent toward the earth. Determine the angle *AOB* so that the altitude of the shuttle at point *B* is 121 km. (*Hint*. Point A is the apogee of the elliptic descent orbit.)

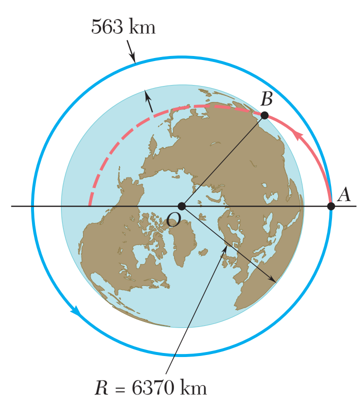


Fig. *P12.135*

Computer Problems

12.C1 Block B of weight 80 N is initially at rest as shown on the upper surface of a 200 N wedge A which is supported by a horizontal surface. A 18 N block C is connected to block B by a cord, which passes over a pulley of negligible mass. Using computational software and denoting by μ the coefficient of friction at all surfaces, calculate the initial acceleration of the wedge and the initial acceleration of block B relative to the wedge for values of $\mu \geq 0$. Use 0.01 increments for μ until the wedge does not move and then use 0.1 increments until no motion occurs.

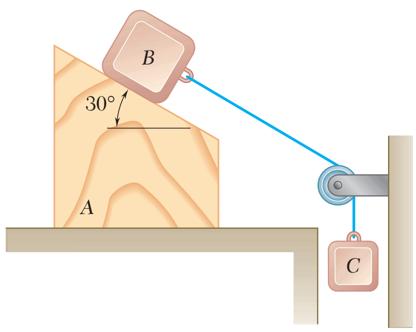


Fig. P12.C1

12.C2 A small 0.50-kg block is at rest at the top of a cylindrical surface. The block is given an initial velocity $\mathbf{v}_0$ to the right of magnitude 3 m/s, which causes it to slide on the cylindrical surface. Using computational software calculate and plot the values of θ at which the block leaves the surface for values of μ_k, the coefficient of kinetic friction between the block and the surface, from 0 to 0.4 using 0.05 increments.

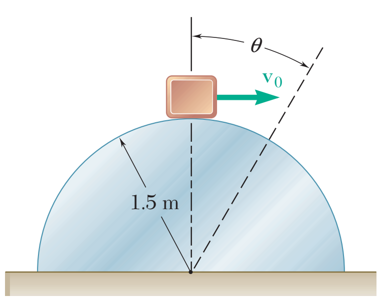


Fig. P12.C2

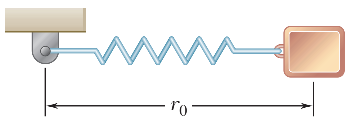


Fig. P12.C3

12.C3 A block of mass m is attached to a spring of constant k. The block is released from rest when the spring is in a horizontal and undeformed position. Knowing that $r_0 = 1.2$ m use computational software to determine (*a*) for $k/m = 15\ s^{-2}$, $20 s^{-2}$, and $25\ s^{-2}$, the length of the spring and the magnitude and direction of the velocity of the block as the block passes directly under the point of suspension of the spring, (*b*) the value of k/m for which that velocity is horizontal.

12.C4 An airplane has a weight of 266893 N and its engines develop a constant thrust of 55603 N during take-off. The drag **D** exerted on the airplane has a magnitude $D = 0.060v^2$, where D is expressed in lb and v is the speed in m/s. The airplane starts from rest at the end of the runway and becomes airborne at a speed of 76.2 m/s. Determine and plot the position and speed of the airplane as functions of time and the speed as a function of position as the airplane moves down the runway.

12.C5 Two wires AC and BC are tied at point C to a small sphere which revolves at a constant speed v in the horizontal circle shown. Calculate and plot the tensions in the wires as functions of v. Determine the range of values of v for which both wires remain taut.

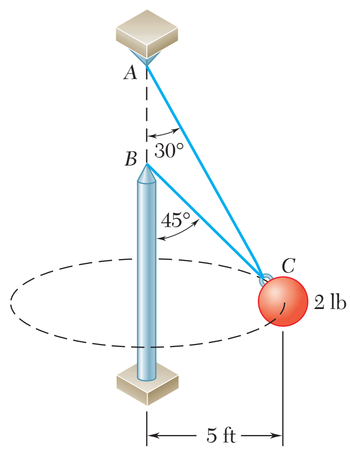


Fig. P12.C5

12.C6 A 4.5 kg bag is gently pushed off the top of a wall at point A and swings in a vertical plane at the end of a rope of length $l = 1.5$ m. Calculate and plot the speed of the bag and the magnitude of the tension in the rope as functions of the angle θ from 0 to 90°.

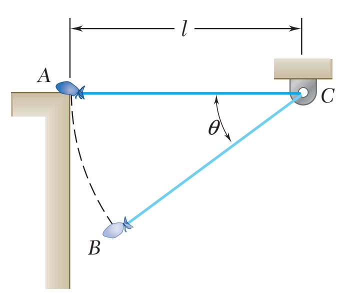


Fig. P12.C6

12.C7 The two-dimensional motion of particle B is defined by the relations $r = t^3 - 2t^2$ and $\theta = t^3 - 4t$, where r is expressed in m, t in seconds, and θ in radians. Knowing that the particle has a mass of 0.25 kg and moves in a horizontal plane, calculate and plot the radial and transverse components and the magnitude of the force acting on the particle as functions of t from 0 to 1.5 s.

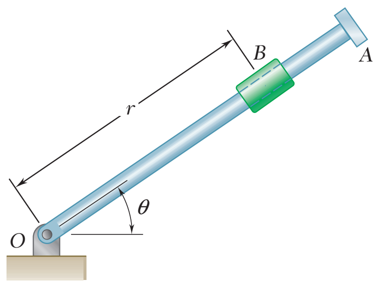


Fig. P12.C7