

Sequences and Series

Opener

Pre-Calculus 11, pages 4–5

Suggested Timing

30–60 min

Materials

- centimetre grid paper
- ruler

Blackline Masters

BLM 1–2 Chapter 1 Prerequisite Skills
BLM U1–1 Unit 1 Project Checklist

Key Terms

sequence	geometric sequence
arithmetic sequence	common ratio
common difference	geometric series
general term	convergent series
arithmetic series	divergent series

What's Ahead

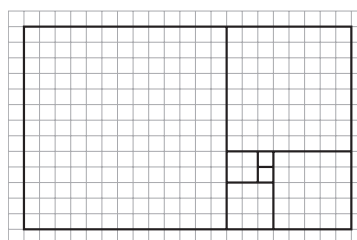
In this chapter, students explore the concepts of arithmetic and geometric sequences and series, as well as infinite geometric series. Through guided discovery and exploration, students differentiate between arithmetic and geometric sequences and series, as well as between finite and infinite geometric series. Students learn to identify the first term, common difference, number of terms, and n th term of an arithmetic sequence. They also write a general equation for the given sequence and solve for any missing values. Students learn to identify the first term, common ratio, number of terms, and n th term of a geometric sequence. Using this information, they write a general equation for the sequence and solve for any missing values. Given an arithmetic or geometric series, students write a general equation for the sum of a finite series, and solve for the first term, common difference or ratio, number of terms, or general term of any series. Students determine if a given geometric series is convergent or divergent. Students also learn to apply the concepts and skills of sequences and series to solve problems.

Planning Notes

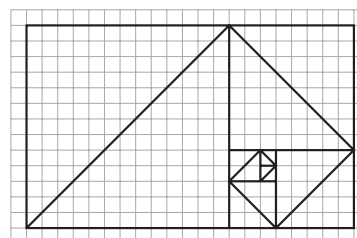
Begin Chapter 1 by having students construct a Fibonacci spiral. Have student rotate a sheet of centimetre grid paper to a landscape orientation.

Provide them with the following steps:

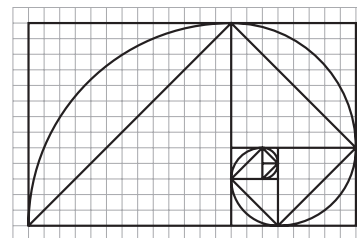
- Begin in the right bottom third of the page and use a ruler to draw a 1 cm by 1 cm square.
- Below the first square, draw another 1 cm by 1 cm square. To the left side of these two squares, draw a 2 cm by 2 cm square.
- Below these squares draw a 3 cm by 3 cm square. To the right of the squares, draw a 5 cm by 5 cm square.
- Continue to add squares in a counterclockwise direction following the pattern of side lengths 1, 1, 2, 3, 5, 8, 13, ... , similar to the diagram on page 4 in the student resource.



- Draw diagonals in every square, moving in a counterclockwise direction.



- Using the diagonals as guides, draw a smooth, spiralling curve from the smallest, one-unit square outward through to the largest square.



The proportional relationship of the squares quickly begins to approach the Golden Proportion of $1 : 1.618...$

Promote a discussion with students using leading questions:

- List the side lengths of each square you drew. What relationship do you see between any two consecutive numbers and the following number?
- What rule determines the side length of the next square?

- Generalize the relationship by using t_1 to represent the first term, t_2 the second term, and so on. What notation would you use for the third term? the n th term? the term before the n th term? two terms before the n th term?
- What generalization can you make using this term notation for the value of the n th term of a Fibonacci sequence?

Organize the class into small groups to discuss the introduction. Then, have them do one of the following:

- Ask each group to do an Internet search on Fibonacci and share their most interesting discovery with the class.
- Have each group watch an online video and share what they discovered with the class. (See the Web Link that follows in this Teacher's Resource for suggested online videos.)

Have students define the Key Terms listed in the student resource. Referring to the photo collage in the opener, ask students to discuss how each of these photos demonstrates or supports one of the Key Terms. Have them share their understanding or experience with any of the Key Terms.

Unit Project

You might take the opportunity to discuss the Unit 1 project described in the Unit 1 opener. Throughout the chapter, there are Project Corner boxes that provide information related to the unit project. These features are not mandatory but are recommended because they provide some background for the final report for the Unit 1 project assignment.

If you are going to develop a project rubric with the class, you may want to start now. See pages 92 and 93 in this Teacher's Resource for information on working with students to develop a class rubric.

Chapter Summary

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. If they have used Foldables™ before, you may wish to have them select a style they found useful to keep their notes in for Chapter 1. Discuss other methods of summarizing information. For example, many students may have used different types of graphic organizers, such as a mind map, concept map, spider map, Frayer model, and KWL chart. Discuss which one(s) might be useful in this chapter.

Once the class has discussed several different summary methods, encourage students to use a summary method of their choice. Allowing personal choice in this way will increase student ownership in their work. It may also encourage some students to experiment with different summary techniques.

Give students time to develop the summary method they have chosen. Ask them to include some method of keeping track of what they need to work on; discuss the advantage of doing this.

Meeting Student Needs

- Consider having students complete the questions on **BLM 1–2 Chapter 1 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- You may wish to post the student learning outcomes for the entire chapter in the classroom, colour-coding the outcomes by section in the chapter. Ensure that students understand the outcomes as written, and be prepared to rewrite some into language they understand. Students can then refer to the outcomes as they work through the chapter. This will help them to self-assess their progress and to identify areas of weakness.
- Encourage students to become familiar with the Key Terms. Consider having them create flashcards or electronic dictionaries. Students might also create posters for the classroom with the definition, and an example, of each Key Term.
- Hand out to students **BLM U1–1 Unit 1 Project Checklist**, which provides a list of all of the requirements for the Unit 1 project.

ELL

- Encourage students to create their own vocabulary dictionary for the Key Terms using written descriptions, examples, and diagrams.

Enrichment

- Tell students that there is a method of encoding data into computer bits that involves using Fibonacci numbers. Challenge students to create an arithmetic sequence and use the encoding method below to code the sequence into bits. Ask them if they notice any patterns.
Have them follow these steps to encode an integer x (for example, 7):
 - Determine the largest Fibonacci number equal to or less than x (for 7, it is 5). Subtract this number from x ($7 - 5 = 2$). Keep track of the difference (2).

- Each Fibonacci number can be represented by its position, according to this sample list:

Position	1	2	3	4	5	6
Fibonacci Number	1	3	3	5	8	13

Let the position of the Fibonacci number that you subtracted be represented by n (5 is in position 4). Write 1 as the n th digit of the code (■ ■ ■ 1).

- Repeat the previous steps, substituting the difference for x , until you reach a remainder of 0. (Since $2 - 2 = 0$ and 2 is in position 2, write 1 as the second digit of the code: ■ 1 ■ 1).
- Place 1 after the last 1 in your code (■ 1 ■ 11), and put 0 in all remaining blanks (01011). This is your completed code.

To decode, remove the last 1 (0101).

Assign the remaining bits the

Fibonacci numbers, 1, 2, 3, 5, 8, 13, ...:

Bit	0	1	0	1
Fibonacci Number	1	2	3	5

Then, add the numbers assigned to the 1s ($2 + 5 = 7$).

Career Link

You may wish to have students who are interested in learning more about biomedical engineers research the career, including the training and qualifications required, and employment opportunities. Have students report their findings to the class. Discuss how this career connects to the chapter.

Web Link

For online videos related to the Fibonacci sequence, go to www.mhrprecalc11.ca and follow the links.

1.1

Arithmetic Sequences

Pre-Calculus 11, pages 6–21

Suggested Timing

100–120 min

Blackline Masters

BLM 1–3 Chapter 1 Warm-Up
 BLM 1–4 Section 1.1 Extra Practice
 TM 1–1 How to Do Page 11 Example 1 Using TI-Nspire™
 TM 1–2 How to Do Page 11 Example 1 Using TI-83/84
 TM 1–3 How to Do Page 21 #28 Using TI-Nspire™
 TM 1–4 How to Do Page 21 #28 Using Microsoft® Excel™

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

Specific Outcomes

RF9 Analyze arithmetic sequences and series to solve problems.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–7, 9, 14 or 17, 19, 26, 27
Typical	#1–10, 11 or 12, 13 or 14, three of 15–23, 26–28
Extension/Enrichment	#8, 11, 12, 15, 18–22, 24–26, 28

Planning Notes

Have students complete the warm-up questions on **BLM 1–3 Chapter 1 Warm-Up** to reinforce prerequisite skills needed for this section.

With the class, discuss Halley’s Comet and see if students can determine the next time the comet will be visible from Earth. Ask questions such as the following:

- How many years passed from the first recorded comet sighting to the second sighting?
- How many years from the second sighting to the third sighting?
- How did Halley determine that the next time the comet would appear, Earth would be in 1758?
- Can you predict the next time that Halley’s Comet will appear?

Tell students that by the end of this activity, they will be able to predict the next appearance of the comet and develop a sequence indicating in which years the comet will appear in the next 1000 years.

Investigate Arithmetic Sequences

The staircase example gives students an opportunity to examine a number of different arithmetic sequences and provides a concrete method of generating these sequences. The goal of the investigation is to help students determine what special attributes of the sequences make them arithmetic. Students should work in pairs to complete the investigation and then share their findings with the class. As you summarize as a class, you may wish to open up the discussion to the following topics:

- Are there any other types of sequences?
- If arithmetic sequences have a rule that consecutive terms have a common difference, can you think of a way to produce a set of sequences using a similar but different rule? Explain.
- Produce three different sequences using your rule.

Ask the following questions to help students understand the two-step staircase example:

- How many cubes are used to build the first two steps?
- How many cubes are used to build the second and third steps?
- How many cubes are used to build the third and fourth steps?

Have students fill in the remaining squares on the table and list all the values as a sequence of numbers. Ask if they notice any special attributes that determine how each term is related to the next term and to the previous term.

You may wish to have students complete the two-step staircase example and answer #4 and 5 before completing the three-step staircase and the staircase summary tables.

For #4, some students may have difficulty seeing the pattern and may benefit from the following leading questions:

- What pattern do you see in the sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21?
- What is the first term?
- What is the difference between terms?
- How many terms are there?
- What is the n th term?

- Can you think of a way to represent each of the terms using their term number? Explain.
 - Can you list the value of the term next to its term number? Explain. What are the term numbers of 17, 19, and 21?
- For #5, have students describe and explain how they determined the 11th and 12th terms using words, algebra, or a diagram. To lead students to make this discovery, ask if they can use information like the value of the first term, n th term, and difference of terms to determine the 11th term and then the 12th term.
- For #6 to 8, have students explain their strategy. Prompt them with the following questions:

- What features of the sequences are similar to the two-step staircase: first term? n th term? difference between terms? number of terms?
- What features are different?
- Which feature defines the sequences as being arithmetic?

Suggest to students that they use the two-step staircase as a model to complete the three-step, four-step, five-step, and six-step staircase tables. Then, have students describe their strategy for determining the number of cubes in each of these staircases.

For #9, students are required to see the relationship between the term number and the number of differences from the first term. If they are having difficulty, have them use a table like the following example:

Term #	t_1	t_2	t_3	t_4	t_n
1	3	5	7	9	
2	3	$3 + 2$	$3 + 2 + 2$	$3 + 2 + 2 + 2$	
3	$3 + (0)2$	$3 + (1)2$	$3 + (2)2$	$3 + (3)2$	

Have them fill in columns for t_5 to t_7 . Prompt their thinking with questions such as the following:

- Do you see any relationship between the term number and the number of differences you add? Explain.
- If there are 100 terms, how many differences would you add to the first term?
- Determine the 100th term. How can you express this relationship using n as the variable for the term number?

Meeting Student Needs

- Read through the outcomes for this section with the class. Ensure that students understand the terminology used in the outcomes and what they will learn by the end of the section.

- Have students perform the operations to determine the sequence in the opening paragraph. Discuss why the interval between sightings of Halley's Comet is not always 76 years. It is important for students to understand that the numbers are a description of the natural world and that sometimes there are anomalies in the natural world.
- For the Investigate, have students make the staircases with blocks or cubes, or draw the staircases and count the blocks.
- Have students create and complete copies of the tables found in the investigation.
- It may be beneficial for students to work with a partner through the investigation.

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- Make sure students make the connection between the photo of Halley's Comet in the student resource and the term *Halley's Comet*.

Gifted

- Tell students that an arithmetic sequence is a list of numbers with a common difference. Challenge them to create a way of testing the following hypotheses.
 - The sum of two arithmetic sequences is another arithmetic sequence. (For the first sequence, assign n for the first number and $n + d$ for the next number. For the second sequence, assign $n + 1$ for the first number and $n + 1 + d$ for the next number. Then, determine the sum. Note that some students may notice that this is not a true generalized test of the hypothesis.)
 - The product of two arithmetic sequences is another arithmetic sequence. (For the first sequence, assign n for the first number and $n + d$ for the next number. For the second sequence, assign $n + 1$ for the first number and $n + 1 + d$ for the next number. Then, determine the product.)

Suggest that students test each hypothesis using arithmetic sequences. Then, have them test with another pair of sequences. If the hypothesis appears to be true, ask them to use a general case to summarize their findings.

Common Errors

- Some students may have difficulty understanding that arithmetic sequences have a common difference between terms.
- R_x** Students should get in the habit of using at least three sets of consecutive terms to determine the common difference.

- Some students may not be able to see that any term in the Investigate can be determined by adding the first term and a product of one less than the term number and the common difference.

R_x Lead students to see the relationship by using one-on-one discussions about the connection between the sum of the first term and one less than the term number. For example, you might ask

- If you use n to represent the term number, how would you write one less than the term number?
- How would you represent the product of one less than the term number and the common difference?

- If you use t_1 to represent the first number, how would you write the sum of the first term and the product of one less than the term number and the common difference?

- Some students may not understand how to use subscripts to define specific terms in the sequence.
- R_x** Give students practice by, for example, asking them to represent the fifth term of a sequence using t to represent a term and a subscript to represent the term number.

Answers

Investigate Arithmetic Sequences

1.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	3	5	7	9	11	13	15	17	19	21

2.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	6	9	12	15	18	21	24	27	30	33

3.

Term	Number of Steps in the Staircase				
	2	3	4	5	6
1	3	6	10	15	21
2	5	9	14	20	27
3	7	12	18	25	33
4	9	15	22	30	39
5	11	18	26	35	45
6	13	21	30	40	51

4. Add 2 cubes to get each next value.

5. To find the 11th term, add 2 to the 10th term. To find the 12th term, add 2 to the 11th term.

6. For a three-step staircase, multiply the term number by 3 and add 3. For a four-step staircase, multiply the term number by 4 and add 6. For a five-step staircase, multiply the term number by 5 and add 10. For a six-step staircase, multiply the term number by 6 and add 15.

7. a) Yes

b) Each set of consecutive terms has a common difference.

8. a) Add a value equal to the number of steps.

b) Yes

9. a) Determine a pattern, then develop a formula to represent the pattern using a variable to represent the number of terms.

b) $t_{100} = 3 + (n - 1)2$

$t_{100} = 3 + (99)(2)$

$t_{100} = 201$

c) $t_n = t_1 + (n - 1)d$, where t_1 is the first term, n is the number of terms, and d is the difference between terms

Assessment

Supporting Learning

Assessment as Learning

Reflect and Respond

Have students complete the Reflect and Respond questions with a partner or individually.

- Encourage students to use the investigation to assist them in answering the questions.
- Suggest that students use more than one method when answering #5. They could compare their strategies with another student or group.
- Provide manipulatives (such as linking cubes) where possible.
- If students are still struggling, complete Example 1 as a class and then ask them to return to the Reflect and Respond questions. To assess understanding, ask them for the 50th term.
- Discuss #9 as a class to allow students to hear multiple strategies and reasoning for the solutions.

Link the Ideas

With the class, discuss the definitions. Discuss which terms they are already familiar with and which ones are new to them.

Have students consider the sample sequence 10, 16, 22, 28, Ask them what relationship they notice between the term number and the number of common differences added to that term. If they have problems answering this question, have them refer to the following list in the student resource and note what is added in each expression.

$$\begin{array}{ll} t_1 = t_1 & \text{Add } 0. \\ t_2 = t_1 + d & \text{Add } d. \\ t_3 = t_1 + 2d & \text{Add } 2d. \\ \vdots & \\ t_n = t_1 + (n - 1)d & \text{Add } (n - 1)d. \end{array}$$

Another approach to doing these types of questions is to use logic. For example, students can look at the term number and the number of common differences.

Example 1

Encourage students to get into the habit of using t_1 to represent the first term, using d to represent the common difference, and then writing the general formula. By substitution, they should then be able to determine the general term. This strategy will assist them with any sequence or series question that they solve by helping them to determine which formulas they can use. It also allows them to get a concrete and visual understanding of the relationship between the terms of a sequence and any given term or the sum of a series. Encourage them to practise this strategy with the Your Turn question for Example 1 and the Check Your Understanding questions.

You might take the opportunity to work with students in small groups or individually to discuss the algebraic phrases used in parts a) and b) of Example 1. The ability to communicate with algebraic statements is one of the differences between Pre-Calculus 11 and Foundations of Mathematics 11.

You might have students explore how technology can be used to support their answers. You may wish to have students use **TM 1–1 How to Do Page 11 Example 1 Using TI-Nspire™** or **TM 1–2 How to Do Page 11 Example 1 Using TI-83/84**.

Example 2

In this example, students are given at least three terms and the n th term of the sequence and are asked to identify the number of terms in the sequence.

Encourage them to continue the habit of using t_1 to represent the first term and d to represent the common difference. You may wish to guide students to understand that they can determine the common difference by subtracting any three consecutive terms (for example, $d = t_4 - t_3$, $d = t_3 - t_2$, and $d = t_2 - t_1$), and that if the values are the same, the sequence is arithmetic. They can then write the formula $t_n = t_1 + (n - 1)d$. By substituting for t_1 , d , and t_n , they can solve for n , the term number.

Students may benefit from a one-on-one or small-group discussion in which they share how they determined the first term, the common difference, and the n th term of the sequence given. Then, they share how they would use the given data to solve for n .

Encourage students to use the same process for the Your Turn that was used for Example 2, by listing known values and using the general formula to solve for the term number.

Example 3

This example challenges students to determine the first term, t_1 , and the common difference, d , given two non-consecutive terms of an arithmetic sequence. Method 2 requires students to produce two equations involving two variables and solve using a linear system of equations. Some students may need assistance in recalling how to solve linear systems. Also, some students may benefit from having a one-on-one or small-group discussion to help them with the algebraic logic in the solution to Example 3.

Example 4

This example requires students to list a set of terms of a sequence using either the general term, the graphing function of a calculator, or a table of values on a spreadsheet.

Method 2 illustrates a graph drawn using a spreadsheet. You may wish to refer students to **TM 1–1 How to Do Page 11 Example 1 Using TI-Nspire™**, **TM 1–2 How to Do Page 11 Example 1 Using TI-83/84**, and **TM 1–4 How to Do Page 21 #28 Using Microsoft® Excel™**, which show students how to make graphs using technology.

Note that the graph in the student resource shows a cost of \$65 when the number of hours worked is zero. You may wish to discuss with students what this amount means.

Key Ideas

Students may benefit from writing down the Key Ideas in their own words prior to completing the Check Your Understanding questions. They should get into the habit of using the Key Ideas as a reminder of what skills and knowledge are most important in each section. If they write down these key concepts, they will always have a quick reference when needed.

Meeting Student Needs

- When developing the formula to determine the general term of an arithmetic sequence, be sure that students understand what t_n , t_1 , n , and d represent. Post several examples of arithmetic sequences and have students point out the various parts. Work through t_2 , t_5 , and so on to check that students understand what these variables represent.
- For Example 1, you may wish to have students research arts groups or centres that are relevant to their community or culture. For example, the Centre des artistes visuels de l'Alberta is a centre for Francophone artists, and the Métis Artists' Collective is an artists' group that promotes Métis arts and culture.
- For Example 2, consider having students research the importance of the musk-ox and caribou to Aboriginal groups. For example, some Inuit use the musk-ox's soft underwool called *qiviut* (pronounced kiv-ee-ut) for weaving shawls, sweaters, gloves, hats, scarves, and other items.
- For Example 3, Method 2, ensure students know how to solve a system of linear equations using elimination. You may wish to help students recall their previously learned understanding of elimination and substitution.
- You may wish to demonstrate Example 4, Method 2, on a projector with a graphing calculator or a similar computer-generated graph. Students will benefit from the visual representation of the solution.

ELL

- Some students may need assistance in understanding the terms *performing arts group* and *community events leader* in Example 1. It may help if you break down the terms into individual words to guide students what the composite terms mean.
- Some students will not be familiar with musk-ox or caribou. Show students pictures of each of these animals as you repeat the names of them.
- For students who do not know the term *furnace technician* in Example 4, show them a picture and describe what tasks this job involves.
- Make sure students include the following in their vocabulary dictionaries: *arithmetic sequence*, *term of a sequence*, *general term*, *finite sequence*, *infinite sequence*, *common difference*, *first term*, *number of terms*, and *nth term*. Encourage them to include verbal descriptions, diagrams, and/or examples. Also, have them include the variables t_1 , n , d , and t_n , and what they represent.

Common Errors

- Some students may find it challenging to identify the variables for the equation.
- R_x** Encourage students to develop a habit of writing down the known information by listing t_1 , d , and/or n , and then writing the equation for t_n . Suggest that they always identify the target or goal of each question by indicating which variable is unknown, for example, $n = ?$ They can then substitute the known data into the equation and solve for the unknown value.
- Some students may not recognize when they need to use a system of equations to solve a problem.
- R_x** Help students to understand that whenever two variables are unknown, they should use a system of equations to solve a problem.
- Some students may not recognize a sequence as arithmetic.
- R_x** Students should identify the difference for at least three pairs of consecutive t terms. This practice will help them to see the pattern.

Answers

Example 1: Your Turn

- a) $t_n = 70 + 5(n - 3)$, where n is the child's age
b) 105 cm

Example 2: Your Turn

38 months

Example 3: Your Turn

$$t_1 = 54, d = -4, t_n = -4n + 58$$

Example 4: Your Turn

\$505

Assessment	Supporting Learning
Assessment for Learning	
<p>Example 1 Have students do the Your Turn related to Example 1.</p>	<ul style="list-style-type: none"> • Students may benefit from verbalizing and writing out the sequence so that it is visually available to them. • Review the meaning of the variables in the equation. • Encourage students to always list the known and unknown values so that they can visualize what they are solving for. • Complete calculator work only after you are certain students understand what they are solving for and can estimate an answer.
<p>Example 2 Have students do the Your Turn related to Example 2.</p>	<ul style="list-style-type: none"> • Have students list the next three values in the sequence so they are engaged in understanding and generating numbers to fit a pattern. • Have students verbally describe what they are trying to find. • Have them list the known and unknown values. • You may wish to have students complete an additional calculation for Example 2 before moving onto the Your Turn question.
<p>Example 3 Have students do the Your Turn related to Example 3.</p>	<ul style="list-style-type: none"> • It may benefit students to review what t_3 and t_8 represent. Ask students what they are looking for in the question. • Have students work in a small group to write a system of equations. Then, have students work individually to verbally identify what variable they are solving for. • It may benefit students to write out the sequence in order to visualize the pattern. • Some students may find it helpful to plot the discrete points on a graph to assist in the linear visualization and identification of specific term values. • Ask students how they can verify their answer.
<p>Example 4 Have students do the Your Turn related to Example 4.</p>	<ul style="list-style-type: none"> • Review with students what it means to find a general term. • Encourage students to record the known and unknown values. • Writing part of the sequence may assist visual learners.

Check Your Understanding

Practise

Students will find #1 to 7 to be very similar to the examples in the student resource. Encourage students to practise the habit of writing down known information from the question. They can then use the equation they formulate, when appropriate, to solve for missing values (t_1 or d) or unknown terms.

For #5, students need to recognize that they have been given enough terms to determine t_1 and d . You may wish to ask them leading questions:

- What is the first term? What variable do we use to represent the first term?
- How can you determine the common difference?
- Using the first term, t_1 , the common difference, d , and the n th term, can you substitute these values into the general equation and solve for n ? Explain.

For #7, you may need to meet with students individually or in small groups to discuss how they can use a graph to identify the terms of an arithmetic sequence. Have students list the ordered pairs shown on the graph.

Then, guide them with prompts:

- What do you notice about the values of the first term of each ordered pair?

- What number system do they represent?
- In arithmetic sequences, what number system do the term numbers represent?
- From the list of ordered pairs, do the x values or y values represent the values of the sequence?
- What is the first term?
- What is the common difference?
- How many terms are there?
- Can you express this sequence using a general term? Explain.

Students need to recognize that arithmetic sequences, when graphed, form linear graphs. They also need to recognize the similarity between the general term of the sequence and the slope-intercept form of a linear function. Have students write down the general term of the given sequence and then the slope-intercept form of a linear function. Students may require prompts to help them:

- Do you remember how to express the slope-intercept form of a linear equation? ($y = mx + b$)
- What does m represent?
- What does b represent?
- Determine and simplify the general term of the sequence given in the graph. Does your general term look similar to a linear equation?
- What is the slope from your general term? What is the y -intercept?

Apply

This set of questions requires students to use their new knowledge and skills in different ways or apply them to more concrete applications. Encourage students to work in pairs so that they can brainstorm and help each other make connections between the variables used in arithmetic sequences and the given information.

For #8, students need to recognize that the value of n must be a whole number. If they use the given information to solve for n and determine that n is not a whole number, they can assume that the given term (34) is not a part of the sequence. Instead of giving students this information directly, guide them toward this understanding. Ask students:

- What number system have we been using for the values of n in the general term?
- Solve for n in part A of the question. Is n a whole number?
- Is 34 a term of this sequence? Explain.
- What is the term number for 34?
- Repeat these steps for parts B to D. Did you determine any solutions for n , where n is not a whole number?
- If n is not a whole number, do you think that 34 is a term of this sequence? Why or why not?

For #9 and 10, you may wish to remind students of the process they have been practising of listing the known values and then using the general equation to solve the problem.

For #11 and 12, students must recognize that the expressions listed are consecutive terms of an arithmetic sequence and that the common difference of the sequence is a constant determined by subtracting two consecutive terms. Some students may experience difficulty extending a concrete process to an abstract situation. Have them practise identifying the common difference when there are three numerical terms and then apply that process to #11 and 12.

For #13, students use their knowledge of arithmetic sequences to solve a concrete application problem. Have students check that they have determined the correct perimeter for each figure by comparing their results with a classmate's.

For #14, students perform operations using time values. You may need to remind students that these expressions are not decimal values and that the next value after 8:59 is 9:00 not 8:60. In part d), the sum of 4:24 and 7:52 is 11:76, but this represents the time 12:16. Discuss these points with students individually or in small groups.

For #15, students follow a similar process as in previous questions. Assist students by using the following prompts:

- What is the area of the artwork?

- What term of an arithmetic sequence does the area represent?
- Which variable of the general equation represents the work completed on the first day?
- If the progress on the artwork forms an arithmetic sequence, what variable from the general equation represents the work completed each day?

For #16, you may wish to assist students with their thinking by asking questions such as the following:

- On which day of her program did Susan perform 11 sit-ups? How would you write this using sequence notation?
- On which day of her program did Susan perform 29 sit-ups? How would you write this using sequence notation?
- If $t_6 = 11$ and $t_{15} = 29$, how can we determine the values of t_1 and d ?
- Which example from the student resource does this question remind you of?
- Can you use Example 3 to help you determine the values of t_1 and d ? Explain.

For #17, students must make a connection between the variables in the general equation of an arithmetic sequence and the number of carbon atoms or hydrogen atoms. Use leading questions to assist students with their thinking:

- Given the table, which variable, t_1 , d , n , or t_n , represents the number of carbon atoms?
- Which variable represents the number of hydrogen atoms?

Once students make this connection, they should be able to list the variables and determine the general term.

The next question, #18, requires students to identify terms within a certain range of values. To help students determine the n th term within the range, encourage them to use multiples of 28. Ask them the following questions:

- What is the largest multiple of 28 that is less than 1000?
- How did you determine this value?

If students have difficulty, ask the following questions:

- What is the result when you divide 1000 by 28?
- What is the largest number less than 1000 that is divisible by 28?

To assist students in determining the first term within the given range, have them use multiples of 7. Use the following prompts:

- What is the smallest number greater than 500 that is divisible by 7?
- If you begin a new sequence starting at 504, what are the next three terms of the sequence?

- What is the largest number divisible by 7 that is less than 600?
- Which variable, t_1 , d , n , or t_n , does this value represent?

Students should then be able to apply their new skills to complete the table.

You may wish to have students work on #19 in pairs or small groups. Have them work together to recognize a relationship between the change in pressure for each foot of descent. Use leading questions to guide students:

- If a 30-ft descent produces a change of 14.7 psi, what pressure change occurs for a descent of 1 ft?
- What is the atmospheric pressure at sea level?
- What will be the change in pressure for each foot of descent?
- What variable will you use to represent the pressure at sea level?
- What variable will you use to represent the total change in descent?

Students should then use their knowledge of arithmetic sequences and linear graphs to answer the question.

Because of their importance in Inuit culture, beluga whales are a popular subject in Inuit art. Although hunters are discouraged from harvesting female belugas and calves, beluga whales are a source of nutrition and an important part of the Inuit diet. Traditionally, the whale fat was used as fuel for lamps, for heating, and in cooking. Today, whale meat is enjoyed in a number of ways.

To help students visualize the problem in #20, suggest that they sketch a diagram of the quadrilateral and label the sides. By working in pairs or small groups, they should be able to brainstorm a way to label the sides using sequence notation. Consider guiding them by telling them that they may need to use a system of equations to solve this problem.

For #21 and 22, encourage students to use a table to develop the terms of an arithmetic sequence.

Students use their skills and knowledge of arithmetic sequences to solve #23. As necessary, ask them leading questions to get them started:

- What is the first term of the sequences?
- What is the common difference of the sequences?
- How many terms are in the sequences?
- Do you know the value of the n th term?
- Do you know the sum of the sequences?
- Can you use this information to solve the problem? How?

Extend

It may be beneficial for students to draw a diagram for #24 to help them recognize how the radius increases with each wheel.

For #25, encourage students to consider using a 24-h clock. Discuss how operations involving time values are different from operations involving decimal numbers. Have students share how these operations differ.

Create Connections

Make an enlarged copy of #26 for each student. Have students cut out the squares with symbols inside and place them on top of the given answer squares. Ask students to write one or two questions of their own that could be answered using the symbols.

For #27, suggest to students that they use their graphic organizers to summarize their understanding of arithmetic sequences.

The Mini Lab in #28 can be used as an interactive activity for students to see what happens to graphs when the first term changes and the common difference changes. This activity will help them to see the relationship between the first term and the y -intercept, and between the common difference and the slope of the graph. Have students work in pairs or groups.

Project Corner

The Project Corner box presents students with information about minerals and possible examples of arithmetic sequences. It might benefit students to have a class discussion about some of the examples, and then have students work in pairs to generate a sample arithmetic sequence from the data given. This process will assist them in knowing what to look for in their own research and how they might display the sequence. Emphasize to students that these are examples and that they may choose a natural resource that is not listed in the student resource.

Some students will research and collect information that may or may not be arithmetic. Explain to students that as the chapter progresses, the class will review the suggested examples in each Project Corner box, but that each student may use a different type of sequence or series.

Meeting Student Needs

- Provide **BLM 1–4 Section 1.1 Extra Practice** to students who would benefit from more practice.
- Have a class discussion about #3. Ensure that students understand exactly what each solution represents.
- For #7d), some students may need a reminder of how to determine the slope of a line when given the graph.
- In #10, some students may find it challenging to work with a common difference of $-3y$. Take the time to illustrate similar examples.
- Students may wish to have square tiles available to assist with #13. Alternatively, grid paper may be helpful for visual learners.
- For #14, you may want to have students research Cheryl Tooshkenig Mitchell, who has broken new ground with her successes as a First Nations golfer.
- For #15, suggest that students research other artists of Inuit wall hangings. You might have them write their own problems based on the wall hangings they find during their research. Students can then solve each other's problems. You may wish to mention that Lucy Ango'yuaq is a second-generation wall-hanging artist. Her mother, Mary Kuutsiq Mariq, was also well known for her wall hangings. In this piece, Lucy uses a number of embroidery stitches, including chain, fly, cross herringbone, and stem.
- For #17, you may wish to have students find out more alkane names.
- For #22, you may wish to discuss extraneous information. Students need to focus on the following

information: 1985 is year 1, 2007 is the last year (so $n = 22$), the initial population is 1657, and the n th population as 1048. Students should immediately recognize that d will be negative.

- When discussing the Project Corner box, encourage students to consider a resource that might be important to their family or cultural group. For example, according to *Native Participation in British Columbia Fisheries — 2003* by Michelle James (Ministry of Agriculture, Food and Fisheries: November 2003), the Nisga'a Fishery caught 1.1% of the value of the commercial salmon catch in 2000, 1.3% in 2001, and 1.8% in 2002.
- You may wish to have students use **TM 1–3 How to Do Page 21 #28 Using TI-Nspire™** or **TM 1–4 How to Do Page 21 #28 Using Microsoft® Excel™**.

ELL

- For those students who are not familiar with the sport of golf, have another student who has knowledge of golf explain the game and what *tee-off time* is.
- The language in #17, 23, and 24 may be challenging to some students. Consider not assigning these questions to new English language learners or have them work with a partner who can help them understand the questions.
- Some students may not be familiar with the following terms: *beluga whale, aquarium, water pressure, beekeeper, rotation, solar eclipse, phase, and symbols*. Use a combination of descriptions, examples, and pictures to assist in student understanding.

Assessment	Supporting Learning
Assessment for Learning	
<p>Practise and Apply</p> <p>Have students do #1–7, 9, 14 or 17 and 19. Students who have no problems with these questions can go on to the remaining questions.</p>	<ul style="list-style-type: none"> • It may benefit some students to compare their responses with a partner's. • For #6, students may use the work they completed in the Your Turn or from the examples as a quick review. • For #9, some students could be encouraged to write down 16 dashed lines and fill in the 16th number. From here, they could work to the left by subtracting 7. This may help students to understand the sequence initially, but encourage them to use the formula to solve for the same first value. • Encourage visual learners to always list some of the numbers from a sequence. You might provide a guideline for when it is acceptable to write out a sequence to find a term and when it is not (the length of the sequence being the main deciding factor). • For #14, review the 24-h clock with students. For visual learners, assign #16 instead. • It may assist students in analysing a problem to review the meaning of m and b in $y = mx + b$.
Assessment as Learning	
<p>Create Connections</p> <p>Have all students complete #26 and 27.</p>	<ul style="list-style-type: none"> • To assist students with #26, have them create their own arithmetic sequence with a positive value for d and then a negative value for d. They should identify the values for t_1, d, and n. Have them refer to this example to select the responses from the list. • Encourage students to write their own descriptions, definitions, and examples for the graphic organizer in #27. Use their work to assess their understanding.

Arithmetic Series

Pre-Calculus 11, pages 22–31

Suggested Timing

60–90 min

Materials

- counting disks
- centimetre grid paper
- ruler

Blackline Masters

BLM 1–3 Chapter 1 Warm-Up

BLM 1–5 Section 1.2 Extra Practice

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

Specific Outcomes

RF9 Analyze arithmetic sequences and series to solve problems.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2a)–c), 3b), c), e), 4–6, 7a), 8, 10, 11, 13, 15, 18, 22
Typical	#1–7, 9–11, 13, 15, 16, 18, 22
Extension/Enrichment	#8, 12, 14–23

Planning Notes

Have students complete the warm-up questions on **BLM 1–3 Chapter 1 Warm-Up** to reinforce prerequisite skills needed for this section.

The information about Carl Gauss can open a discussion about arithmetic series. This opener ends with an opportunity for students to work through Gauss’s discovery by answering the question, “What do you think Gauss did next?”

Students may require a few leading questions to finish the problem for Gauss:

- How many terms of 101 did Gauss get?
- What is the product of the number of terms and 101?
- How many sums of 1 to 101 does this product represent?
- How can you get just one sum of 1 to 101?

- What is the sum of all whole numbers from 1 to 100? After completing Gauss’s discovery, some students may want to write a formal description of it.

$$\begin{aligned}
 S_{100} &= 1 + 2 + 3 + \dots + 99 + 100 \\
 + S_{100} &= 100 + 99 + 98 + \dots + 2 + 1 \\
 \hline
 2S_{100} &= 101 + 101 + 101 + \dots + 101 + 101 \\
 2S_{100} &= 100(101) \\
 2S_{100} &= 10\,100 \\
 S_{100} &= \frac{10\,100}{2} \\
 S_{100} &= 5050
 \end{aligned}$$

Investigate Arithmetic Series

Have students work in pairs on this Investigate. At the end of each part, encourage each student pair to share with the class what they discovered. It may be useful for students to use counting disks, checkers, or coins in order to have a concrete model to work with.

Most students should be able to answer the questions in #1 and 2. You may wish to pose these prompts:

- How many disks are there altogether?
- What is the sum of the numbers 1 to 5?
- How does the number of disks compare to the sum of the numbers 1 to 5?
- How much larger is the area of a rectangle than the area of a right triangle with similar sides?

Work with students in pairs or individually to help them develop Gauss’s method. Use leading questions such as the following:

- Do you notice any relationship between the height and the width of the rectangle?
- How much greater is the height than the width?
- If you represent the width by n , how would you represent the height?
- How would you represent the area of a rectangle with a width of n and a height of $n + 1$?
- Since this area is twice the sum of numbers from 1 to n , how can you represent the sum of numbers 1 to n ?

Have students write down this relationship and use it as a method of adding n consecutive numbers.

Some students may want to produce a formal proof of Gauss’s method for adding n consecutive integers. Ask students if they can use Gauss’s method to determine the sum of numbers 1 to n in the same way as they did for the sum of numbers 1 to 100. Have them replace 100 with n , 99 with $n - 1$, and 98 with $n - 2$.

Have them complete the proof by replacing each question mark with $n + 1$.

$$\begin{aligned}
 S_n &= 1 + 2 + 3 + \dots + n - 1 + n \\
 + S_n &= n + n - 1 + n - 2 + \dots + 2 + 1 \\
 \hline
 2S_n &= (?) + (?) + (?) + \dots + (?) + (?) \\
 2S_n &= n(?) \\
 S_n &= \frac{n(?)}{2}
 \end{aligned}$$

Ask students to compare this equation with the one developed earlier.

For Part B, provide students with centimetre grid paper and have them follow the directions as listed in the Investigate. After completing Part B, have students form groups to compare the work they have completed. At the end of the investigation, encourage the groups to share their answers and explain them to the class.

Meeting Student Needs

- Ensure that students understand the outcomes for this section and what they are expected to learn.
- Draw students' attention to the Key Terms for this section. Some students may wish to make flashcards. If students created posters for the chapter opener, display the posters with terms that are relevant to this section.
- If counting disks are not available for the Investigate, students may use grid paper and draw a circle inside each square. In either case, a visual representation may be beneficial to all students.

- Have students use sequences and series to describe patterns they may be familiar with. For example, challenge students to describe the weaving pattern of the Métis sash using sequences and series. Make sure you have a sash, or a picture of a sash, on hand to show to students.

ELL

- Assist students in understanding the meaning of *ascending order* and *descending order*. Illustrate with simple examples, such as 1, 2, 3, ... and 10, 9, 8,
- Before students work through the Investigate, make sure they understand *counting disk*, *segment*, *spiral*, and any other terms they may not be familiar with.

Common Errors

- Some students may confuse arithmetic sequences and arithmetic series.
- R_x** Discuss with students the different targets of sequences and series. Ask:
 - Are you trying to determine a specific term?
 - Are you trying to determine the sum of all the terms?
- Some students may be unable to decide which sum formula to use.
- R_x** Have students write down the known values for t_1 , d , n , and t_n , and then use the appropriate formula based on the known information.

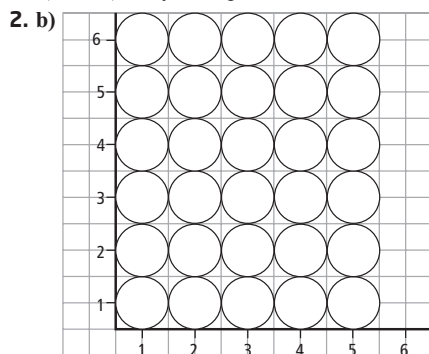
Web Link

For information about other discoveries made by Carl Gauss, go to www.mhrprecalc1.ca and follow the links.

Answers

Investigate Arithmetic Series

1. b) 15 c) They are equal.



c) length = 6 disks, width = 5 disks d) 30
e) The area is twice as great as the sum.

3. To find the sum of n consecutive integers, multiply n by $(t_1 + n)$ and then divide by 2. $t_1 + n$ represents the length of one side of the rectangle and n represents the length of the other side. So, $n(t_1 + n)$ represents the area of the rectangle.
4. This method is similar to Gauss's method because he multiplied the sum of the lowest and highest values by the number of integers and divided by two.
6. a) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14
b) 105
c) Example: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 = 105$
7. a) yes
b) 15, there are 14 terms.
8. $\frac{n(n+1)}{2} = \frac{20(21)}{2}$
 $\frac{n(n+1)}{2} = 210$

Assessment	Supporting Learning
Assessment as Learning	
<p>Reflect and Respond Have students complete the Reflect and Respond questions.</p>	<ul style="list-style-type: none"> • It may be beneficial to have students work in pairs. • Students having difficulty with #3 should apply the rectangle method to determine the sum of consecutive numbers that do not start at 1. This might help students make the connection to Gauss's method. Students will be able to form the rectangle by rotating and translating the pattern of disks. Suggest that students pattern their work after #1 and 2 in the Investigate. • For #7, suggest that students write out the sequence to provide them with a visual representation. • As a class, discuss the various strategies that students used to answer the questions. Discussion will help learners to recognize alternate approaches.

Link the Ideas

Some students may have difficulty understanding the concept of the sum of the first n terms.

Given the series $2 + 4 + 6 + 8 + \dots$, have students label the sum of the first term S_1 , the sum of the first two terms S_2 , the sum of the first three terms S_3 , and the sum of the first four terms S_4 .

The formal proof for the general formula for determining the sum of n terms may be challenging for some students. A similar but different proof can be developed using a concrete example. Have students rewrite the proof in the student resource by replacing t_1 with 5, d with 3, and n with 6. Once students have followed the formal proof with concrete values, they may be able to follow and discuss the abstract version of the formal proof.

Refer students to the last green side bar in the Link the Ideas, which asks about directly deriving the formula using Gauss's method. You may wish to guide students to write the following derivation:

The general arithmetic sequence may be written as $t_1, (t_1 + d), (t_2 + 2d), (t_1 + 3d), \dots, (t_n - d), t_n$

For this sequence, t_1 is the first term

d is the common difference

n is the number of terms

t_n is the general term

Use Gauss's method.

The corresponding series is represented as

$$\begin{aligned}
 S_n &= t_1 + (t_1 + d) + (t_1 + 2d) + \dots + (t_n - d) + t_n \\
 S_n &= t_n + (t_n - d) + (t_n - 2d) + \dots + (t_1 + d) + t_1 \\
 \hline
 2S_n &= (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \dots + (t_1 + t_n) + (t_1 + t_n) \\
 \frac{2S_n}{2} &= \frac{n(t_1 + t_n)}{2} \\
 S_n &= \frac{n}{2}(t_1 + t_n)
 \end{aligned}$$

Example 1

You may wish to ask the following questions to guide students as they work on this example:

- What are the first four terms of this series?
- What is the common difference of these terms?
- What is the n th term of this series?
- If the firefly flashes twice in the first minute, four times in the second minute, and six times in the third minute, what does the variable n represent?
- How many terms are in this series?
- Does listing the values t_1, d, n , and t_n help for substituting into the formula?
- Do you get the same answer when using the two different formulas?

To assist students with the Your Turn question, use prompts like the following:

- Do any of the values for t_1, d, n , or t_n change from those used in Example 1? Explain.
- Are any of the variables missing?
- What formula can you use to solve this question, without knowing the n th term?

Example 2

Students may find the algebra in this example challenging. Have them work in pairs and read the solution out loud to each other to help them understand the flow of logic. They may need leading questions to guide them through the solution:

- Why would you use the formula $S_n = \frac{n}{2} [2t_1 + (n - 1)d]$ for this question?
- Given $S_2 = 13$, how many terms are added together to get 13?
- What is the value of n ?
- Why can you substitute 13 for S_2 and 46 for S_4 ?
- Do you remember how to solve a system of equations? Explain how.

- Once you know the first term and the common difference, how can you determine the first six terms of the series?
- Is there a way that you can determine the n th term once you know the values of t_1 , d , and n , without writing out all of the terms?

For the Your Turn question, you may want to encourage some discussion of which method to use. With 20 terms, students would not write out the entire series to determine the n th term. Since the formula

$S_n = \frac{n}{2} (t_1 + t_n)$ requires the n th term, most students would likely use the formula $S_n = \frac{n}{2} [2t_1 + (n - 1)d]$.

Ask the following questions to assist students in their thinking:

- Using series notation, how can you express that the sum of the first two terms is 19? that the sum of the first four terms is 31?
- Which of the two formulas can you use to produce two equations involving t_1 and d ?
- What are the values of t_1 and d ?
- What are the first six terms of the series?
- Can you determine the 20th term given the values of t_1 , d , and n ?
- How will you determine the sum of the first 20 terms of this series?

Key Ideas

Encourage students to write down the Key Ideas, including the formulas, in their own words before beginning the Check Your Understanding questions.

Meeting Student Needs

- Keep the formulas for the sum of an arithmetic series displayed in the classroom. Ensure that students understand the four variables used in the formula.
- For Example 1, you might have students research the significance of the firefly or how it got its flashing signal in stories from Aboriginal oral histories. Alternatively, you may wish to encourage students to make up their own story. See the Web Link that follows in this Teacher's Resource. If possible, invite a community Elder to tell this story to the class.
- It is important for students to use both formulas in Example 1. Have a second example available for students to practise prior to moving on to Example 2.
- Have students use both formulas for the Your Turn at the end of Example 2.

ELL

- Clarify the difference between a sequence and a series. Otherwise, students may find it challenging to know when to use a formula related to sequences and a formula related to series. It may be helpful to compare and contrast the two terms.
- Make sure students add the following term to their vocabulary dictionary: *arithmetic series*. Encourage them to include verbal descriptions, diagrams, and/or examples. Also, have them include S_n and what it represents.
- For students who are not familiar with fireflies in Example 1, show a photo and provide a description. Make sure they understand what it means for a firefly to “flash.”
- For the Key Ideas, make sure students know what *associated* means. Explain that this means something *goes with* something else.

Enrichment

- Have students determine the sum of the following pattern: $1 + 5 + 9 + 13 + \dots + 45 + 49 + 53$. (378)
- Ask students to consider a Rubik's Cube™. Ask them what numbers could be assigned to each coloured square that would result in the sum of the sides being an arithmetic series. (There are nine coloured squares on each side of a Rubik's Cube, so each side sum is the product of one square and 9. As long as the value of the squares is a series, the sum will also be a series.)

Common Errors

- Some students may choose an inappropriate formula for determining the sum of a series.
- R_x** By writing down the values of t_1 , d , n , and t_n , students will recognize which formula they can use based on the data that are known. They should form a habit of writing down the formula before doing any substitution.
- Some students may not know when to use a system of equations.
- R_x** By writing down the known values of t_1 , d , n , and t_n , they will know when two variables are unknown. They will then be aware that they may need to use a system of equations to solve for the missing terms.

Web Link

For some practice questions related to arithmetic series, go to www.mhrprecalc11.ca and follow the links.

To read a story from the Anishnabe oral history about thunderbirds and fireflies, go to www.mhrprecalc11.ca and follow the links.

Answers

Example 1: Your Turn

1806 flashes in 42 min

Example 2: Your Turn

$$8 + 11 + 14 + 17 + 20 + 23, S_{20} = 730$$

Assessment	Supporting Learning
Assessment for Learning	
<p>Example 1 Have students do the Your Turn related to Example 1.</p>	<ul style="list-style-type: none"> • It may be helpful for some students to work in pairs. • For the Your Turn problem, have students list the known and unknown values. Ask: <ul style="list-style-type: none"> – Are they different from the ones in the original set? – If so, how? • To check for understanding, you may wish to ask students if both formulas could have been used to solve this question. Ask them to verbalize or write out their thinking. Have them compare their responses with those of a classmate.
<p>Example 2 Have students do the Your Turn related to Example 2.</p>	<ul style="list-style-type: none"> • Some students may benefit from working in pairs. • It may be helpful to prompt students with leading questions: <ul style="list-style-type: none"> – What information do we get from the example? – How does this apply to the Your Turn question? • Some students may need assistance in recalling how to solve systems of equations.

Check Your Understanding

Practise

The first six questions provide opportunities for students to develop and practise arithmetic series skills. They allow students to get into the habit of writing down known information and the appropriate formula, and then substituting and solving. Ensure that students take the time to develop these habits and skills.

Apply

For #7, students must determine the first term and n th term within a given range of values in order to solve the question. Have students share with the class the method they used to determine the n th term of the series.

For #8, you may wish to prompt students with the following questions:

- In 24 h, how many times will the clock chime once? twice?
- How many times will the clock chime from midnight to noon?
- If you know how many times it will chime in 12 h, how can you determine the number of times it will chime in 24 h?

To assist students with #9, use prompts:

- What is the common difference between the number of circuits the pilot flew each day?
- How can you determine the first term of the series?

Have students write down the first five terms of the series and solve parts a) and b) of the problem. Then, have them generalize by determining the total number of circuits by the end of the n th day.

For #10, remind students through leading questions that they can use their skills in solving systems of equations. Ask leading questions such as these ones:

- If you know the second term and fifth term of a sequence, can you determine the values of the first term and the common difference? Explain.
- What do we call the process of solving for two variables using two equations?
- Are the values of t_1 and d the same or different for an arithmetic series?

For #11, the following questions will assist students in their thinking:

- Have you seen a problem like this one solved in the student resource? If so, where? (Example 2)
- How might you use a similar approach to solving this question?

To guide students as they work on #12, ask them leading questions:

- What are the first three terms?
- What is the first term?
- What is the common difference?
- How might you use these values and the general formula for the sum of a series to express S_n in terms of n ?
- Using an arithmetic series, what is the total distance travelled by the object?
- Using Galileo's formula, what is the total distance travelled by the object?
- Are the two total distances you determined the same or different?
- Have you demonstrated algebraically that the sums are equivalent? Explain.

Students use their skills and knowledge of arithmetic series to solve #13. As necessary, ask them leading questions to get them started:

- What is the first term of the series?
- What is the common difference of the series?
- How many terms are in the series?
- Do you know the value of the n th term?
- Do you know the sum of the series?
- Can you use this information and the formulas for sum of an arithmetic series to solve the problem? How?

For #14, you may wish to assist students in seeing the relationship between the series of $1 + 2 + 3 + 4 + 5$ and the number of connecting lines. Have them draw six points in the shape of a hexagon, and label the points 1 to 6, as in the diagram in the student resource. Have them start at point 1 and draw lines to every other point. Ask them how many lines they drew. Have them draw lines from point 2 to points 3 to 6. Ask them how many new lines they drew. Have them draw lines from point 3 to points 4 to 6. Again, ask them how many new lines they drew. Continue until students have drawn all of the connecting lines. Ask students guiding questions:

- Why are no new lines drawn from point 6?
- What is the number of lines drawn from each point? Write your answer as a sequence, and then as a series.
- Is this list similar to the series written in the student resource? How?
- What relationship do you notice between the largest number of lines drawn from any point and the number of points?

For #15, you may wish to ask leading questions:

- What is special about consecutive terms of an arithmetic sequence?
- Can you use this special attribute to solve for x ? How?

Extend

These questions require students to be creative and apply learned concepts and skills. You may wish to organize the class into small groups. Challenge each group to solve the problems and then share their solutions with the rest of the class.

In #16, there is a complicating factor of overlapping rings. Have students brainstorm how to compensate for the overlap of rings. Have them draw the first four rings. Then, have them label the length of the outside diameters and the length from the top of one ring to the top of the next ring. Have them use this information to list the first three terms of the series. Ask students:

- Is there any ring that does not have an overlap?
- How can you compensate for the length of the last ring?

For #17, have students use an arithmetic series of their choice to test and justify their answers to this question.

For #19, you might wish to suggest that students make a table that includes the time, term number, and number of bushels harvested. The table will help students develop a plan to solve the problem.

For #20, have students work in groups to brainstorm solutions, and then have them share their results with the rest of the class.

Create Connections

Consider having students attempt the questions individually and then work with a partner to share and discuss their solutions.

Project Corner

The Project Corner box presents students with information about diamond mining and possible examples of arithmetic series. It might benefit students to have a class discussion about some of the examples and then work in pairs to generate a sample arithmetic series from the data given. Provide target years or quantities that they can use to generate sums. This process will assist them in knowing what to look for in their own research and how they might display the series. Emphasize to students that these are examples and that they may choose a natural resource that is not listed in the student resource.

Some students will research and collect information that may or may not be arithmetic. Explain to students that as the chapter progresses, the class will review the suggested examples in each Project Corner box, but that each student may use a different type of sequence or series.

Meeting Student Needs

- Provide **BLM 1–5 Section 1.2 Extra Practice** to students who would benefit from more practice.
- Students should be reminded that the use of the t_n formula is not restricted to section 1.1. Students will need to use that formula as well as the two S_n formulas in this Check Your Understanding section.
- The Practise section is useful for students to understand the basic parts of the formula. Encourage most students to complete all parts of #1 to 6. For those students who find these questions to be a challenge, it may be better for them to work through only the first two parts of each question. Students confident with these questions need complete only the last two parts.

- Some students may need to be instructed to determine the last multiple of 4 (or 6) in the sequence before completing #7.
- Since #11 involves solving a system of two equations, you may need to assist some students with setting up the two equations.
- After completing #13, students may wish to organize their own Canstruction® Competition either within their class or as a joint effort with the student council. They could study the mathematics related to this section of the chapter while completing a community project to assist food banks in their area.

ELL

- You may need to assist students with the following terms in the Check Your Understanding section: *grandfather clock, chime, circuits* (flown by a plane), *airfield, respectively, competition, architects, engineers, display, designs, handshakes, interlocking, peg, outer, doubling, arrangement, formation, and array*. Use a combination of descriptions, diagrams, and examples to help them with their understanding.
- In #12, make sure students understand that *second second* refers to the next unit of time after the first unit of time. It may be helpful to use a clock to show the meaning.
- Some students may find the language in #19 challenging. Consider not assigning this question to new English language learners.
- For #23, show students the pictures of bowling pins and snooker balls in triangular formations that are provided in the student resource.

Assessment	Supporting Learning
Assessment for Learning	
<p>Practise and Apply Have students do #1, 2a)–c), 3b), c), e), 4–6, 7a), 8, 10, 11, 13, and 15. Students who have no problems with these questions can go on to the remaining questions.</p>	<ul style="list-style-type: none"> • Some students may benefit from working with a partner. • For all questions, suggest that students list known and unknown values and the formulas that they could use. • Visual students will benefit from writing out the first three or four terms of a sequence or series. • Students who find #1 to 3 challenging can refer back to Example 1 for review. Asking them to verbalize which are the known values from the question may be a sufficient prompt for them to solve the question. • Students who find #4 to 6 challenging can refer back to Example 2. • For #8, it may benefit students to identify the pattern of a 24-h clock. For example, ask students how many times it will strike 2:00. • For #10, students may find it easier to translate the words into sequence and series notation. For example, “sum of the first 25 terms” = S_{25}. Translating will help students identify the known and unknown values and the formula to use.
Assessment as Learning	
<p>Create Connections Have all students complete #22. Students who have no difficulty can try additional problems.</p>	<ul style="list-style-type: none"> • Have students complete the question individually and then share their response with a partner. • Some students may have a preferred formula and therefore have difficulty with working through both solutions. Ask students if both formulas can always be used. Ask them to identify similarities and differences.

1.3

Geometric Sequences

Pre-Calculus 11, pages 32–45

Suggested Timing

90–150 min

Materials

- coins

Blackline Masters

BLM 1–3 Chapter 1 Warm-Up
BLM 1–6 Section 1.3 Extra Practice

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

Specific Outcomes

RF10 Analyze geometric sequences and series to solve problems

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–10, 12 or 14, one of 15–19, 25
Typical	#1–10, 12 or 14, two of 15–19, 20, 25
Extension/Enrichment	#7, 11, 19–27

Planning Notes

Have students complete the warm-up questions on **BLM 1–3 Chapter 1 Warm-Up** to reinforce prerequisite skills needed for this section.

Consider using computer searches to help students explore the Fibonacci sequence. See the Web Link that follows in this Teacher’s Resource for different approaches to describing the Fibonacci sequence. If you would prefer that students explore on their own, have them complete a computer search on the Fibonacci sequence. There are many online videos available on this topic.

The question in the student resource, “What makes this sequence geometric?” offers an opportunity to discuss the difference between arithmetic sequences and geometric sequences. Challenge students to identify a mathematical operation that defines terms of an arithmetic sequence and also terms of a geometric sequence.

Investigate a Geometric Sequence

Organize students to work in pairs or small groups of three to four. Have them record their discoveries and then share them with the class. They may wish to use a tree diagram to help them record and organize the possible outcomes, especially once they work beyond three coins.

To help students complete the investigation and develop an understanding of geometric sequences, interact with each group and use leading questions:

- What is the value of the first term?
- What relationship exists between terms of an arithmetic sequence?
- What relationship do you see between terms of a geometric sequence?
- Do you think that this will produce a general formula that is different from the general formula for arithmetic sequences? Why?

You may want to extend this investigation to using a set of standard six-sided dice. Ask students to collect data and then answer #3 to 5, replacing the word *coin* with *die*.

Meeting Student Needs

- At the beginning of the section, post the learning outcomes for students to see. Invite students to share possible applications for these outcomes.
- Help students by working through #1 of the Investigate together, especially the tree diagram for #1c).
- Students may need assistance to complete #5. Generate the answer together as a class, explaining the terms of the given formula.

ELL

- Students may need help with the following terms: *architectural feat*, *capture spiral*, *hub*, *preceding*, and *prediction*. Use a combination of description, examples, and visuals to assist them in their understanding.

Enrichment

- Present the following challenge to students: A geometric sequence contains fractions with 1 as the numerator. The second and third terms of the sequence have a denominator of 8 and 32, respectively. The fourth term is $\frac{1}{64}$ times the first. Determine the initial fraction and the fourth term.
 $\left(\frac{1}{2} \text{ and } \frac{1}{128}\right)$

Gifted

- Challenge students with the following problem:
 Suppose an animal population increased each month in a geometric sequence, 20, 40, 80, 160, 320, and decreased each month in an arithmetic sequence, 10, 20, 30, 40, 50. Determine the generalized expression for each sequence. Also, determine the population change each month and see if there is a pattern in the population change. (The expression for the geometric sequence is $t_n = 20(2)^{n-1}$, and the arithmetic sequence is $t_n = 10 + (n-1)10$. The population change is 10, 20, 50, 120, which is neither a geometric nor an arithmetic pattern.)

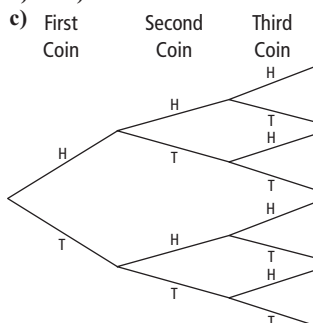
Web Link

For a video showing how a spider's web forms a geometric sequence, and for videos about the Fibonacci sequence, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate a Geometric Sequence

1. a) 2 b) 4



2.

Number of Coins, n	Number of Outcomes, t_n	Expanded Form	Using Exponents
1	2	(2)	2^1
2	4	(2)(2)	2^2
3	8	(2)(2)(2)	2^3
4	16	(2)(2)(2)(2)	2^4
⋮	⋮	⋮	⋮
n	2^n	(2)(2)(2)...(2) n times	2^n

3. a) 2, 4, 8, 16
 b) Each term is 2 times the previous term. Yes, in an arithmetic sequence, a constant is added to each term; here, each term is multiplied by a constant.
 c) 32, 64; multiplied each term by 2
 d) Multiply the previous term by 2 to produce the next term.
4. b) The quotient of each division is 2, which is the same as the constant multiplier in 3c).
5. a) Yes, any two consecutive terms have the same ratio.
 b) $t_n = 2(2)^{n-1}$
 c) Substitute $n = 20$ into $t_n = 2(2)^{n-1}$ and evaluate.
 $t_n = 2(2)^{20-1}$
 $t_n = 2(2)^{19}$
 $t_n = 1\,048\,576$

Assessment	Supporting Learning
Assessment as Learning	
<p>Reflect and Respond Have students complete the Reflect and Respond questions.</p>	<ul style="list-style-type: none"> It may be beneficial to have students work with a partner. Students may need help in recalling how to draw a tree diagram for the coin toss outcomes. Most students will not have difficulty completing the table. However, it is important to differentiate between those students who completed the table using their understanding rather than using patterning. Students who complete the table in #2 using patterning will likely give an answer of 2^n for #5b). Encourage them to consider how the fact that t_1 is 2 will affect the value of n in each of the next steps. Provide students with coins, counters, or linking cubes to model their thinking. For #5c), ensure that students use their formula from #5b) and not 2^{20}.

Link the Ideas

Have a class discussion about the major concepts described in the Link the Ideas. Some prompts might include:

- Can you determine the common ratio of any two terms of a geometric sequence?
- If you have the 3rd term and 6th term, how many common ratios separate these terms?

Have students refer to the list of terms and products of t_1 and r . Ask them:

- What relationship exists between the term number and the exponent of each term?
- Does the formula $t_n = t_1 r^n$ satisfy the relationship discussed above?

Students should work in pairs to complete the examples. This will give them an opportunity to brainstorm and compare if they have difficulties or questions.

Example 1

Discuss the example with the class or with small groups, helping students by asking leading questions:

- The bacteria split from one cell to two to four and so on. If there are ten bacteria at the start, what will be the next three terms of the sequence?
- How can you determine the common ratio?
- Can you use any two successive terms to determine the ratio? Explain.
- Give an example of how you determined the ratio using two terms other than the first two.
- For Example 1, what is the value of t_1 in the formula $t_n = t_1 r^n$? What is the value of r ?
- By substitution, what is the general equation of this geometric sequence?

For the Your Turn, coach students by asking:

- There are three bacteria to start in the sample, and they behave as in Example 1. What are the first four terms of the sequence?
- What is the 1st term? What variable will it replace?
- What is the common ratio? What variable will it replace?
- What is the general term of this sequence?

Example 2

This example requires students to understand that the common ratio is 67% of the original length. To assist them, ask questions like the following:

- If the original photo has a length of 25 cm, what is the shortest possible length of the first copy? How did you determine this length?
- What is the shortest possible length of the second copy? How did you determine this length?

- Write down the lengths of the original photo, first copy, and second copy as a sequence. What is the common ratio of this sequence?
- What is the general equation for this sequence?
- Using your calculator, determine all of the terms of this sequence to one decimal place. How many terms are in the sequence?
- Use the general equation to identify the value of the 6th term. Is it the same value as the one you determined using your calculator?

For the Your Turn, suggest that students write down the first three terms of the sequence. Assist students using the following prompts:

- What is the 1st term of the sequence?
- What is the common ratio?
- If there are eight reductions, how many terms are there in total?
- What is the shortest possible length of the 8th reduction?

Example 3

Students are required to determine two unknown variables, given two different terms of a geometric sequence. This is not a linear system of equations and students may be confused as to how to solve for the two variables without using the skills developed in grade 10.

Before students begin Method 1, discuss the signs of the terms. Ask students what they can predict about the value of r . (It must be negative.)

For Method 2, students should be familiar with the substitution method from their work with linear systems. Ask students to share which variables are common in both equations and which one they can isolate without using radicals or roots. Use questions to prompt students' thinking:

- Can you substitute this expression for t_1 in the other equation? How?
- Once you have solved for r , how can you determine the value of t_1 ?

For the Your Turn question, you may wish to have students work in pairs. Encourage one student to use Method 1 and the other student to use Method 2. Have them compare and discuss their results. For Method 2, you might prompt students' thinking with leading questions:

- If the 2nd term is 28, what is the value of n ?
- Can you express the equation of the 2nd term using the general formula?
- Can you express the 5th term using the general formula?
- How can you use the substitution method to solve for t_1 and r ?

Example 4

In part b), students are required to determine the common ratio of the sequence, given the 1st term and n th term. You may wish to coach students by using leading questions:

- What is the 1st term?
- What is the n th term?
- What is the number of terms?
- What operation do you need to use to isolate the power r^{87} ?
- What is the opposite operation of squaring a number?
- What is the opposite operation of cubing a number?
- What is the opposite operation of taking a number to the power of 87?

For the Your Turn, ensure that students see the advantage of listing all known information when they begin to solve a problem: the 1st term, common ratio, number of terms, and n th term. If any of these values are unknown, they need to be determined. Coach by asking leading questions:

- What is the 1st term?
- What is the n th term?
- If 26.6 million is the 1st term and occurs in 1990, and 38.4 million is the n th term and occurs in 2025, how many terms are in this geometric sequence?
- Can you think of two different ways to determine the number of terms?
- Knowing the 1st term, n th term, and number of terms, how can you determine the common ratio of this sequence?

Key Ideas

Have students write down the Key Ideas in their own words and refer to them when required.

Meeting Student Needs

- Post the formula to determine the general term of a geometric sequence. Discuss the variables in order to establish an understanding of each part of the formula.
- For Example 1, ensure students make the connection between the starting number of bacteria (10) and the 10 found in the general term of the sequence.
- Check students' solution to the Your Turn of Example 1 before proceeding with Example 2.
- For Example 2, some students may want to verify the results of the sample solution by reducing the size of the actual length of the photograph by 67% five consecutive times. Similarly, you may wish to have them do the Your Turn using the method in the example and then using the longer method. Discuss

how developing and using the formula shown can be a time-effective way of working through this kind of question.

- For Example 2, emphasize the question, "Why is the number of terms in this case?" Many students will not make the connection without assistance.
- For Example 3, students may wish to try other methods. For example, some students may prefer to determine the answer using Guess and Check.
- For Example 4, students may need assistance recalling how to use a calculator to determine root 87 of a number.
- Students may be interested to visit the music room. Invite students with experience with a piano to play various notes. If the school has an upright piano, remove the panel so that students can see the strings as they vibrate to create the pitch.
- Consider having students research a musical instrument that is relevant to their culture or community. Have them find out the frequency of the notes for the instrument and whether the frequencies form a geometric sequence.
- For the Your Turn for Example 4, you might have students complete the problem based on population changes in their community. You may wish to provide population data or have students conduct the research themselves.
- For the Key Ideas, create a poster to put on display that shows how to determine the common ratio.
- Provide a quick assessment when the class reaches the Key Ideas. Put various questions on the board and have students identify the known values of the variables as well as the unknown value(s).

ELL

- Assist students in understanding the following terms: *consecutive*, *enlargements*, *reductions*, *frequency* (of musical notes), *population projection*, and *annual growth rate*. Encourage them to add these terms to their vocabulary dictionary.
- Example 1 contains quite a bit of scientific language. You may wish to have ELL students work through this example with another student who can guide them through the language.

Common Errors

- Some students may confuse geometric sequences and arithmetic sequences.

R_x Coach students to get in the habit of determining the common ratio for at least three sets of terms.

- Some students may mistakenly use the term number as the exponent in the general equation.

R_x Coach students to get in the habit of writing down all known information, as well as the general formula, $t_n = t_1 r^{n-1}$, at the start of solving each question. It will help them to remember the general equation and remind them that the exponent is $n - 1$, not n .

- Some students may have difficulty solving a geometric system of equations.

R_x Have students practise choosing the method they will use to solve these types of equations. Some students may need a set of up to ten questions that require them to solve geometric systems of equations using the method of their choice.

Answers

Example 1: Your Turn

$$t_n = 3(2)^{n-1}, t_1 = 3, r = 2$$

Example 2: Your Turn

0.71 cm

Example 3: Your Turn

$$t_1 = 7; r = 4; 7, 28, 112$$

Example 4: Your Turn

1.011. Assume that the population continues to grow at the same rate.

Assessment	Supporting Learning
Assessment for Learning	
Example 1 Have students do the Your Turn related to Example 1.	<ul style="list-style-type: none"> • Some students may benefit from working in pairs. • For the Your Turn, have student continue the practice of listing the known and unknown values before substituting into the general formula. • It might be necessary to discuss some basic exponent laws. A common error for students is to multiply the base numbers, resulting in $t_n = 20^{n-1}$. You may wish to check for understanding before moving on by providing additional values and having students write the general term; for example, $t_1 = 4, b = 3$ and $t_1 = 2, b = 6$.
Example 2 Have students do the Your Turn related to Example 2.	<ul style="list-style-type: none"> • Some students may benefit from working in pairs. • Have students verbally describe what they are trying to determine. • Ask students to verbalize how the 60% will be used in the question and what form it needs to take. You could quickly check for understanding by asking what would change if it was 6%.
Example 3 Have students do the Your Turn related to Example 3.	<ul style="list-style-type: none"> • Some students may benefit from working in pairs. • Have students verbally describe what they are trying to determine. • Encourage students to write out both equations before solving. • Some students may need assistance in recalling how to use substitution to isolate a variable.
Example 4 Have students do the Your Turn related to Example 4.	<ul style="list-style-type: none"> • Some students may benefit from working in pairs. • Have students verbally describe what they are trying to determine. • It may be necessary to remind students about opposite operations as they solve their equation. • Some students may need additional assistance in using technology to find the root.

Check Your Understanding

If possible, provide students with some class time to practise and solve problems. This allows an opportunity for students to get help should they experience difficulty. Have students work in pairs or small groups to provide them with an opportunity to brainstorm and discuss the questions.

Practise

These questions help students to develop a process of identifying geometric sequences, determining and

listing known values from the question, and then solving. Remind students to develop a plan by asking themselves the following set of questions:

- What is the 1st term?
- What is the n th term?
- What is the common ratio?
- What is the number of terms?
- What is the equation to determine the general term?
- What am I trying to determine?

For #2, have students write down the full solutions, not just fill in the table.

Apply

This set of questions requires students to apply concepts and skills they have learned and practised to more complex, real questions. Once again, consider having students form pairs or small groups so that they can work cooperatively to solve these questions.

For #10, coach by using prompts such as these sample questions:

- If jeans fade by 5% of their original colour with each washing, how much of the colour remains after one washing?
- Write down the first three terms of this sequence. What is the common ratio of these three terms?

For #11, coach students to determine the number of terms in this sequence:

- Write down the years from 2004 to 2010. How many years did you write down?
- How many terms will be in this sequence?
- What relationship exists between the difference of the years 2010 and 2004, and the number of terms?

When discussing #13, you might want to mention that St-Pierre-Jolys started Les Folies Grenouilles in 1970 as a means of celebrating the town's rich Francophone heritage during the year of Manitoba's centennial.

Since #15 is similar to #11, use similar coaching to assist students in determining the number of terms.

In #17, students may find it challenging to determine the value of the common ratio. This question is very similar to Example 2. Coach by asking students to list the last three arc lengths and to use them to determine the common ratio.

For #19, consider the following types of questions to assist students in identifying the number of terms and the common ratio:

- If a person's kidney filters out 18% of the medicine every 2 h, what percent of the medicine remains in the body after 2 h?
- How would you write that percent as a decimal?
- What is the common ratio of this sequence?
- If a kidney filters out 18% every 2 h, what is the time difference between each term of the geometric sequence?
- How many terms will there be after 6 h? after 12 h?

Students may need some coaching to complete part b). Ask them if they can write out all of the terms of this sequence until there is only 20 mL of medicine left in the body.

Students interested in the coiled baskets in #21 may wish to learn more about this basket-making technique common to many North American Aboriginal groups. Refer students to the Web Link at the end of this section.

Extend

This set of questions requires students to go beyond a basic understanding of geometric sequences and to use other mathematical skill sets or to work more abstractly to answer the questions. Once again, working in pairs or small groups may help students as they solve these questions. Have the pairs or small groups meet at the end of a set time period to share how they solved the questions.

For #22, students are required to work abstractly and, therefore, may have difficulty getting started. Coach using questions such as these ones:

- If a, b, c, \dots forms an arithmetic sequence, what is the difference between terms a and b ? between terms b and c ?
- What relationship exists for these differences?
- How can you write this relationship using the variables a, b , and c ?
- If $6^a, 6^b, 6^c, \dots$ forms a geometric sequence, what is the ratio of the 1st term and 2nd term?
- What is the ratio of the 2nd term and 3rd term?
- What relationship exists between the ratios of the first two terms and the second two terms?
- How can you write this relationship in terms of a, b , and c ?
- What do you notice about the relationship between the variables a, b , and c in both the arithmetic sequence and the geometric sequence?

For #23, students may need to be reminded of the basic definition of geometric sequences. Assist them by asking what relationship exists between two consecutive terms of a geometric sequence.

For #24, you may wish to ask students leading questions to help them recognize that the term numbers do not correspond to the fret numbers:

- What is the 1st term of this sequence? What distance does it represent?
- What term number represents the distance from the bridge to the 1st fret?
- What term represents the distance to the 2nd fret?
- Are the fret numbers and term numbers the same?

Create Connections

Organize students into groups to answer these questions. Have each group choose a question by lottery, and then explain the question and justify their solution.

Another approach to these questions is to have a contest with points awarded for completing the questions within a certain time limit, or for having the most complete solution, or for coming up with the most unique solution. Award prizes to the winning groups.

For #25, to assist students in determining the common ratio, ask them how much of the water remains if the aquarium is losing 8% of the water each day. Some students may wish to assess which method is correct, provide justification for their choice, and then double-check their choice by doing the question the long way (i.e., by reducing the amount in the aquarium by 8% on each day).

Project Corner

The Project Corner box for section 1.3 presents students with some information related to forestry. Students should be familiar enough with percents to determine how each set of data could generate a sequence. It might benefit students to have a class discussion about some of the data. Have students work in pairs to generate a sample geometric sequence from the data given. This process will assist them in knowing what to look for in their own research and how they might display the sequence. Emphasize to students that these are examples and that they may choose a natural resource that is different from what is presented here, or they may explore a business related to the natural resource. They could research major sawmills in their province and predict future growth or scaling back.

Some students will research and collect information that may or may not be geometric. Explain to students that as the chapter progresses, the class will review the suggested examples in each Project Corner box, but that each student may use a different type of sequence or series.

Meeting Student Needs

- Provide **BLM 1–6 Section 1.3 Extra Practice** to students who would benefit from more practice.
- For the Check Your Understanding, you may wish to create a station approach. Put #10 to 12 on cards or posters and spread them around the room. Place desks at each station. The number of desks can be determined by dividing the number of students in the class by the number of questions assigned. Students

move from station to station. As they complete each question, they have a different group of students to work with. The solutions or some prompts for the solutions could be placed on the back of each question card to provide assistance when you are working with another group.

- If you include #9 as a station, ensure to make a labelled diagram available.
- Request that students research the costs associated with wind turbines as mentioned for #11. Have students answer the following question: If a single 660-kW wind turbine can look after 250 homes, why are we not using them more?
- For #13, it might be interesting for students to research St-Pierre-Jolys Les Folies Grenouilles. See the related Web Link that follows in this Teacher's Resource.
- For #15, you may wish to have students research the Arctic Winter Games. See the Web Link that follows in this Teacher's Resource. Alternatively, students might research similar events that are relevant to their community or culture, such as the Francophone Games.
- For #21, you may wish to invite a community Elder to the class to show students how to make a coiled basket.
- You may wish to use #25 as a small-group assessment. Students could write a brief explanation of the errors found in each solution.

ELL

- There are a number of terms that may be unfamiliar to students: *fades, annually, featured, slow-pitch tournament, parade, fireworks, competition, maximum, performance, sledge, pendulum, primary function, kidneys, filter, impurities, prescribing, dosage, frequency, medicine, charge, battery, twist, bridge and fret (on a guitar), evaporation, puzzle, row, and column*. For the Check Your Understanding, you may wish to partner students with a student with strong English language skills who can help them with the vocabulary.
- It may be better not to assign #11, 12, and 14 to new English language learners since #11 and 14 contain a lot of vocabulary that may be unfamiliar, and #12 requires a lot of reading.
- Ensure that students note the difference between the meaning of frequency in Example 4 and #19.

Web Link

For more information about St-Pierre-Jolys Les Folies Grenouilles, go to www.mhrprecalc11.ca and follow the links.

For more information about coiled baskets, go to www.mhrprecalc11.ca and follow the links.

For more information about the Arctic Winter Games, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment for Learning	
<p>Practise and Apply Have students do #1–10, 12 or 14, one of 15–19. Students who have no problems with these questions can go on to the remaining questions.</p>	<ul style="list-style-type: none"> • It may benefit some students to work with a partner and compare their responses when they are finished a question. • For #1 to 4, students may use the work they completed in the Your Turns or the examples as a quick review. • For #7, ask students to identify what the 4th term should be and then ask how they can use the answer to find the values for y. • Encourage students to estimate the value for r in #8. They can use this to check whether their first three terms are reasonable values. • For #9, visual learners may find it easier to label each of the heights for the ball with the term number and the value of r being used. Encourage students to use the general formula to find the 6th term and then use the value of r they have determined to check manually whether the answer is correct. • The terminology in #14 is quite scientific. If students find the language challenging, it may have an impact their success. Assign #12 in this case. • Perhaps provide a guideline for when it is acceptable to write out a sequence to find a term. For all questions, reinforce that students should list the known and unknown values before starting.
Assessment as Learning	
<p>Create Connections Have all students complete #25.</p>	<ul style="list-style-type: none"> • To assist students with #25, suggest that they work with a partner and solve the question using their preferred approach. Students can then compare their responses with those in the question. • For students attempting #27, encourage them to write out the sequence but also to show all their work for each new calculation. The work should assist them in determining the pattern.

1.4

Geometric Series

Pre-Calculus 11, pages 46–57

Suggested Timing

120–150 min

Materials

- ruler
- isometric dot paper

Blackline Masters

BLM 1–3 Chapter 1 Warm-Up
 BLM 1–7 Section 1.4 Extra Practice
 TM 1–5 How to Do Page 49 Example 1a) Using TI-Nspire™
 TM 1–6 How to Do Page 49 Example 1a) Using TI-83/84

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

Specific Outcomes

RF10 Analyze geometric sequences and series to solve problems

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–10, 12, 14, 20–22
Typical	#1–10, 12, 14, 20–22
Extension/Enrichment	#11, 12, 15–19, 20–22

Planning Notes

Have students complete the warm-up questions on **BLM 1–3 Chapter 1 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, read the opening paragraph and look at the examples of fractal patterns in nature. Invite students to share additional examples that they can think of. Have them share ideas about how each example shows a fractal pattern.

Investigate Fractals

Have students build the fractal tree and fill in the table individually. As a class, discuss the determined values in the table to make sure that all students have completed the table accurately before they move on to #3 to 6. Have students form small groups of two or three to discuss and respond to the investigation questions. Once the groups have completed their work, gather as a class to discuss the groups' results.

Note that in #3, students should be able to identify whether it is a geometric sequence. You may wish to ask guiding questions:

- What is the definition of a geometric sequence?
- What is a common ratio?
- How can you find the common ratio?
- What is the minimum number of term pairs that you need to test to determine if the sequence is geometric?

For #5, have students to add the first five terms of the sequence. Ask them what the total number of branches is for the first five terms.

For #6, ask students what process is necessary to add 100 terms. Once they determine that it is an inefficient method, you may wish to take students' thinking further by asking:

- Is there another process for determining the sum of the terms of a geometric sequence? Explain.
- Do you think that there is a method you can use that is similar to Gauss's sum of arithmetic sequences? Explain.

Meeting Student Needs

- You may wish to have some students research the work of Benoit Mandelbrot and share their findings with the class. See the Web Link that follows in this Teacher's Resource for more information on Benoit Mandelbrot.
- Encourage students to share fractal patterns that are familiar to them. Prompt student thinking by asking where these patterns may be found in plants and trees in their region and in items relevant to students' cultural background.
- Remind students that a series is a sequence expressed as a sum of n terms. Also, assist them in recalling the meaning of each symbol such as r , n , etc.
- Allow students to work in pairs. Encourage discussion and cooperative learning.

- Post the final results. Students could display their fractal tree as well as their table created for #2.
- Some students might benefit from using a computer program that generates the fractals. Not only will it assist those who are not strong at drawing, it will provide an effective visual for repeated iterations that could not be drawn easily by hand.

ELL

- Assist students in understanding fractals by showing them visual representations.
- Terms such as *relatively*, *generated*, *self-similarity*, *buried*, *branches*, *trunk*, and *suitable* may be unfamiliar to students. Demonstrate their meaning using a combination of visuals, descriptions, and examples.

Gifted

- Present students with the following challenge: For the geometric sequence 1, 2, 4, 8, 16, 32, ..., if the sum of the first n terms of this sequence is a prime number, this sum multiplied by the n th term is a perfect number. For example, the sum of the first three terms of the series ($1 + 2 + 4$) is 7, which is a prime number. The sum 7 multiplied by 4 (the third term in the series) equals 28, which is

a perfect number. A perfect number is a positive integer that is the sum of its proper positive divisors ($28 = 1 + 2 + 4 + 7 + 14$). Determine another prime number and perfect number using this geometric sequence. (The first four perfect numbers are generated by the formula $2^{n-1}(2^n - 1)$):

- For $n = 2$: $2^1(2^2 - 1) = 6$
- For $n = 3$: $2^2(2^3 - 1) = 28$
- For $n = 5$: $2^4(2^5 - 1) = 496$
- For $n = 7$: $2^6(2^7 - 1) = 8128$

Common Errors

- Some students may fill in the tables with incorrect values.
- R_x** Guide students to build the fractal tree correctly and large enough so that they can determine the values for the table.

Web Link

For more information about fractals and for examples of fractals, go to www.mhrprecalc11.ca and follow the links.

For more information about Benoit Mandelbrot, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Fractals

2.

Stage	1	2	3	4	5
Number of New Branches	1	2	4	8	16

3. Yes, a geometric sequence has been generated. First term: 1; common ratio: 2; general term: $t_n = 2^{n-1}$

4. a) Yes b) Yes

5. Example: Add terms 1 to 5: $t_1 + t_2 + t_3 + t_4 + t_5$.

6. No, it would not be suitable because it would take too long to add 100 terms.

Assessment	Supporting Learning
Assessment as Learning	
Reflect and Respond Have students complete the Reflect and Respond questions.	<ul style="list-style-type: none"> • It may be beneficial to have students work with a partner. • A computer program for drawing the fractal may help those students with the required drawing.

Link the Ideas

Work through this section with the whole class. Display a copy of the fan-out diagram. Use the diagram and ask leading questions to help students discover relationships and to help them develop a method for determining the sum of terms of a geometric series. Challenge students to determine a similar formula to Gauss's formula for the sum of an arithmetic series. From an inductive example, like the fan-out example from the student resource, $S_n = 1 + 2 + 4 + 8$, students should be able

to build and develop a process to determine the sum for a specific case. Students should then be able to follow and understand the general case.

Students may need some coaching to understand why we multiply by 2, the common ratio. Ask questions to guide their thinking:

- What is the common ratio of the series?
- What happens if you multiply each term by the common ratio?

- What do you notice about the terms of the new sequence?
- Is there a way to eliminate common terms? How?

To help students understand the progression, you may wish to explain the steps.

- Subtract ① from ②.

$$\begin{array}{r} 2S_4 = \quad 2 + 4 + 8 + 16 \\ -S_4 = \quad 1 + 2 + 4 + 8 \\ \hline (2-1)S_4 = -1 + 0 + 0 + 0 + 16 \end{array}$$

- Isolate S_4 by dividing by $(2 - 1)$.

$$\begin{aligned} S_4 &= \frac{16 - 1}{2 - 1} \\ S_4 &= 15 \end{aligned}$$

Note that the expression $(2 - 1)$ is the difference of the two sums. Ask students:

- What does the value 2 represent from the given series?
- If the common ratio was 5 instead of 2, what value would you multiply the sum by?
- How would you represent the difference of the two sums?

The student resource does not simplify $(2 - 1)$, so that when you discuss the general case, it is clear that $(2 - 1)$ represents $(r - 1)$. Coach students to understand the purpose of developing the general case by asking leading questions:

- Why do we determine a general case solution for the sum?
- Instead of a specific answer, what will the general case solution provide?

As you guide students through the development of the general case, refer back to the specific case and ask coaching questions.

Remind students to form a habit of:

- listing known values for t_1 , r , n , t_n , and/or S_n
- identifying what is unknown
- identifying the target of the question
- choosing and writing down the appropriate formula
- substituting known values
- solving for the unknown value

Have students try the examples on their own first to see if they can solve them without help. Then, discuss the methods in the student resource. Point out that there are many ways to solve series questions.

Example 1

You may wish to use the following prompts to guide students.

- What is the first term?
- What is the common ratio?

- What is the number of terms?
- What are you trying to determine?
- Write down the general formula for the sum. Can you solve for the sum of ten terms using this formula?

For the Your Turn, have students work in pairs to complete these questions. Have students each solve the questions and then compare and discuss their answers.

For part a), use the following prompts:

- What is the first term?
- What is the common ratio?
- How many terms are in the series?

To assist students with part b), ask students how many terms are in the series.

Example 2

This example requires students to recognize that they are missing the number of terms, n . The example suggests two different methods to solve the question. The first involves a two-step approach in which you determine the number of terms using the general equation for a geometric sequence and then use the derived general sum formula. The second method manipulates the general sum formula to use the n th term, t_n , instead of n and allows students to solve for the sum directly. Most students should be able to follow and understand the first method. They may need help understanding the derivation used in the second method. Work through this derivation with students, asking leading questions when possible to help them connect with why certain operations were performed.

- Why are both sides of the equation multiplied by r ?
- What relationship does this produce?
- How does $S_n = \frac{t_1(r^n - 1)}{r - 1}$ change to $S_n = \frac{t_1 r^n - t_1}{r - 1}$? Why?
- Do you need the value of n to solve the new general formula for the sum of a geometric series? Explain.

For the Your Turn, you might have students solve the questions using both methods and then have them identify their preferred method.

Example 3

This example requires students to determine known information from a word problem.

Have students attempt the question on their own before discussing the solution in the student resource. Remind students that when they work on questions like these ones they should ask themselves whether they can use one of the two geometric sum formulas to solve the problem.

Key Ideas

Have students write the Key Ideas in their own words and use them when required as they work on the Check Your Understanding questions.

Meeting Student Needs

- You may want to develop the formula using two or three specific examples where the common ratio is 3 or 4.
- Post the formula, including the definitions of the parts of the formula as shown in the blue box in the Link the Ideas in the student resource. Also display the alternate formula from Example 2, Method 2, once the class reaches this part of the section.
- Method 1 of Example 2 requires knowledge of exponents. You may need to revisit the exponent rules used in this example.
- For Example 3, have students research board games that are relevant to their community or culture. For example, they may be interested in learning about Francophone Scrabble™ and the World Francophone Scrabble Championship. Also, they could find information about the Aboriginal board game called *musinaykahwhan metowaywin* (Playing Leader). You may wish to have a community Elder visit the classroom to show some traditional games.

- Invite students to suggest Key Ideas covered in this section. For example, they might explain the formula used for the sum of the first n terms of a geometric sequence.
- You may wish to provide students with **TM 1–5 How to Do Page 49 Example 1a) Using TI-Nspire™** or **TM 1–6 How to Do Page 49 Example 1a) Using TI-83/84**.

ELL

- Terms such as *district, emergency, superintendent, assistant, contacted, derive, organization, promote, tournaments, participate, eliminated, champion, and matches* may be unfamiliar to students. Assist students with their meaning using a combination of visuals, descriptions, and examples.
- It may be better for some students if the Link the Ideas is presented without the context of the fan-out system.
- Have students include the term *geometric series* in their vocabulary dictionaries.
- You may wish to show students the game of Scrabble™ to help them understand what the game involves. Consider allowing students to play a game of Scrabble™, either in English or in their first language if it uses the same alphabet system.

Answers

Example 1: Your Turn

a) 16 400 b) $\frac{21\,845}{256}$ or $85\frac{85}{256}$

Example 2: Your Turn

a) $\frac{87\,381}{64}$ or $1365\frac{21}{64}$ b) -5462

Example 3: Your Turn

511 matches

Assessment	Supporting Learning
Assessment for Learning	
Example 1 Have students do the Your Turn related to Example 1.	<ul style="list-style-type: none"> It may be beneficial for some students to work in pairs. For the Your Turn, have students continue the practice of listing the known and unknown values before substituting into the general formula. This practice will also help students identify which general formula to use. Some students may benefit from brief recall of the order of operations for solving with the formula.
Example 2 Have students do the Your Turn related to Example 2.	<ul style="list-style-type: none"> For some students, it may be beneficial to work in pairs. Have students verbally describe what they are trying to determine. In this example, students determine n. If students find the concept challenging, have them use Method 1 only. Method 2 presents an alternate formula for the sum of a geometric sequence.
Example 3 Have students do the Your Turn related to Example 3.	<ul style="list-style-type: none"> Some students may benefit from working in pairs. Have students verbally describe what they are trying to find. Once they have listed their known and unknown values, have students express what the question is asking and how they are going to represent the unknown value.

Check Your Understanding

Practise

This set of questions allows students to practise basic skills in solving geometric series questions. It is important that students get help when needed. Organize students into pairs or small groups. Provide assistance to any group having difficulty, which has the added benefit that all members of the group will receive coaching at the same time. Have students try each question individually and then share their solution with the others in the group. At the end of the Practise, have the class share any helpful hints for solving the questions.

For #1, coach using the following prompts:

- What is the definition of a geometric sequence?
- What is the difference between a sequence and a series?
- How can you determine if a series is geometric?

For #3 to 8, make sure that students use the known and target values to determine which of the two geometric sum formulas will solve the question.

Expect students to show all of their work when solving geometric sum questions. Ensure that they do the following:

- write down the general sum formula they have chosen to use
- list the known values and target value from the question
- substitute known values to solve for the target value

Apply

This set of questions requires students to determine basic information from each question. Using this information, students must decide whether the question involves a sequence or series, and then solve for the required sum or term value. Have students work in small groups, solving each question individually and then sharing their solutions with their group. At the end of the session, discuss the set of questions as a class, sharing any unique solutions and going over the questions that any group could not solve.

For #10, have students draw a diagram to model the path the tennis ball travels, up to the time that it hits the floor the sixth time. Ask leading questions:

- How many times does the ball travel down?
- What is the first distance the ball travels down?
- How many times does the ball travel up?
- What is the first distance the ball travels up?
- What is the total distance the ball travels down?
- What is the total distance the ball travels up?
- What is the total distance the ball travels?
- Can you determine another logical way to solve this problem? Explain.

For #11, ask students guiding questions such as the following:

- By how much does Celia increase her distance each month?
- If 100% represents Celia's initial distance, how can you represent Celia's total distance as a percent after it increases by 10%?
- How does this percent relate to the value of 1.1?
- Why does the series include the value of 1.1?

For the Mini Lab in #12, provide each student with isometric dot paper to construct the diagrams described in the question. Remind students to show their work for parts c) and d).

Extend

This set of questions requires students to apply their understanding of geometric sequences or series. In most cases, they will also need to use skill sets learned in grade 10.

Consider having students choose a panel of judges from the class. Set up a competition or game that involves the Extend questions. Award points to each group based on a time limit, the quality of answers, and/or the uniqueness of solutions. Reward the winning group at the end of class (for example, put a picture of the group on display under the title Mathematicians of the Week). This activity may open up a discussion about what makes an answer better or more unique than another answer.

For #18, guide students with prompts:

- If a , b , and c are terms of a geometric series, what is the first term of the series?
- Using geometric term notation, what are the second term and third term of the geometric series?
- Can you write the sum of the three terms using series notation? Explain.
- Can you write the product of the three terms using series notation? Explain.
- Can you solve for either t_1 or r given these relationships? Explain.
- Can you use a system approach to solve for either t_1 or r ? Explain.

Create Connections

This set of questions helps students to summarize their understanding of the concepts covered up to this point in the chapter.

Have students put #20 and 22 in a math portfolio, or display them in the classroom, or use them as a summary for this section.

For #22, lead a class discussion using guiding questions:

- Consider the difference between the value of the sum of five terms and the value of the fifth term. Does Tom's solution seem reasonable?
- What error did Tom make in solving this question?
- How can you avoid this type of error when solving geometric sequence and series questions?

Project Corner

The Project Corner box for section 1.4 introduces students to oil discovery, both in eastern and western Canada. Oil is a topic that is much in the news with direct impact on the global economy. Students are likely familiar with the media coverage given to oil prices receive the daily news and stock market reports. It may be beneficial to discuss why oil impacts the world's economy and how changes in production affect the dollar. It may be challenging for students to put into perspective the production rate of oil in barrels per day.

For students interested in oil as their natural resource for the project, you may suggest that they do some initial research to understand the production rates of oil in barrels per day and how the size of an oil field can help in predicting production and supply rates. Students will need to use their understanding of rates to generate a sequence and a series. Encourage students to look at the natural resource as a commodity for the world market. This may be a possible topic for cross-curricular exploration with social studies. Discuss with other teachers possible links that such an investigation might have to their course. Any links might present an opportunity for students to complete a joint project.

It might benefit students for you to take up a class discussion of some of the major locations of oil production. Students might use this discussion to guide them to focus on a particular province, company, or type of oil that they may wish to study. As students talk about oil, the discussion of coal reserves and its impact on the global economy may come up. Encourage students who might be interested in this area to follow much the same investigative approach that was suggested for oil. Both of these areas would provide many venues and areas for students to collect data. Encourage them to look for graphs that might show increases that are geometric. Population growth as a result of oil drilling and production may be one area of focus.

Some students will research and collect information that may or may not be geometric. Explain to students that as the chapter progresses, the class will review the suggested examples in each Project Corner box, but that each student may use a different type of sequence or series.

Meeting Student Needs

- Provide **BLM 1–7 Section 1.4 Extra Practice** to students who would benefit from more practice.
- It may be beneficial to complete #1 and 2 orally as a class.
- Allow students to work in pairs for #9 to 15. Post large sheets of paper around the classroom. Encourage students to place questions or comments about particular questions on these sheets. Have a class discussion about these questions and comments.
- Encourage artistic students to complete #12.
- Some students may wish to learn more about Koch snowflakes and use them in a work of art. Refer these students to the Web Link at the end of this section.
- If possible, provide a computer program to assist students in drawing the fractals.
- For #12, you may wish to have students research legends from various cultures about snowflakes. Alternatively, have students research legends about monarch butterflies for #21.
- For #14, have students research wampum belts, which the Haudenosaunee (also known as Iroquois) peoples made from sacred shell beads. For more information about the wampum, see the Web Link that follows in this Teacher's Resource. Alternatively, students could research the beading designs of the Woodlands and/or Plains peoples.

ELL

- Terms such as *committee*, *notify*, *corresponding*, *marathon*, *advertising*, *campaign*, *metropolitan*, *Indigenous*, *predominate*, *religions*, *successive*, *prescribe*, *intervals*, *metabolizes*, *gradually*, *infection*, *ampicillin*, *tablet*, *generation*, and *assumptions* may be unfamiliar to students. Assist students with their meaning using a combination of visuals, descriptions, and examples.
- Show students pictures of a tennis ball, snowflakes, bead working, and a monarch butterfly to help them visualize what these terms mean.

Common Errors

- Some students may confuse sequences and series.
- R_x** Students should get in the habit of reading and rereading a question to determine if the target answer is a specific term value or the sum of terms.

Web Link

For information about the wampum, go to www.mhrprecalc11.ca and follow the links.

For more information about Koch snowflakes and for ideas about how to use them in an art project, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment for Learning	
Practise and Apply Have students do #1–10, 12, and 14. Students who have no problems with these questions can go on to the remaining questions.	<ul style="list-style-type: none">• It may benefit some students to work in pairs and compare responses as they finish each question.• In pairs, have them complete part a) of #1 to 5 to ensure they understand what to do. Clarify and coach where necessary.• Some students may find it challenging to determine which formula to use in #3 to 8. Encourage them to list the known and unknown values to help identify which formula is appropriate.• Encourage students to draw diagrams and write sequences wherever necessary.• Visual and kinesthetic learners may benefit from the use of a computer program. Challenge pairs of students to generate different fractals and identify the known values.
Assessment as Learning	
Create Connections Have all students complete #20–22.	<ul style="list-style-type: none">• Encourage students to write summative explanations, examples, and diagrams into the Foldable or graphic organizer that they started at the beginning of the chapter.• The flowchart in #21 is a useful Assessment as Learning tool for students to complete and have on hand for quick reference.• Alternately, you may wish to complete #21 as a class and create a larger version that can be hung on the wall for student reference as they complete the review and practice test at the end of the chapter.

Infinite Geometric Series

1.5

Pre-Calculus 11, pages 58–65

Suggested Timing

100–120 min

Materials

- ruler
- centimetre grid paper
- scissors

Blackline Masters

BLM 1–3 Chapter 1 Warm-Up
BLM 1–8 Section 1.5 Extra Practice

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

Specific Outcomes

RF10 Analyze geometric sequences and series to solve problems

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 3a), b), 4, 5a), c), 6, 7, 9, 12, 14, 20–22
Typical	#1, 2, 3a), b), 4, 5a), c), 6–11, 13, one of 15–17, 20–22
Extension/Enrichment	#11, 14, 18–22

Planning Notes

Have students complete the warm-up questions on **BLM 1–3 Chapter 1 Warm-Up** to reinforce prerequisite skills needed for this section.

Have students do an Internet search to look up as many of Zeno’s paradoxes as they can find. Organize students into small groups. Have each group take one of Zeno’s paradoxes and either support or refute it in a classroom debate. For information about how to run a classroom debate, see the Web Link that follows in this resource.

Investigate an Infinite Series

Have students work individually on #1 to 7 of the investigation. Organize students into small groups to

work through the Reflect and Respond questions of the investigation. Gather as a whole class at the end of a given time period and discuss groups’ findings.

Students will enjoy an opportunity to use their pencils and rulers as they complete #1. It may be helpful to provide students with centimetre grid paper for this activity. After students complete #2 to 7, discuss with the class the concept of infinity and relate the discussion to Zeno’s paradox of motion. Ask students how the discussion changes when you consider physical limitations as opposed to an abstract model, where infinite divisions are possible.

For #9, refer students back to their results from #8 to determine the value of $\left(\frac{1}{2}\right)^x$ as x increases to infinity.

As this is students’ first formal exposure to exponential functions and their graphs, it may be helpful to indicate that an exponential function is of the form $y = a^x$, where a is a constant value. In #8 and 9, encourage students to graph the term number as the independent variable and the partial sum of the terms as the dependent variable. It might be useful for students to manually graph the first six terms of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$. Ask students what they notice about each graph and whether each partial sum appears to approach a particular value as the number of terms gets larger. The following should be apparent from the graphs: As x increases for the function $y = \left(\frac{1}{2}\right)^x$, the partial sum levels off and appears to approach a value of 1. The function $y = 2^x$ appears to continually increase. Note that levelling off is characteristic of infinite geometric series in which $-1 < r < 1$.

Students may need some coaching to understand that either geometric sum formula works and it is for convenience

that we use $S_n = \frac{t_1(1-r)}{1-r}$.

- What is the value of $\frac{(-1)}{(-1)}$?
- What is the result of multiplying $r^x - 1$ by -1 ?
- What is the result of multiplying $r - 1$ by -1 ?
- What is the result of multiplying $\frac{t_1(r^x - 1)}{(r - 1)}$ by $\frac{(-1)}{(-1)}$?
- Why use the formula $\frac{t_1(1-r^x)}{(1-r)}$ when $r < 1$?

For #10a), students may need to be reminded that $r = \frac{1}{2}$. Ask students to examine what happens to the value of r^x as x gets larger. For #10b), coach students with guiding questions:

- What sum formula will you use if $r = \frac{1}{2}$?
- If you replace the value of r^x by your answer in part a), what is the value of $1 - r^x$?
- What is the new formula for the sum of the terms of an infinite geometric series?

Meeting Student Needs

- You may wish to illustrate Zeno's paradox of motion. If the hallway has a tile floor, have a student stand at one end and have the rest of the class stand at the other end. Count the number of tiles between the student and the class. The student should then move half the distance toward the class, then half the remaining distance. Continue this process until it is too difficult for the student to move accurately. Switch to using a quarter or other small object to demonstrate covering half the distance. Invite students to imagine that the remaining distance has been magnified and is now covering the length of the hallway. Students should have a greater understanding of Zeno's paradox. Ask them to explain the limitations of this demonstration.
- For the Investigate, some students may benefit from actually cutting the paper each time, labelling each as a part of the whole. If they have difficulty, they can

compare the piece of paper remaining to the original to determine which fraction of the whole is left. Have them label each part with $\frac{1}{2}$, then $\frac{1}{4}$, then $\frac{1}{8}$, etc. before cutting in half again.

- It may be helpful to students to complete #8 to 10 as a class.

ELL

- Students may not be familiar with such terms as *philosopher*, *paradoxes*, *indicating*, *physical limitations*, *indefinitely*, and *modify*. Assist them with their understanding using a combination of visuals, descriptions, and examples.

Gifted

- Tell students that the sum to infinity is a finite value. The sum of the first n terms of a geometric series tends to n when n tends to infinity. The sum to infinity exists only when a series is convergent. The sum to infinity is given by the expression $S = \frac{t_1}{1-r}$ if $-1 < r < 1$. Have students use the expression to test possible values of a sum to infinity.

Web Link

For information about how to hold a classroom debate, go to www.mhrprecalc11.ca, and follow the links.

To find out more about infinite geometric series, go to www.mhrprecalc11.ca, and follow the links.

Answers

Investigate an Infinite Series

2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

3. $\frac{1}{64}, \frac{1}{128}$

4. Geometric, there is a common ratio of $r = \frac{1}{2}$.

5. $t_x = \left(\frac{1}{2}\right)^x$

6. Yes, the term value will continue to decrease but never disappear.

7. The remaining area will approach 0.

8. a) As x increases $\left(\frac{1}{2}\right)^x$ decreases and approaches 0.
b) No

9. a) As x increases the sum approaches 1.

b) No, it will never actually reach 1 because there will always be a small section of the paper unshaded.

10. a) 0

b) Given $S_x = \frac{t_1(1-r^x)}{1-r}$ and since as $r^x \rightarrow 0$ then $1 - r^x \rightarrow 1$.

Therefore, $S_x = \frac{t_1(1)}{1-r}$ or $\frac{t_1}{1-r}$.

c) 1

Assessment	Supporting Learning
Assessment as Learning	
<p>Reflect and Respond</p> <p>Have students complete the Reflect and Respond questions.</p>	<ul style="list-style-type: none"> • Have students complete #1 to 7 and then discuss the results as a class. Have them complete #8 individually. Because some students may not understand the sum of an infinite geometric series, have them work in small groups to complete #9 and 10. Before students move on in section 1.5, it is important that they understand that the sums can get smaller and converge on a number or that they can get larger and diverge. • Check that students understand inspection methods that can be used to determine whether a geometric sequence is convergent or divergent (such as using the r value).

Link the Ideas

You might wish to have students use their graphing calculators to demonstrate the difference between convergent and divergent series. Students can use a geometric series of their choice. Have them determine the general equation for the series, then graph the series by replacing t_n with y , and n with x . Have them graph each of the following situations: $r > 1$, $r < -1$, $0 < r < 1$, and $-1 < r < 0$. Ask students the following types of questions:

- What happens to each graph as x increases?
- Which graphs continue to increase or decrease without end?
- Which graphs appear to approach a specific value of y ?
- Which graphs appear to converge to a specific value?
- Which graphs appear to diverge from the x -axis?

Discuss the Link Your Ideas section with the class keeping in mind what they discovered using their graphing calculators.

Example 1

Have students attempt part a) of the example on their own. Discuss the solution by asking students to identify t_1 and r from the given series and then to identify how they determined the solution. Ask them if the result seems reasonable. Repeat with part b).

For the Your Turn, use coaching questions:

- What is the value of the first term?
- What is the common ratio?
- Which value, t_1 or r , will you use to determine if the series converges or diverges?
- What values of r define the series as being convergent?
- Which type of series, convergent or divergent, has a sum?
- What formula determines the sum of an infinite convergent series?

Example 2

For part a), guide students with the following types of prompts:

- What fraction of the largest square is the largest shaded square?
- What fraction of the largest square is the second largest shaded square? third largest shaded square? fourth largest shaded square?
- Do these fractions represent an infinite geometric sequence? How do you know?
- How would you write this sequence as an infinite geometric series?

For part b), you may wish to guide students with the following questions:

- Is there a relationship between the total shaded area of the largest square and the series you wrote for part a)? Explain.
- How can you determine the total shaded area of the largest square?
- What is the first term, t_1 ?
- What is the common ratio, r ?
- What is the sum of this infinite series?

For the Your Turn, have students attempt the question individually, and then discuss it with the class as a whole. Use a set of coaching questions similar to the following:

- What is the first term, t_1 ?
- What is another way to write the first term?
- What is the common ratio, r ?
- What is another way to write the common ratio?
- Why would you write the first term and common ratio as fractions to determine the sum of the infinite geometric series? What happens if you do not write them as fractions?

Key Ideas

Have students write the Key Ideas in their own words to use as they complete the Check Your Understanding questions.

Meeting Student Needs

- Define *converge* and *diverge* before going through the Link the Ideas.
- Encourage students to create a table to compare and contrast convergent series and divergent series. Teach the two concepts simultaneously, placing one comment in each column before moving on to the next point.
- Emphasize that a divergent series never approaches a fixed value and has no sum.
- Place the formula for the sum of a convergent series on the board, emphasizing that it can be used only for series that approach a particular value.

ELL

- Encourage students to include the following terms in their vocabulary dictionary: *convergent series*, *fixed value*, and *divergent series*. Also, have them include S_∞ and what it represents.

Common Errors

- Some students may not know how to determine the sum of a convergent series.

R_x Have students record the value of r for each question and use the Key Ideas to remind them what values of r can be used to determine the sum.

- Some students may improperly convert each term of the series from decimals to fractions.

R_x Have students compare the number of decimals in the decimal form and the number of zeros in the denominator of the equivalent fraction form. Ask them what relationship they notice.

Answers

Example 1: Your Turn

a) $\frac{5}{4}$ b) no sum, series is divergent

Example 2: Your Turn

$$0.\overline{584} = \frac{584}{1000} + \frac{584}{1\,000\,000} + \frac{584}{1\,000\,000\,000} + \dots$$

$$0.\overline{584} = \frac{584}{999}$$

Assessment	Supporting Learning
Assessment for Learning	
<p>Example 1 Have students do the Your Turn related to Example 1.</p>	<ul style="list-style-type: none"> Have students continue to list the known and unknown values. It is good practice for students to identify whether the r value they write each time is greater than 1 or less than 1. This should help them to distinguish the divergent from the convergent. There may be some value in coaching students on the less than ($<$) and greater than ($>$) symbols. As a mnemonic for the less than symbol, tell them that the Ls go together (less than and left pointing).
<p>Example 2 Have students do the Your Turn related to Example 2.</p>	<ul style="list-style-type: none"> If students are having difficulty understanding the Example 2 Your Turn, they should approach it as a pattern. However, they need to work toward an understanding of how to derive a sequence from a repeating decimal. It may benefit some students to get coached in recalling their understanding of place value. They should also have a good understanding of dividing fractions, both with a pencil and paper and with a calculator.

Check Your Understanding

Practise

Have students complete these questions in class individually. Then, have them compare and discuss their answers with a partner. It is important that all students understand the concepts of convergent and divergent before they continue with the rest of the questions.

Apply

Have students attempt these questions in class individually. Then, have students compare and discuss their answers with a partner. Most of these questions require students to solve for either t_1 , r , or some value of x that will allow them to determine t_1 or r . Coach students by having them list the known values from the given information and solve for the unknown value. Some students may need to be reminded to consider the definitions for infinite and convergent series. Some students may need the hint to solve for $(1 - r)$, and then solve for r . At the end of the Apply questions, discuss the questions as a class, and have students share their understanding and the methods they used to solve the questions.

For #11, coach students to write down the complete formula for the sum of infinite geometric series, including the restrictions for r . To help them, ask leading questions:

- If $-1 < r < 1$, what range of values for r satisfies the statement?
- If $r = \frac{x}{3}$, what range of values for $\frac{x}{3}$ satisfies the statement $-1 < \frac{x}{3} < 1$?
- What values of x make the statement true?

For #15, use prompts like the following questions:

- If the ball falls 16 m on the first leg of its journey, how far will it bounce back up?
- How far will it fall on the second leg of its journey?
- How far will it bounce back up?
- Do these values form a single geometric sequence? Explain.
- Can you split this question into two different geometric series? Explain.
- What will each series represent?

Extend

Have students work in pairs or small groups. Challenge each pair or group to solve the questions and then explain their solutions to the class.

For #19, some students may want to follow the directions and produce the described squares out of a piece of paper. They will need paper, a ruler, and scissors. Have them line up their cut squares and then ask leading questions:

- What is the length of the first set of three squares?
- What is the length of the second and third set of three squares?
- Do these lengths form a geometric sequence? Explain.
- What is the first term?
- What is the common ratio?

For #20, you may wish to ask students guiding questions:

- What do you notice about the relationship between the sums of each infinite geometric series?
- If you choose a value for z , can you determine a value for x ? Explain.
- If x is the first term and x^2 is the second term of a geometric series, what is the common ratio?
- If the ratios must add to 1, what is the common ratio for the second geometric series?
- What is the first term of the second geometric series?

Create Connections

These questions serve as a review of the concepts covered in this section.

For #22, you may wish to provide the following hints:

- Write down the first four terms of each series.
- Use the series to determine the values of t_1 and r .

Ask guiding questions to assist student thinking:

- What variable will you use in the arithmetic and geometric series formulas?
- What do you notice about the formula for the infinite geometric series?

For #23, have students form groups to complete this Mini Lab. Provide students with centimetre grid paper and suggest that they cut the paper into a 32 cm by 32 cm square so that it is easy to divide evenly. Give them 5 to 10 min to complete the Mini Lab. Then, as a class, discuss what they discovered.

Project Corner

The Project Corner box for section 1.5 further explores oil and gas resources. For students who select oil as their natural resource, they may find the additional statistics helpful in collecting data and designing their sequence. It may benefit all students to discuss whether they think all or any of the natural resources can be produced at the same rate indefinitely. Have them give reasons for their thinking.

Encourage students who are interested in petroleum to find out how gas processing is affected by oil and, in turn, the global economy. This could be a cross-curricular assignment, depending on the topics and timing in relation to other courses.

Some students will research and collect information that may or may not be geometric. Explain to students that as the chapter progresses, the class will review the suggested examples in each Project Corner box, but that each student may use a different type of sequence or series.

Meeting Student Needs

- For #8, you may wish to have students research oil wells in their area and any trends in production. Consider having them write a question based on their findings. They can then exchange with a partner and solve their partner's question. Alternatively, you might have students research the impact of oil production on their community. They could then share their findings with the class.
- Have students work in pairs to complete #9 to 11.
- Since #14 and 16 effectively indicate students' level of understanding, you might wish to use these questions as an alternate means of assessment.
- Provide **BLM 1–8 Section 1.5 Extra Practice** to students who would benefit from more practice.

ELL

- Some students may not be familiar with terms such as *produced/production*, *barrels of crude*, *trend*, *initial*, *successive*, *reasonable*, *impact*, *pounds*, *rebounds*, *succeeding*, and *altitude*. Some terms lend themselves well to visual representations to assist student understanding, such as *oil well*, *swing*, *pendulum*, *pile driver*, *metal post*, and *hot air balloon*. Other terms can be taught using a combination of visuals, descriptions, and examples.
- Note that though some students may have difficulty with #12, it may be due to lack of language understanding rather than lack of mathematical understanding.

- Since #23 is a Mini Lab, have group members assist students who have difficulty with the language in the procedure.

Common Errors

- Some students may confuse convergent and divergent series.
- **R_x** Have students write down the formula for an infinite geometric series and include the statement of restrictions on r : $-1 < r < 1$.

Assessment	Supporting Learning
Assessment for Learning	
<p>Practise and Apply Have students do #1, 2, 3a), b), 4, 5a), c), 6, 7, 9, 12, and 14. Students who have no problems with these questions can go on to the remaining questions.</p>	<ul style="list-style-type: none"> • For #1 to 5, encourage students to use the examples as models. Suggest that they complete part a) of each question and have someone check their work before they move on to the other parts. • Remediate as necessary and clarify any misunderstandings. • Encourage students who cannot identify a convergent from a divergent sequence/series to use one of the formulas to check their thinking. • For #6, 7, and 12, encourage students to write out the first three terms of the sequence so that they have a visual from which to work. Although a diagram is provided for #12, it may help students to identify the known and unknown values by writing out the initial terms of the sequence.
Assessment as Learning	
<p>Create Connections Have all students complete #21 and 22.</p>	<ul style="list-style-type: none"> • Create Connections #21 offers students a chance to explain their thinking. You might have them write it into the Foldable or graphic organizer that they started at the beginning of the chapter. • The Mini Lab in #23 is useful for Assessment for Learning or Assessment of Learning since it combines all the topics of the chapter and would provide valuable insights into your students' abilities and thinking. Approach the Mini Lab by having students of varied abilities form groups of four. Students will likely enjoy the interactive process of this brief exercise. Also, it provides a powerful visual of a sequence being built. Encourage students to write the sequence symbolically beside each square for comparison. One set could be mounted on the wall for students as a reminder of a visual sequence. For students who require more engagement, give them the option to see the process using a computer program. However, be certain that students have clear understanding of how the sequence and squares are related.

Chapter 1 Review

Pre-Calculus 11, pages 66–68

Suggested Timing

60–90 min

Materials

- 0.5 centimetre grid paper
- ruler

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- BLM 1–4 Section 1.1 Extra Practice
- BLM 1–5 Section 1.2 Extra Practice
- BLM 1–6 Section 1.3 Extra Practice
- BLM 1–7 Section 1.4 Extra Practice
- BLM 1–8 Section 1.5 Extra Practice

Planning Notes

Have students who are not confident discuss strategies with you or another classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. Provide an opportunity for students to get extra help on questions they need help with. They can use their list to help them prepare for the practice test.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 1–4 Section 1.1 Extra Practice**, **BLM 1–5 Section 1.2 Extra Practice**, **BLM 1–6 Section 1.3 Extra Practice**, **BLM 1–7 Section 1.4 Extra Practice**, and **BLM 1–8 Section 1.5 Extra Practice**.
- Before students begin the questions, invite them to list in their notebook as many Key Ideas as they can remember from the chapter. Of particular importance are the similarities and differences between arithmetic sequences, arithmetic series, geometric sequences, geometric series, and infinite series; the meaning of

t_n , S_n , t_1 , r , etc. Have a class discussion once students have written all they remember. Divide the board into five columns (one for each section of the chapter) and write down student comments under each heading.

- If it has not been done already, post all learning outcomes. Invite students to ask questions about any outcomes that they do not understand.
- Prior to beginning the chapter review, hand out or have students create a formula page that they can use for the final assessment.
- Individualize the chapter review. Have students choose three questions from each section to begin. Correct the questions and analyse errors. Encourage students to request assistance for the questions they are unable to complete successfully. Students can then choose more questions based on the results attained.

ELL

- Encourage students to refer to their vocabulary dictionaries as they work on the questions.
- Use a combination of visuals, descriptions, and examples to help student understand such terms as *contact*, *neighbourhood*, *concert hall*, *culture*, *bacteria*, *subsequent*, *spiral*, *adjacent*, and *indefinitely*.
- For #14, show students a picture of Mickey Mouse to familiarize them with the reference.
- For #15, show students a number of different types of flowcharts to help them in their understanding of this term.

Enrichment

- Suggest that students compare the features of geometric patterns and arithmetic patterns. Have them create a table that shows similarities and differences.

Gifted

- Some students may already be familiar with the skills handled in this review. To provide enrichment and extra challenge for gifted students, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment for Learning	
<p>Chapter 1 Review</p> <p>The Chapter 1 Review is an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource.</p>	<ul style="list-style-type: none"> • You may wish to provide concrete and kinesthetic learners with tiles to assist them in completing #10. • Have students revisit any section that they are having difficulty with prior to working on the chapter test.

1

Chapter 1 Practice Test

Pre-Calculus 11, pages 69–70

Suggested Timing

30–45 min

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BLM 1–9 Chapter 1 Test

indicate which questions they need no help with, a little help with, and a lot of help with. Have students first complete the questions they know they can do, followed by the questions they know something about. Finally, suggest to students that they do their best on the remaining questions. Ensure that students are coached on those questions that they indicated they need help with.

This practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–10.

Planning Notes

Have students start the practice test by writing the question numbers in their notebook. Have them

Study Guide

Question(s)	Section(s)	Refer to	The student can ...
#1	1.1	Example 1	✓ determine t_1 , d , n , or t_n in a problem that involves an arithmetic sequence
#2, 4	1.1	Example 1 Example 3	✓ determine the values of t_1 , d , n , or S_n in an arithmetic sequence ✓ solve a problem that involves an arithmetic sequence
#3	1.4	Example 1	✓ determine t_1 , r , n , or S_n involving a geometric series
#5, 9	1.3	Example 2	✓ solve a problem that involves a geometric sequence
#6	1.3	Example 1 Example 2	✓ solve a problem that involves an arithmetic sequence
#7	1.1 1.3	Link the Ideas	✓ justify an example of an arithmetic sequence ✓ justify an example of a geometric sequence
#8	1.1 1.3	Example 1 Example 2 Example 4	✓ determine t_1 , d , n , or t_n in a problem that involves an arithmetic sequence
#10	1.1 1.3	Examples 1 and 3 Examples 1, 3, and 4	✓ determine t_1 , r , n , or S_n involving an arithmetic or geometric sequence
#11	1.1	Example 2 Example 4	✓ determine t_1 , r , n , or S_n involving a geometric sequence
#12	1.1 1.2	Examples 2 and 4 Example 1	✓ determine t_1 , r , n , or t_n in a problem that involves a geometric sequence

Assessment	Supporting Learning
Assessment as Learning	
Chapter 1 Self-Assessment Have students use their responses on the practice test and work they completed earlier in the chapter to identify skills and concepts they may need to reinforce.	<ul style="list-style-type: none"> Before the chapter test, coach them in areas in which they are having difficulties.
Assessment of Learning	
Chapter 1 Test After students complete the practice test, you may wish to use BLM 1–9 Chapter 1 Test as a summative assessment.	<ul style="list-style-type: none"> Consider allowing students to refer to the Foldable or graphic organizer that they started at the beginning of the chapter.

Unit 1 Project

Pre-Calculus 11, page 71

Suggested Timing

60–90 min

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BLM U1–1 Unit 1 Project Checklist

Mathematical Processes

- Communication (C)
- Connections (CN)
- Mental Math and Estimation (ME)
- Problem Solving (PS)
- Reasoning (R)
- Technology (T)
- Visualization (V)

General Outcome

Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

- RF9** Analyze arithmetic sequences and series to solve problems
RF10 Analyze geometric sequences and series to solve problems

Planning Notes

If students have not yet chosen a natural resource for their project, take time to review the information and pictures on this page. You might also have them review the information and pictures on page 3, as well as each of the Project Corners in Chapter 1. With these images and data in mind, brainstorm the natural resources students could research. Encourage them to choose a resource that they find interesting.

Depending on the class, you may wish to have individual students choose a resource, or have teams of students choose the same resource. These resource teams could share research, but develop their own separate presentations at the end of the unit.

During their project work, students may wish to discuss their research and sequence or series ideas with each other. Peer coaching will assist many students in improving their data analysis skills.

Remind students that they have until the end of the unit to finalize this project, but that they should start collecting data and considering how they might present it. Students will finalize their presentation after they have completed Chapter 2.

Assessment	Supporting Learning
Assessment for Learning	
<p>Unit 1 Project</p> <p>This part of the unit project gives students an opportunity to research data that will allow them to apply and demonstrate their knowledge of:</p> <ul style="list-style-type: none"> • arithmetic sequences and series • geometric sequences and series • infinite geometric series <p>Encourage students to start researching their chosen resource now and to explore ideas for some of the series or sequences.</p>	<ul style="list-style-type: none"> • You may wish to have students use the part of BLM U1–1 Unit 1 Project Checklist that provides a list of the required components for the Chapter 1 part of the Unit 1 Project. • Discussing the Project Corner boxes at the end of each section of Chapter 1 and using the data to develop different types of series and sequences will assist students in developing appropriate sequences and series for the data they research. • Make sure students realize that they need to use this part of the project to demonstrate what they have learned about the different types of sequences and series discussed in Chapter 1. Their project is still in development. It will be finalized and presented at the end of the unit.

