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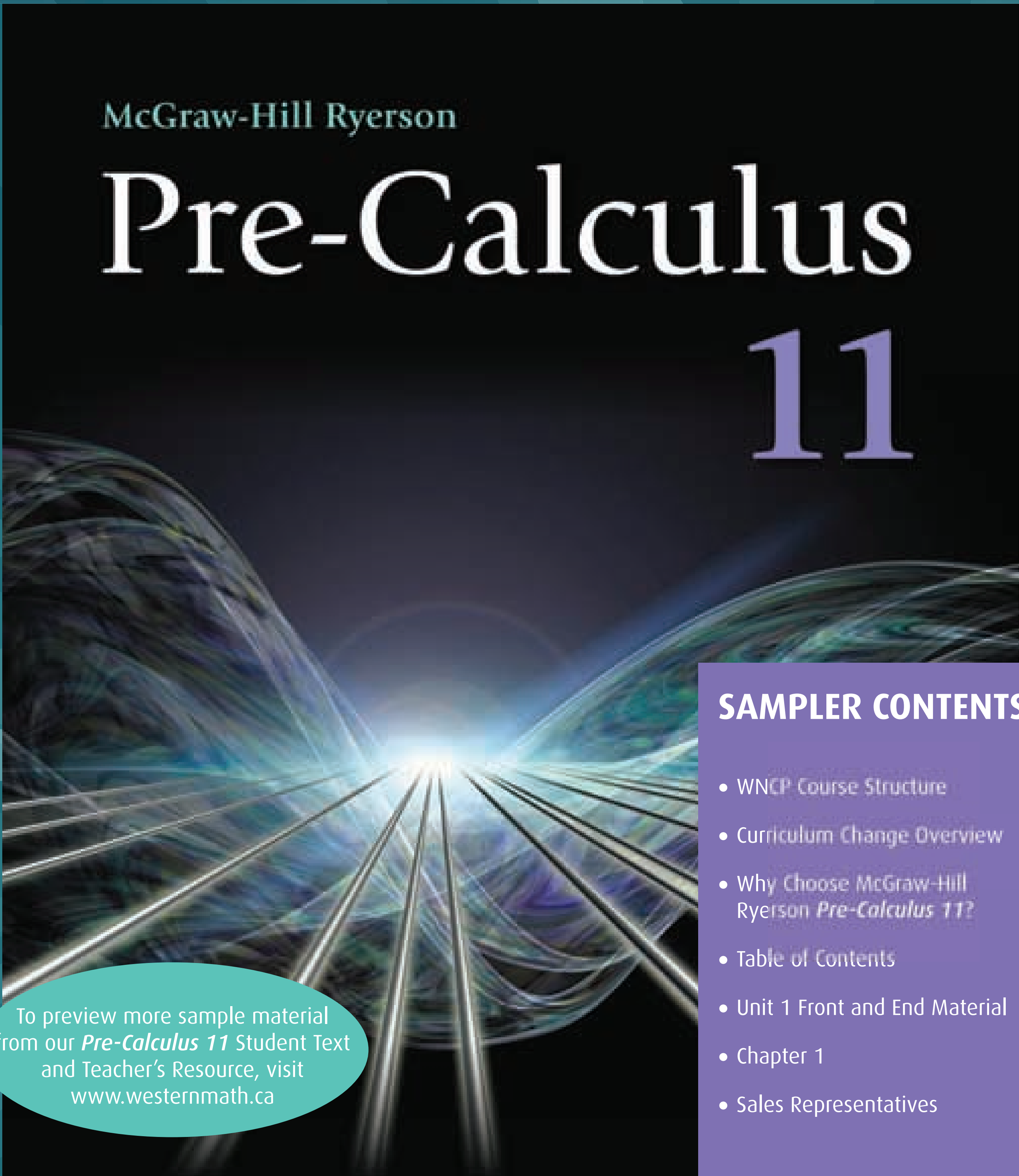

To preview more sample material from our *Pre-Calculus 11* Student Text and Teacher's Resource, visit www.westernmath.ca

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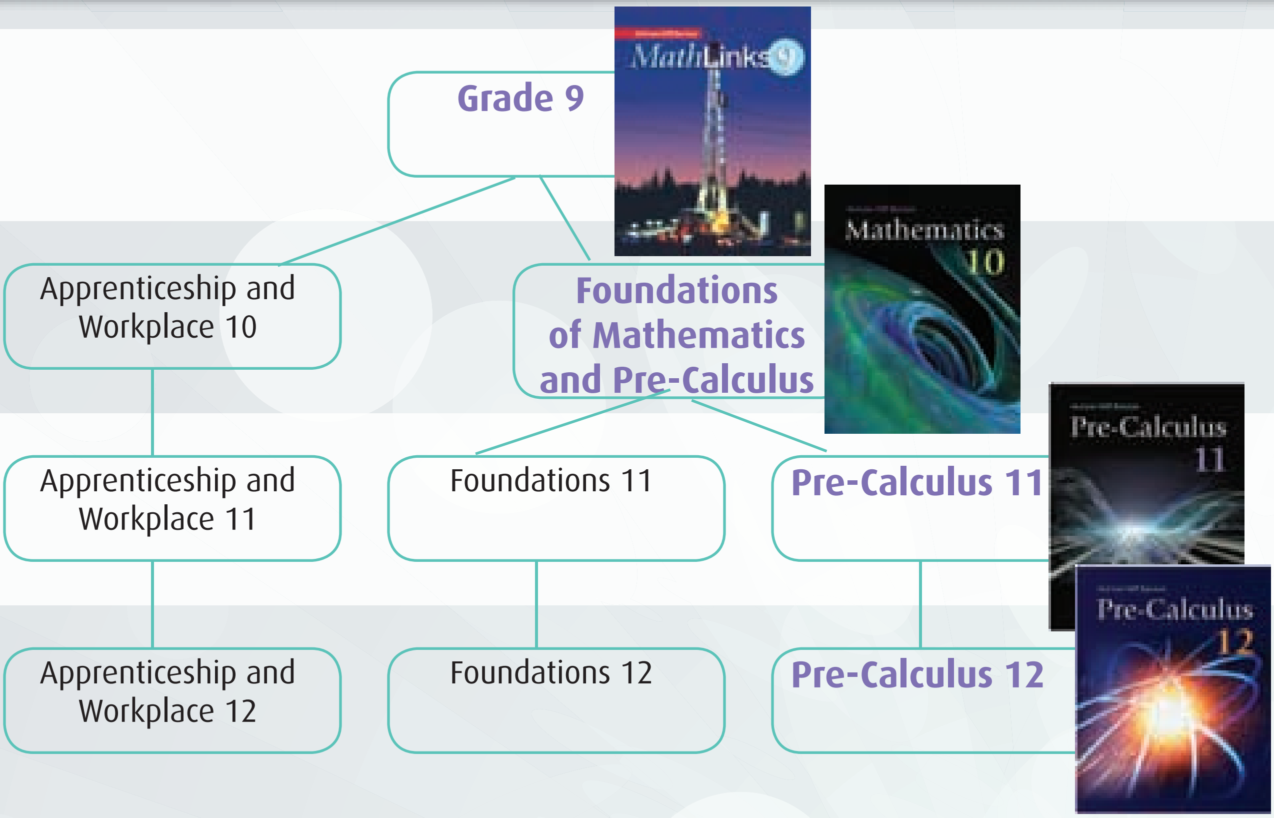
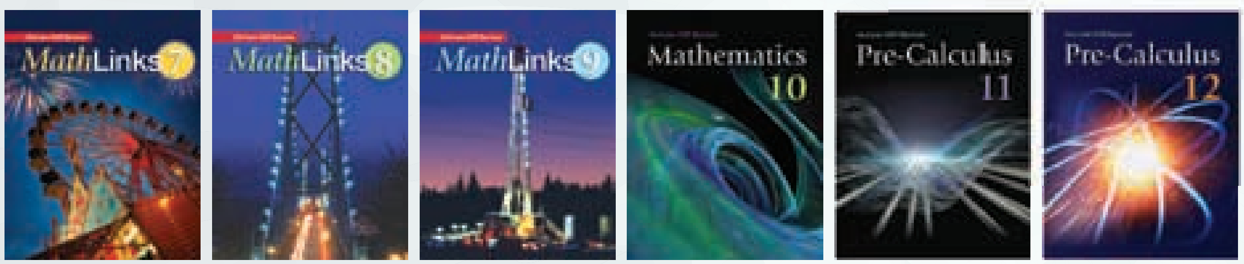
WNCP Course Structure



McGraw-Hill Ryerson offers a unique continuum for grades 7-12



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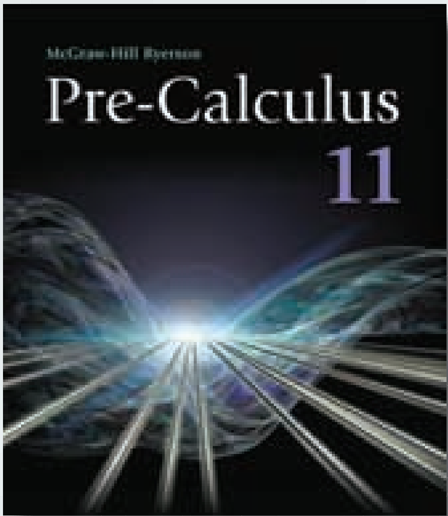
For more information on these products please visit: www.mathlinks.ca and www.westernmath.ca

Curriculum: How much has changed?



1996: 32 outcomes
Of these, **24** have been deleted or significantly changed.

2008: 20 outcomes
Of these, **15** are either significantly changed or are new.

Out of 52 outcomes in total for 1996 and 2008, there have been changes in **39** of them.

This is a **75%** change altogether.

New Topics

- Linear-quadratic and quadratic-quadratic systems
- Arithmetic sequences and series **Moved from Foundations of Mathematics and Pre-Calculus 10**
- Rational expressions (equivalent forms, non-permissible values, and operations) and equations **Moved from Foundations of Mathematics and Pre-Calculus 10**
- Cosine and Sine Laws **Moved from Foundations of Mathematics and Pre-Calculus 10**
- Geometric growth/number patterns **Moved from Foundations of Mathematics and Pre-Calculus 10**
- Sine and cosine for angles from 90° to 180° **Moved from Pre-Calculus 10**
- Sine and cosine laws **Moved from Foundations of Mathematics and Pre-Calculus 10**
- Geometric sequences and series **Moved from Pre-Calculus 12**
- Reciprocal functions (linear and quadratic only) **Moved from Pre-Calculus 12**
- Absolute value functions **Moved from Pre-Calculus 12**

Deleted Topics

- Financial/consumer math
- Reasoning/logic/proof
- Systems of linear equations in three variables
- Circle geometry **Moved to Grade 9**
- Systems of linear equations in two variables **Moved to Foundations of Mathematics and Pre-Calculus 10**
- Remainder Theorem and Factor Theorem **Moved to Pre-Calculus 12**
- Operation on functions and compositions of functions **Moved to Pre-Calculus 12**
- Non-linear equations (solving algebraically or graphically)
- Graph and analyze polynomial and rational functions **Moved to Pre-Calculus 12**

✔ **Pre-Calculus 11 is based on current content within the WNCP curriculum and the needs of Pre-Calculus 11 students and teachers**

Why choose McGraw-Hill Ryerson *Pre-Calculus 11*?

1 **Selected by the WNCP; developed by Western Educators**

McGraw-Hill Ryerson was selected as the exclusive publisher by the WNCP to develop custom resources for the new Pre-Calculus, Grades 11 & 12 courses. Throughout development these resources are rigorously reviewed and vetted by western educators and the WNCP.

2 **A Comprehensive Resource Package**

McGraw-Hill Ryerson leads the way with a complete curriculum solution. The McGraw-Hill Ryerson Pre-Calculus program is a carefully blended mix of print and digital resources (available in English and French) to meet all learning and teaching needs.

3 **Well-constructed Examples and Worked Solutions**

Hundreds of Examples and Worked Solutions with thousands of questions and problems offering teachers and students choice and flexibility for years to come.

4 **Variety and Choice in Assessment**

The McGraw-Hill Ryerson ***Pre-Calculus 11*** program offers a variety of assessment opportunities and strategies to accommodate the diversity of students' needs and to enhance learning.

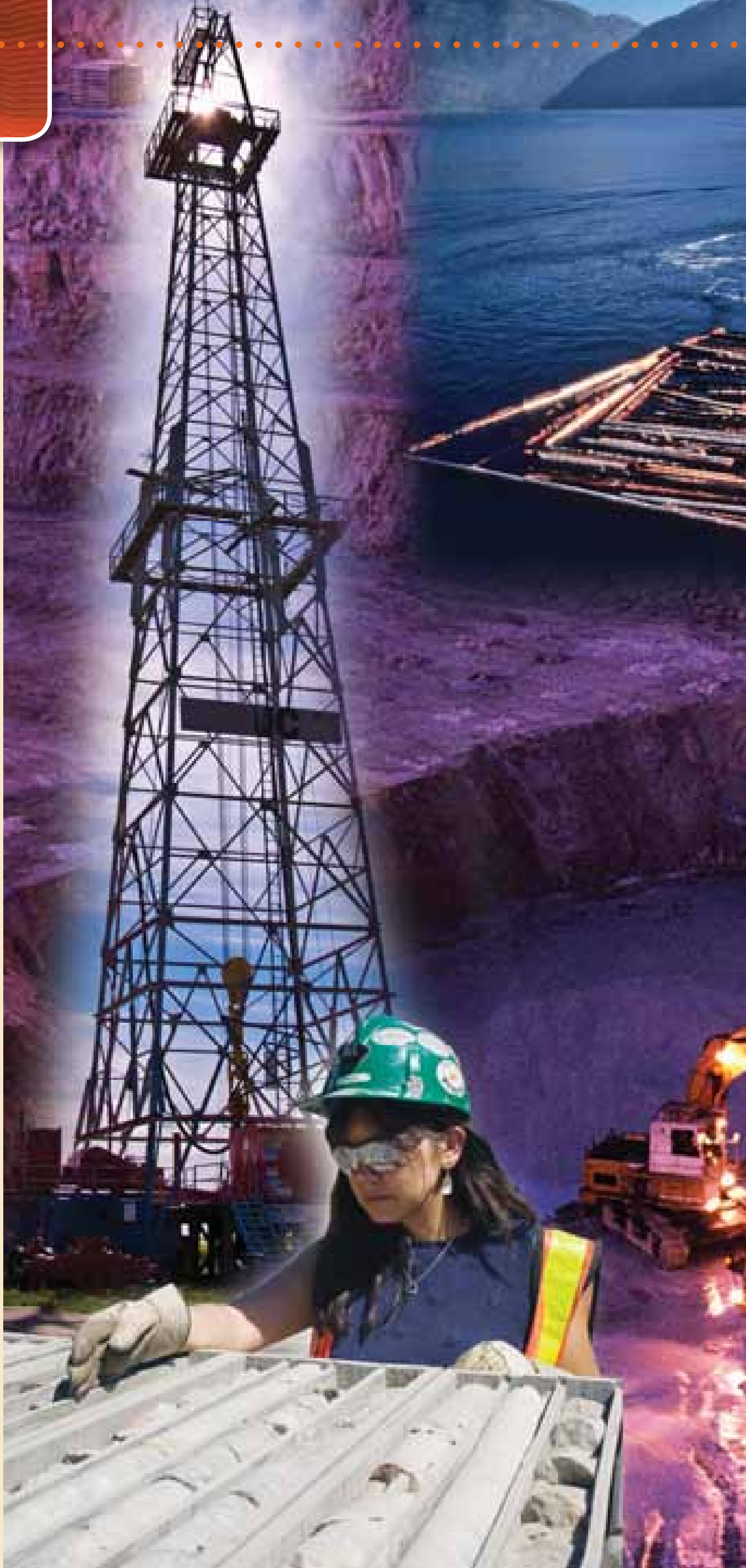
5 **Engage 21st Century Learners**

Meaningful and relevant contexts engage students throughout the student text, while the digital and online pieces provide unique opportunities to engage students and allow for learning, reviewing, and self-assessing 24/7.

Table of Contents

Patterns

Many problems are solved using patterns. Economic and resource trends may be based on sequences and series. Seismic exploration identifies underground phenomena, such as caves, oil pockets, and rock layers, by transmitting sound into the earth and timing the echo of the vibration. Surveyors use triangulation and the laws of trigonometry to determine distances between inaccessible points. All of these activities use patterns and aspects of the mathematics you will encounter in this unit.



Looking Ahead

In this unit, you will solve problems involving...

- arithmetic sequences and series
- geometric sequences and series
- infinite geometric series
- sine and cosine laws

Unit 1 Project — Canada's Natural Resources

Canada is a country rich with natural resources. Petroleum, minerals, and forests are found in abundance in the Canadian landscape. Canada is one of the world's leading exporters of minerals, mineral products, and forest products. Resource development has been a mainstay of Canada's economy for many years.

In this project, you will explore one of Canada's natural resources from the categories of petroleum, minerals, or forestry. You will collect and present data related to your chosen resource to meet the following criteria:

- Include a log of the journey leading to the discovery of your resource.
- In Chapter 1, you will provide data on the production of your natural resource. Here you will apply your knowledge of sequences and series to show how production has increased or decreased over time, and make predictions about future development of your chosen resource.
- In Chapter 2, you will use skills developed with trigonometry, including the sine law and the cosine law to explore the area where your resource was discovered. You will then explore the proposed site of your natural resource.

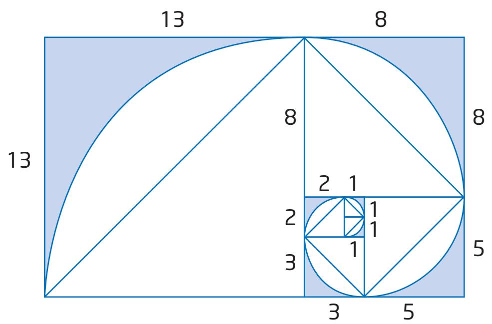
At the end of your project, you will encourage potential investors to participate in the development of your resource. Your final project may take many forms. It may be a written or visual presentation, a brochure, a video production, or a computer slide show. Or, you could use the interactive features of a whiteboard.

In the Project Corner box at the end of most sections, you will find information and notes about Canada's natural resources. You can use this information to help gather data and facts about your chosen resource.

CHAPTER 1

Sequences and Series

Many patterns and designs linked to mathematics are found in nature and the human body. Certain patterns occur more often than others. Logistic spirals, such as the Golden Mean spiral, are based on the Fibonacci number sequence. The Fibonacci sequence is often called Nature's Numbers.



The pattern of this logistic spiral is found in the chambered nautilus, the inner ear, star clusters, cloud patterns, and whirlpools. Seed growth, leaves on stems, petals on flowers, branch formations, and rabbit reproduction also appear to be modelled after this logistic spiral pattern.

There are many different kinds of sequences. In this chapter, you will learn about sequences that can be described by mathematical rules.

Web Link

To learn more about the Fibonacci sequence, go to www.mhrprecalc11.ca and follow the links.

Key Terms

sequence	geometric sequence
arithmetic sequence	common ratio
common difference	geometric series
general term	convergent series
arithmetic series	divergent series

Did You Know?

In mathematics, the *Fibonacci sequence* is a sequence of natural numbers named after Leonardo of Pisa, also known as Fibonacci. Each number is the sum of the two preceding numbers.
1, 1, 2, 3, 5, 8, 13, …

Career Link

Biomedical engineers combine biology, engineering, and mathematical sciences to solve medical and health-related problems. Some research and develop artificial organs and replacement limbs. Others design MRI machines, laser systems, and microscopic machines used in surgery. Many biomedical engineers work in research and development in health-related fields. If you have ever taken insulin or used an asthma inhaler, you have benefited from the work of biomedical engineers.

Web Link

To learn more about biomedical engineering, go to www.mhrprecalc11.ca and follow the links.

1.1

Arithmetic Sequences

Focus on…

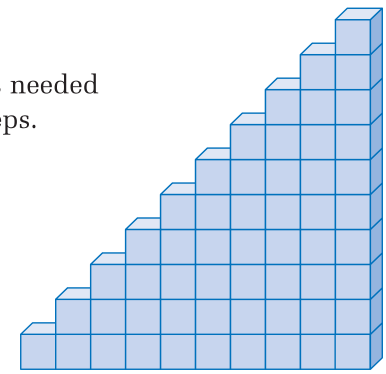
- deriving a rule for determining the general term of an arithmetic sequence
- determining t_1, d, n, or t_n in a problem that involves an arithmetic sequence
- describing the relationship between an arithmetic sequence and a linear function
- solving a problem that involves an arithmetic sequence

Comets are made of frozen lumps of gas and rock and are often referred to as icy mudballs or dirty snowballs. In 1705, Edmond Halley predicted that the comet seen in 1531, 1607, and 1682 would be seen again in 1758. Halley's prediction was accurate. This comet was later named in his honour. The years in which Halley's Comet has appeared approximately form terms of an arithmetic sequence. What makes this sequence arithmetic?

Investigate Arithmetic Sequences

Staircase Numbers

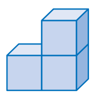
A *staircase number* is the number of cubes needed to make a staircase that has at least two steps. Is there a pattern to the number of cubes in successive staircase numbers? How could you predict different staircase numbers?



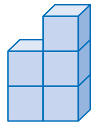
Part A: Two-Step Staircase Numbers

To generate a two-step staircase number, add the numbers of cubes in two consecutive columns.

The first staircase number is the sum of the number of cubes in column 1 and in column 2.

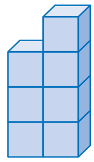


For the second staircase number, add the number of cubes in columns 2 and 3.



2 3

For the third staircase number, add the number of cubes in columns in 3 and 4.



3 4

1. Copy and complete the table for the number of cubes required for each staircase number of a two-step staircase.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	3	5								

Part B: Three-Step Staircase Numbers

To generate a three-step staircase number, add the numbers of cubes in three consecutive columns.

The first staircase number is the sum of the number of cubes in column 1, column 2, and column 3.

For the second staircase number, add the number of cubes in columns 2, 3, and 4.

For the third staircase number, add the number of cubes in columns 3, 4, and 5.

2. Copy and complete the table for the number of cubes required for each step of a three-step staircase.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	6	9								

3. The same process may be used for staircase numbers with more than three steps. Copy and complete the following table for the number cubes required for staircase numbers up to six steps.

	Number of Steps in the Staircase				
Term	2	3	4	5	6
1	3	6			
2	5	9			
3					
4					
5					
6					

4. Describe the pattern for the number of cubes in a two-step staircase.
5. How could you find the number of cubes in the 11th and 12th terms of the two-step staircase?
6. Describe your strategy for determining the number of cubes required for staircases with three, four, five, or six steps.

Reflect and Respond

sequence
- an ordered list of elements

7. a) Would you describe the terms of the number of cubes as a **sequence**?
   b) Describe the pattern that you observed.
8. a) In the sequences generated for staircases with more than two steps, how is each term generated from the previous term?
   b) Is this difference the same throughout the entire sequence?
9. a) How would you find the number of cubes required if you were asked for the 100th term in a two-step staircase?
   b) Derive a formula from your observations of the patterns that would allow you to calculate the 100th term.
   c) Derive a general formula that would allow you to calculate the nth term.

Link the Ideas

Sequences

A *sequence* is an ordered list of objects. It contains elements or terms that follow a pattern or rule to determine the next term in the sequence. The terms of a sequence are labelled according to their position in the sequence.

The *first term* of the sequence is t_1.
The *number of terms* in the sequence is n.
The *general term* of the sequence is t_n. This term is dependent on the value of n.

The first term of a sequence is sometimes referred to as a. In this resource, the first term will be referred to as t_1.

t_n is read as "t subscript n" or "t sub n."

Finite and Infinite Sequences

A *finite sequence* always has a finite number of terms.
Examples: 2, 5, 8, 11, 14
5, 10, 15, 20, …, 100

An *infinite sequence* has an infinite number of terms. Every term is followed by a new term.
Example: 5, 10, 15, 20, …

Arithmetic Sequences

An **arithmetic sequence** is an ordered list of terms in which the difference between consecutive terms is constant. In other words, the same value or variable is added to each term to create the next term. This constant is called the **common difference**. If you subtract the first term from the second term for any two consecutive terms of the sequence, you will arrive at the common difference.

The formula for the **general term** helps you find the terms of a sequence. This formula is a rule that shows how the value of t_n depends on n.

Consider the sequence 10, 16, 22, 28, ….

Terms	t_1	t_2	t_3	t_4
Sequence	10	16	22	28
Sequence Expressed Using First Term and Common Difference	10	10 + (6)	10 + (6) + (6)	10 + (6) + (6) + (6)
General Sequence	t_1	$t_1 + d$	$t_1 + d + d$ $= t_1 + 2d$	$t_1 + d + d + d$ $= t_1 + 3d$

The general arithmetic sequence is $t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots,$ where t_1 is the first term and d is the common difference.

$t_1 = t_1$
$t_2 = t_1 + d$
$t_3 = t_1 + 2d$
$\vdots$
$t_n = t_1 + (n - 1)d$

The *general term* of an arithmetic sequence is
$\boldsymbol{t_n = t_1 + (n - 1)d}$
where t_1 is the first term of the sequence
n is the number of terms
d is the common difference
t_n is the general term or nth term

arithmetic sequence
- a sequence in which the difference between consecutive terms is constant

common difference
- the difference between successive terms in an arithmetic sequence, $d = t_n - t_{n-1}$
- the difference may be positive or negative
- for example, in the sequence 10, 16, 22, 28, …, the common difference is 6

general term
- an expression for directly determining any term of a sequence
- symbol is t_n
- for example, $t_n = 3n + 2$

Example 1

Determine a Particular Term

A visual and performing arts group wants to hire a community events leader. The person will be paid \$12 for the first hour of work, \$19 for two hours of work, \$26 for three hours of work, and so on.

a) Write the general term that you could use to determine the pay for any number of hours worked.

b) What will the person get paid for 6 h of work?

Solution

State the sequence given in the problem.

$t_1 = 12$
$t_2 = 19$
$t_3 = 26$
$\vdots$

The common difference of the sequence may be found by subtracting any two consecutive terms. The common difference for this sequence is 7.
$19 - 12 = 7$
$26 - 19 = 7$

The sequence is arithmetic with a common difference equal to 7. Subtracting any two consecutive terms will result in 7.

Did You Know?

The relation $t_n = 7n + 5$ may also be written using function notation: $f(n) = 7n + 5$.

a) For the given sequence, $t_1 = 12$ and $d = 7$.
Use the formula for the general term of an arithmetic sequence.
$t_n = t_1 + (n - 1)d$
$t_n = 12 + (n - 1)7$ Substitute known values.
$t_n = 12 + 7n - 7$
$t_n = 7n + 5$
The general term of the sequence is $t_n = 7n + 5$.

b) For 6 h of work, the amount is the sixth term in the sequence. Determine t_6.

Method 1: Use an Equation

$t_n = t_1 + (n - 1)d$ or $t_n = 7n + 5$
$t_6 = 12 + (6 - 1)7$ $t_6 = 7(6) + 5$
$t_6 = 12 + (5)(7)$ $t_6 = 42 + 5$
$t_6 = 12 + 35$ $t_6 = 47$
$t_6 = 47$

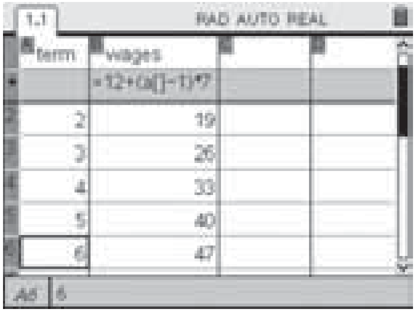
The value of the sixth term is 47.
For 6 h of work, the person will be paid \$47.

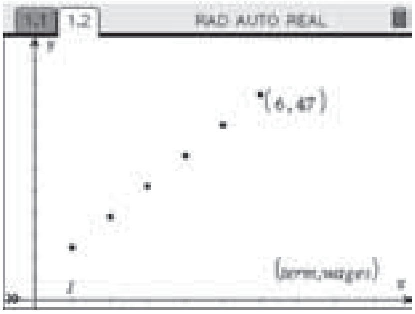
Method 2: Use Technology

You can use a calculator or spreadsheet to determine the sixth term of the sequence.

Use a table.

You can generate a table of values and a graph to represent the sequence.





The value of the sixth term is 47.

For 6 h of work, the person will be paid $47.

Your Turn

Many factors affect the growth of a child. Medical and health officials encourage parents to keep track of their child's growth. The general guideline for the growth in height of a child between the ages of 3 years and 10 years is an average increase of 5 cm per year. Suppose a child was 70 cm tall at age 3.

a) Write the general term that you could use to estimate what the child's height will be at any age between 3 and 10.

b) How tall is the child expected to be at age 10?

Example 2

Determine the Number of Terms

The musk-ox and the caribou of northern Canada are hoofed mammals that survived the Pleistocene Era, which ended 10 000 years ago. In 1955, the Banks Island musk-ox population was approximately 9250 animals. Suppose that in subsequent years, the growth of the musk-ox population generated an arithmetic sequence, in which the number of musk-ox increased by approximately 1650 each year. How many years would it take for the musk-ox population to reach 100 000?

Solution

The sequence 9250, 10 900, 12 550, 14 200, . . ., 100 000 is arithmetic.

For the given sequence,
First term $t_1 = 9250$
Common difference $d = 1650$
*n*th term $t_n = 100\ 000$

To determine the number of terms in the sequence, substitute the known values into the formula for the general term of an arithmetic sequence.

$$\begin{aligned} t_n &= t_1 + (n-1)d \\ 100\ 000 &= 9250 + (n-1)1650 \\ 100\ 000 &= 9250 + 1650n - 1650 \\ 100\ 000 &= 1650n + 7600 \\ 92\ 400 &= 1650n \\ 56 &= n \end{aligned}$$

There are 56 terms in the sequence.
It would take 56 years for the musk-ox population to reach 100 000.

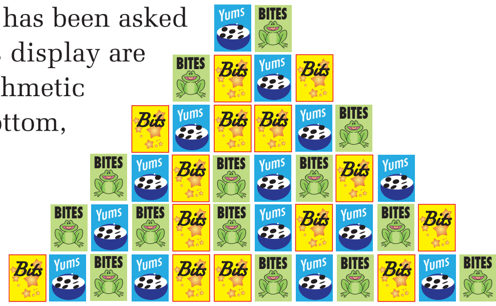
Your Turn

Carpenter ants are large, usually black ants that make their colonies in wood. Although often considered to be pests around the home, carpenter ants play a significant role in a forested ecosystem. Carpenter ants begin with a parent colony. When this colony is well established, they form satellite colonies consisting of only the workers. An established colony may have as many as 3000 ants. Suppose that the growth of the colony produces an arithmetic sequence in which the number of ants increases by approximately 80 ants each month. Beginning with 40 ants, how many months would it take for the ant population to reach 3000?

Example 3

Determine t_1, t_n, and n

Jonathon has a part-time job at the local grocery store. He has been asked to create a display of cereal boxes. The top six rows of his display are shown. The numbers of boxes in the rows produce an arithmetic sequence. There are 16 boxes in the third row from the bottom, and 6 boxes in the eighth row from the bottom.



a) How many boxes are in the bottom row?

b) Determine the general term, t_n, for the sequence.

c) What is the number of rows of boxes in his display?

Solution

a) **Method 1: Use Logical Reasoning**

The diagram shows the top six rows. From the diagram, you can see that the number of boxes per row decreases by 2 from bottom to top. Therefore, $d = -2$

What is the value of d if you go from top to bottom?

You could also consider the fact that there are 16 boxes in the 3rd row from the bottom and 6 boxes in the 8th row from the bottom. This results in a difference of 10 boxes in 5 rows. Since the values are decreasing, $d = -2$.

Substitute known values into the formula for the general term.

$$t_n = t_1 + (n - 1)\,d$$
$$16 = t_1 + (3 - 1)(-2)$$
$$16 = t_1 - 4$$
$$20 = t_1$$

The number of boxes in the bottom row is 20.

Method 2: Use Algebra

Since t_1 and d are both unknown, you can use two equations to determine them. Write an equation for t_3 and an equation for t_8 using the formula for the general term of an arithmetic sequence.

$$t_n = t_1 + (n - 1)d$$

For $n = 3$
$$16 = t_1 + (3 - 1)d$$
$$16 = t_1 + 2d$$

For $n = 8$
$$6 = t_1 + (8 - 1)d$$
$$6 = t_1 + 7d$$

Subtract the two equations.

$$\begin{array}{r} 16 = t_1 + 2d \\ 6 = t_1 + 7d \\ \hline 10 = -5d \\ -2 = d \end{array}$$

Substitute the value of d into the first equation.
$16 = t_1 + 2d$
$16 = t_1 + 2(-2)$
$16 = t_1 - 4$
$20 = t_1$

Is there another way to solve this problem? Work with a partner to discuss possible alternate methods.

The sequence for the stacking of the boxes is 20, 18, 16, ...
The number of boxes in the bottom row is 20.

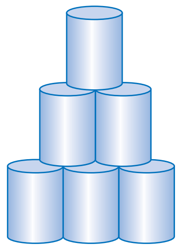
b) Use the formula for the general term of the sequence.
$t_n = t_1 + (n - 1)d$
$t_n = 20 + (n - 1)(-2)$
$t_n = -2n + 22$
The general term of the sequence is $t_n = -2n + 22$.

c) The top row of the stack contains two boxes.
Use the general term to find the number of rows.
$t_n = -2n + 22$
$2 = -2n + 22$
$-20 = -2n$
$10 = n$

The number of rows of boxes is 10.

Your Turn

Jonathon has been given the job of stacking cans in a similar design to that of the cereal boxes. The numbers of cans in the rows produces an arithmetic sequence. The top three rows are shown. There are 26 cans in the 8th row from the bottom and 10 cans in the 12th row from the bottom. Determine t_1, d, and t_n for the arithmetic sequence.



Example 4

Generate a Sequence

A furnace technician charges \$65 for making a house call, plus \$42 per hour or portion of an hour.

a) Generate the possible charges (excluding parts) for the first 4 h of time.

b) What is the charge for 10 h of time?

Solution

a) Write the sequence for the first four hours.

Terms of the Sequence	1	2	3	4
Number of Hours Worked	1	2	3	4
Charges ($)	107	149	191	233

How do you determine the charge for the first hour?

The charges for the first 4 h are $107, $149, $191, and $233.

b) The charge for the first hour is $107. This is the first term.
The common difference is $42.

$t_1 = 107$
$d = 42$

Substitute known values into the formula to determine the general term.

$t_n = t_1 + (n - 1)d$
$t_n = 107 + (n - 1)42$
$t_n = 107 + 42n - 42$
$t_n = 42n + 65$

Method 1: Use the General Term

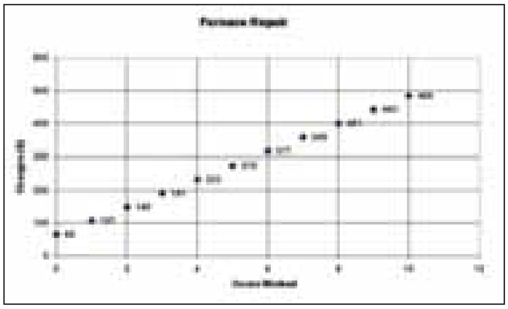
The 10th term of the sequence may be generated by substituting 10 for n in the general term.

$t_n = 42n + 65$
$t_{10} = 42(10) + 65$
$t_{10} = 485$

The charge for 10 h of work is $485.

Method 2: Use a Graph

The general term $t_n = 42n + 65$ is a function that relates the charge to the number of hours worked. This equation $f(x) = 42x + 65$ could be graphed. The slope of 42 is the common difference of the sequence. The y-intercept of 65 is the initial charge for making a house call.



Hours Worked	Charges ($)
0	65
1	107
2	149
3	191
4	233
5	275
6	317
7	359
8	401
9	443
10	485

The terms may now be generated by either tracing on the graph or accessing the table of values.

The charge for 10 h of work is $485.

Your Turn

What is the charge for 10 h if the furnace technician charges $45 for the house call plus $46 per hour?

Key Ideas

- A sequence is an ordered list of elements.
- Elements within the range of the sequence are called terms of the sequence.
- To describe any term of a sequence, an expression is used for t_n, where $n \in \mathrm{N}$. This term is called the general term.
- In an arithmetic sequence, each successive term is formed by adding a constant. This constant is called the common difference.
- The general term of an arithmetic sequence is
  $t_n = t_1 + (n - 1)d$
  where t_1 is the first term
  n is the number of terms ($n \in \mathrm{N}$)
  d is the common difference
  t_n is the general term or nth term

Check Your Understanding

Practise

1. Identify the arithmetic sequences from the following sequences. For each arithmetic sequence, state the value of t_1, the value of d, and the next three terms.

a) 16, 32, 48, 64, 80, ...

b) 2, 4, 8, 16, 32, ...

c) $-4, -7, -10, -13, -16, \ldots$

d) $3, 0, -3, -6, -9, \ldots$

2. Write the first four terms of each arithmetic sequence for the given values of t_1 and d.

a) $t_1 = 5$, $d = 3$

b) $t_1 = -1$, $d = -4$

c) $t_1 = 4$, $d = \frac{1}{5}$

d) $t_1 = 1.25$, $d = -0.25$

3. For the sequence defined by $t_n = 3n + 8$, find each indicated term.

a) t_1 **b)** t_7 **c)** t_{14}

4. For each arithmetic sequence determine the values of t_1 and d. State the missing terms of the sequence.

a) ■, ■, ■, 19, 23

b) ■, ■, 3, $\frac{3}{2}$

c) ■, 4, ■, ■, 10

5. Determine the position of the given term to complete the following statements.

a) 170 is the ■th term of $-4, 2, 8, \ldots$

b) -14 is the ■th term of $2\frac{1}{5}, 2, 1\frac{4}{5}, \ldots$

c) 97 is the ■th term of $-3, 1, 5, \ldots$

d) -10 is the ■th term of 14, 12.5, 11, ...

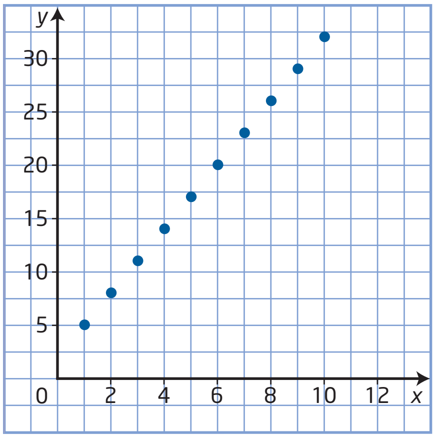
6. Determine the second and third terms of an arithmetic sequence if

a) the first term is 6 and the fourth term is 33

b) the first term is 8 and the fourth term is 41

c) the first term is 42 and the fourth term is 27

7. The graph of an arithmetic sequence is shown.



a) What are the first five terms of the sequence?

b) Write the general term of this sequence.

c) What is t_{50}? t_{200}?

d) Describe the relationship between the slope of the graph and your formula from part b).

e) Describe the relationship between the y-intercept and your formula from part b).

Apply

8. Which arithmetic sequence(s) contain the term 34? Justify your conclusions.

A $t_n = 6 + (n - 1)4$

B $t_n = 3n - 1$

C $t_1 = 12,\ d = 5.5$

D 3, 7, 11, ...

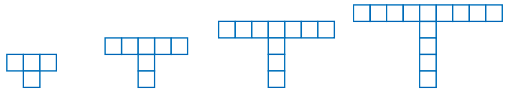
9. Determine the first term of the arithmetic sequence in which the 16th term is 110 and the common difference is 7.

10. The first term of an arithmetic sequence is $5y$ and the common difference is $-3y$. Write the equations for t_n and t_{15}.

11. The terms $5x + 2$, $7x - 4$, and $10x + 6$ are consecutive terms of an arithmetic sequence. Determine the value of x and state the three terms.

12. The numbers represented by x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y.

13. Each square in this pattern has a side length of 1 unit. Assume the pattern continues.



a) Write an equation in which the perimeter is a function of the figure number.

b) Determine the perimeter of Figure 9.

c) Which figure has a perimeter of 76 units?

14. The Wolf Creek Golf Course, located near Ponoka, Alberta, has been the site of the Canadian Tour Alberta Open Golf Championship. This tournament has a maximum entry of 132 players. The tee-off times begin at 8:00 and are 8 min apart.

a) The tee-off times generate an arithmetic sequence. Write the first four terms of the arithmetic sequence, if the first tee-off time of 8:00 is considered to be at time 0.

b) Following this schedule, how many players will be on the course after 1 h, if the tee-off times are for groups of four?

c) Write the general term for the sequence of tee-off times.

d) At what time will the last group tee-off?

e) What factors might affect the prearranged tee-off time?

Did You Know?

The first championship at Wolf Creek was held in 1987 and has attracted PGA professionals, including Mike Weir and Dave Barr.

15. Lucy Ango'yuaq, from Baker Lake, Nunavut, is a prominent wall hanging artist. This wall hanging is called *Geese and Ulus*. It is 22 inches wide and 27 inches long and was completed in 27 days. Suppose on the first day she completed 48 square inches of the wall hanging, and in the subsequent days the sequence of cumulative areas completed by the end of each day produces an arithmetic sequence. How much of the wall hanging did Lucy complete on each subsequent day? Express your answer in square inches.

The Inuktitut syllabics appearing at the bottom of this wall hanging spell the artist's name. For example, the first two syllabics spell out Lu-Si.

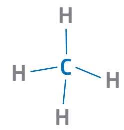
16. Susan joined a fitness class at her local gym. Into her workout, she incorporated a sit-up routine that followed an arithmetic sequence. On the 6th day of the program, Susan performed 11 sit-ups. On the 15th day she did 29 sit-ups.

a) Write the general term that relates the number of sit-ups to the number of days.

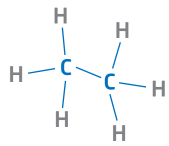
b) If Susan's goal is to be able to do 100 sit-ups, on which day of her program will she accomplish this?

c) What assumptions did you make to answer part b)?

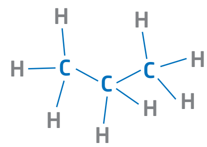
17. Hydrocarbons are the starting points in the formation of thousands of products, including fuels, plastics, and synthetic fibres. Some hydrocarbon compounds contain only carbon and hydrogen atoms. Alkanes are saturated hydrocarbons that have single carbon-to-carbon bonds. The diagrams below show the first three alkanes.



Methane



Ethane



Propane

a) The number of hydrogen atoms compared to number of carbon atoms produces an arithmetic sequence. Copy and complete the following chart to show this sequence.

Carbon Atoms	1	2	3	4
Hydrogen Atoms	4			

b) Write the general term that relates the number of hydrogen atoms to the number of carbon atoms.

c) Hectane contains 202 hydrogen atoms. How many carbon atoms are required to support 202 hydrogen atoms?

18. The multiples of 5 between 0 and 50 produces the arithmetic sequence 5, 10, 15, …, 45. Copy and complete the following table for the multiples of various numbers.

Multiples Of	28	7	15
Between	1 and 1000	500 and 600	50 and 500
First Term, t_1			
Common Difference, d			
nth Term, t_n			
General Term			
Number of Terms			

19. The beluga whale is one of the major attractions of the Vancouver Aquarium. The beluga whale typically forages for food at a depth of 1000 ft, but will dive to at least twice that depth. To build the aquarium for the whales, engineers had to understand the pressure of the water at such depths. At sea level the pressure is 14.7 psi (pounds per square inch). Water pressure increases at a rate of 14.7 psi for every 30 ft of descent.

a) Write the first four terms of the sequence that relates water pressure to feet of descent. Write the general term of this sequence.

b) What is the water pressure at a depth of 1000 ft? 2000 ft?

c) Sketch a graph of the water pressure versus 30-ft water depth charges.

d) What is the y-intercept of the graph?

e) What is the slope of the graph?

f) How do the y-intercept and the slope relate to the formula you wrote in part a)?

Did You Know?

In 2008, the beluga was listed on the near-threatened list by the International Union for the Conservation of Nature.

20. The side lengths of a quadrilateral produce an arithmetic sequence. If the longest side has a length of 24 cm and the perimeter is 60 cm, what are the other side lengths? Explain your reasoning.

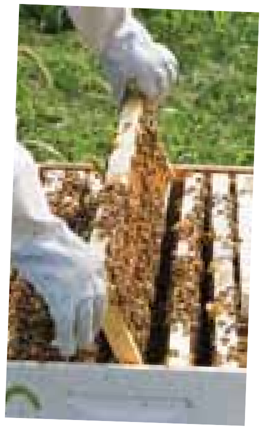
21. Earth has a daily rotation of 360°. One degree of rotation requires 4 min.

a) Write the sequence of the first five terms relating the number of minutes to the number of degrees of rotation.

b) Write an equation that describes this sequence.

c) Determine the time taken for a rotation of 80°.

22. Canadian honey is recognized around the world for its superior taste and quality. In Saskatchewan in 1986, there were 1657 beekeepers operating 105 000 colonies. Each colony produced approximately 70 kg of honey. In 2007, the number of beekeepers was reduced to 1048. Assume that the decline in the number of beekeepers generates an arithmetic sequence. Determine the change in the number of beekeepers each year from 1986 to 2007.

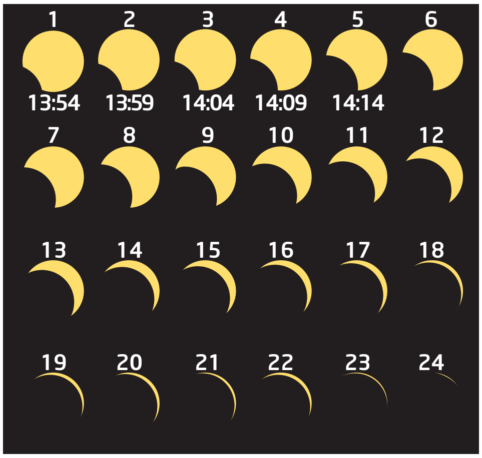


23. The Diavik Diamond Mine is located on East Island in Lac de Gras East, Northwest Territories. The diamonds that are extracted from the mine were brought to surface when the kimberlite rock erupted 55 million years ago. In 2003, the first production year of the mine, 3.8 million carats were produced. Suppose the life expectancy of the mine is 20 years, and the number of diamond carats expected to be extracted from the mine after 20 years is 113.2 million carats. If the extraction of diamonds produces an arithmetic sequence, determine the common difference. What does this value represent?

Extend

24. Farmers near Raymond, Alberta, use a wheel line irrigation system to provide water to their crops. A pipe and sprinkler system is attached to a motor-driven wheel that moves the system in a circle over a field. The first wheel is attached 50 m from the pivot point, and all the other wheels are attached at 20-m intervals further along the pipe. Determine the circumference of the circle traversed by wheel 12.

25. A solar eclipse is considered to be one of the most awe-inspiring spectacles in all of nature. The total phase of a solar eclipse is very brief and rarely lasts more than several minutes. The diagram below shows a series of pictures taken of a solar eclipse similar to the one that passed over Nunavut on August 1, 2008.



a) Write the first five terms of the sequence that relates the time to the picture number. State the values of t_1 and d.

b) Write the general term that defines this sequence.

c) What assumptions did you make for your calculation in part b)?

d) At what time was the sun completely eclipsed by the moon?

Web Link

To learn more about a solar eclipse, go to www.mhrprecalc11.ca and follow the links.

Create Connections

26. Copy and complete the following sentences using your own words. Then, choose symbols from the box below to create true statements. The boxes to the right of each sentence indicate how many symbols are needed for that sentence.

t_1	n	t_n	$<$	d	$\leq$	$\geq$	$>$	$\neq$	0	1	$=$

a) An arithmetic sequence is an increasing sequence if and only if ■ ■ ■.

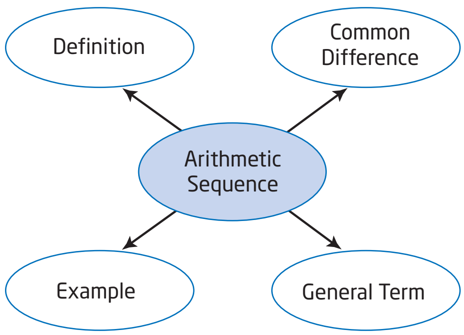
b) An arithmetic sequence is a decreasing sequence if and only if ■ ■ ■.

c) An arithmetic sequence is constant if and only if ■ ■ ■.

d) The first term of a sequence is ■.

e) The symbol for the general term of a sequence is ■.

27. Copy and complete the following graphic organizer by recording the observations you made about an arithmetic sequence. For example, include such things as the common difference and how the sequence relates to a function. Compare your graphic organizer with that of a classmate.



28. **MINI LAB**

Step 1 Create or use a spreadsheet as shown below.

What is the shape of the graph that models the arithmetic sequence? Could an arithmetic sequence graph have any other shape? Explain.

Step 2 Explore the effect that the first term has on the terms of the sequence by changing the value in Cell C2.

a) As the value of the first term increases, what is the effect on the graph? What happens as the value of the first term decreases?

b) Does the graph keep its shape? What characteristics of the graph stay the same?

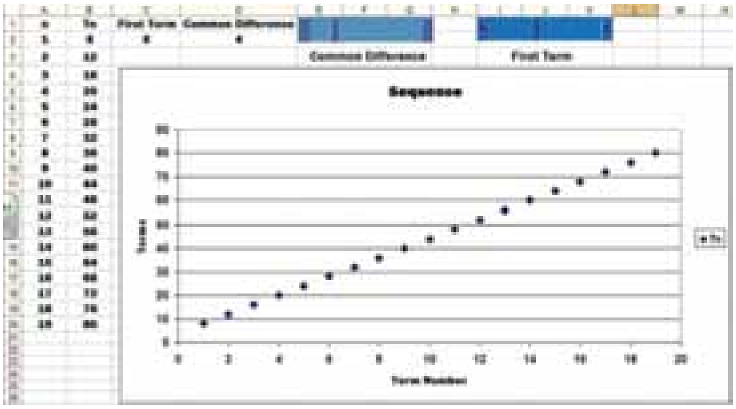
Step 3 Investigate the effect of the common difference on an arithmetic sequence by changing the values in Cell D2.

a) What effect does changing this value have on the graph?

b) How does an increase in the common difference affect the shape of the graph? What happens as the common difference decreases?

Step 4 If a line were drawn through the data values, what would be its slope?

Step 5 What relationship does the slope of the line have to the equation for the general term of the sequence?



Project Corner — Minerals

- A telephone contains over 40 different minerals, a television set has about 35, and an automobile about 15.
- Of the approximately 193 000 metric tonnes of gold discovered, 62% is found in just four countries on Earth. All the gold discovered so far would fit in a cube of side length 22 m.
- Of the approximately 1 740 000 metric tonnes of silver discovered, 55% is found in just four countries on Earth. All the silver discovered so far would fit in a cube of side length 55 m.
- In the average 1360-kg car there are approximately 110 kg of aluminum, 20 kg of copper, 10 kg of zinc, 113 kg of plastics, and 64 kg of rubber.
- Canada is the world's largest potash producer.

1.2

Arithmetic Series

Focus on…

- deriving a rule for determining the sum of an arithmetic series
- determining the values of t_1, d, n, or S_n in an arithmetic series
- solving a problem that involves an arithmetic series

Carl Friedrich Gauss was a mathematician born in Braunschweig, Germany, in 1777. He is noted for his significant contributions in fields such as number theory, statistics, astronomy, and differential geometry. When Gauss was 10, his mathematics teacher challenged the class to find the sum of the numbers from 1 to 100. Believing that this task would take some time, the teacher was astounded when Gauss responded with the correct answer of 5050 within minutes.

Gauss used a faster method than adding each individual term.

First, he wrote the sum twice, once in ascending order and the other in a descending order. Gauss then took the sum of the two rows.

$$\begin{array}{r} 1 + \ \ 2 + \ \ 3 + 4 + \cdots + \ \ 99 + 100 \\ 100 + \ \ 99 + \cdots + 4 + 3 \ + \ \ \ 2 + \ \ \ 1 \\ \hline 101 + 101 + \qquad \cdots \qquad + 101 + 101 \end{array}$$

What do you think Gauss did next?

Web Link

To learn more about Carl Gauss, go to www.mhrprecalc11.ca and follow the links.

Investigate Arithmetic Series

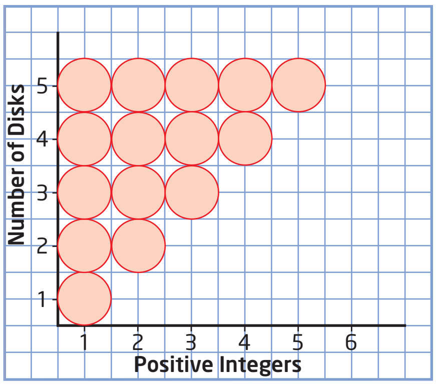
Materials

- 30 counting disks
- grid paper

In the following investigation, work with a partner to discuss your findings.

Part A: Explore Gauss's Method

1. a) Consider the sequence of positive integers 1, 2, 3, 4, 5. Represent each number by a small counting disk and arrange them in a triangular table in which each row represents the integer.



b) What is the sum of the numbers in the sequence?

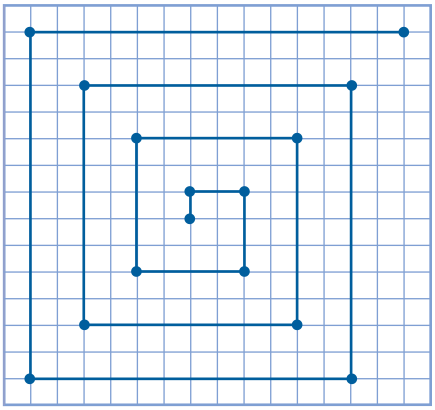
c) How is the sum related to the total number of disks used?

2. a) Duplicate the triangle of disks.
   b) Rotate your new triangle 180°. Join the two triangles together to form a rectangle.
   c) How many disks form the length of the rectangle? the width?
   d) How many disks are in the area of the rectangle?
   e) How is the area of the rectangle related to the sum of the sequence 1, 2, 3, 4, 5?

Reflect and Respond

3. Explain how you could use the results from steps 1 and 2 to find the sum of n consecutive integers. Use the idea of the area of the rectangle to develop a formula that would find the sum of n consecutive integers.
4. Explain how this method is related to the method Gauss used.

Part B: Construct a Squared Spiral



5. Start from the centre of the grid.
   a) Draw a segment of length 1 unit, vertically up.
   b) From the end of that segment, draw a new segment that is 1 unit longer than the previous segment, to the right.
   c) From the end of this segment, draw a new segment that is 1 unit longer than the previous segment, vertically down.
   d) Continue this through 14 segments.
6. a) Record the lengths of the segments as an arithmetic sequence.
   b) What is the total length of the spiral?
   c) Explain how you calculated the total length of the spiral.

Reflect and Respond

7. a) Use Gauss's method to calculate the sum of the first 14 terms of your sequence. Is the sum the same as your sum from step 6?
   b) In using Gauss's method, what sum did you find for each pair of numbers? How many terms were there?
8. Derive a formula that you could use to find the total length if there were 20 segments for the spiral.

Link the Ideas

arithmetic series

- a sum of terms that form an arithmetic sequence
- for the arithmetic sequence 2, 4, 6, 8, the arithmetic series is represented by 2 + 4 + 6 + 8.

In determining the sum of the numbers from 1 to 100, Gauss had discovered the underlying principles of an **arithmetic series**.

S_n represents the sum of the first n terms of a series.

S_n is read as "S subscript n" or "S sub n."

In the series $2 + 4 + 6 + 8 + \cdots$, S_4 is the sum of the first four terms.

You can use Gauss's method to derive a formula for the sum of the general arithmetic series.

The general arithmetic series may be written as
$t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + [(t_1 + (n - 3)d] + [(t_1 + (n - 2)d]$
$+ [(t_1 + (n - 1)d]$

For this series, t_1 is the first term
n is the number of terms
d is the common difference

Use Gauss's method.
Write the series twice, once in ascending order and the other in descending order. Then, sum the two series.

$$\begin{array}{rcccccccccc}
S_n & = & t_1 & + & (t_1 + d) & + \ldots + & [t_1 + (n-2)d] & + & [t_1 + (n-1)d] \\
S_n & = & [t_1 + (n-1)d] & + & [t_1 + (n-2)d] & + \ldots + & (t_1 + d) & + & t_1 \\
\hline
2S_n & = & [2t_1 + (n-1)d] & + & [2t_1 + (n-1)d] & + \ldots + & [2t_1 + (n-1)d] & + & [2t_1 + (n-1)d]
\end{array}$$

$$2S_n = n[2t_1 + (n - 1)d]$$
$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$

The *sum* of an arithmetic series can be determined using the formula
$\boldsymbol{S_n = \frac{n}{2}[2t_1 + (n - 1)d]}$
where t_1 is the first term
n is the number of terms
d is the common difference
S_n is the sum of the first n terms

A variation of this general formula can be derived by substituting t_n for the formula for the general term of an arithmetic sequence.

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$
$$S_n = \frac{n}{2}[t_1 + t_1 + (n - 1)d]$$

Since $t_n = t_1 + (n - 1)d$.

$$S_n = \frac{n}{2}(t_1 + t_n)$$

The *sum* of an arithmetic series can be determined using the formula
$\boldsymbol{S_n = \frac{n}{2}(t_1 + t_n)}$
where t_1 is the first term
n is the number of terms
t_n is the nth term
S_n is the sum of the first n terms

How would you need to express the last terms of the general arithmetic series in order to directly derive this formula using Gauss's method?

Example 1

Determine the Sum of an Arithmetic Series

Male fireflies flash in various patterns to signal location or to ward off predators. Different species of fireflies have different flash characteristics, such as the intensity of the flash, the rate of the flash, and the shape of the flash. Suppose that under certain circumstances, a particular firefly flashes twice in the first minute, four times in the second minute, and six times in the third minute.

a) If this pattern continues, what is the number of flashes in the 30th minute?

b) What is the total number of flashes in 30 min?

Solution

a) Method 1: Use Logical Reasoning

The firefly flashes twice in the first minute, four times in the second minute, six times in the third minute, and so on.

The arithmetic sequence produced by the number of flashes is 2, 4, 6, ….

Since the common difference in this sequence is 2, the number of flashes in the 30th minute is the 30th multiple of 2.

$30 \times 2 = 60$

The number of flashes in the 30th minute is 60.

Method 2: Use the General Term

For this arithmetic sequence,

First term	$t_1 = 2$
Common difference	$d = 2$
Number of terms	$n = 30$

Substitute these values into the formula for the general term.

$$\begin{aligned} t_n &= t_1 + (n - 1)d \\ t_{30} &= 2 + (30 - 1)2 \\ t_{30} &= 2 + (29)2 \\ t_{30} &= 60 \end{aligned}$$

The number of flashes in the 30th minute is 60.

b) Method 1: Use the Formula $S_n = \frac{n}{2}(t_1 + t_n)$

What information do you need to use this formula?

$$\begin{aligned} S_n &= \frac{n}{2}(t_1 + t_n) \\ S_{30} &= \frac{30}{2}(2 + 60) \\ S_{30} &= 15(62) \\ S_{30} &= 930 \end{aligned}$$

Substitute the values of n, t_1, and t_n.

Method 2: Use the Formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

What information do you need to use this formula?

$$\begin{aligned} S_n &= \frac{n}{2}[2t_1 + (n - 1)d] \\ S_{30} &= \frac{30}{2}[2(2) + (30 - 1)(2)] \\ S_{30} &= 15(62) \\ S_{30} &= 930 \end{aligned}$$

Substitute the values of n, t_1, and t_n.

The total number of flashes for the male firefly in one hour is 930.

Which formula is most effective in this case? Why?

Your Turn

Determine the total number of flashes for the male firefly in 42 min.

Example 2

Determine the Terms of an Arithmetic Series

The sum of the first two terms of an arithmetic series is 13 and the sum of the first four terms is 46. Determine the first six terms of the series and the sum to six terms.

Solution

For this series,

$S_2 = 13$

$S_4 = 46$

Substitute into the formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$ for both sums.

For S_2:

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$
$$S_2 = \frac{2}{2}[2t_1 + (2 - 1)d]$$
$$13 = 1[2t_1 + (1)d]$$
$$13 = 2t_1 + d$$

For S_4:

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$
$$S_4 = \frac{4}{2}[2t_1 + (4 - 1)d]$$
$$46 = 2[2t_1 + (3)d]$$
$$23 = 2t_1 + 3d$$

Solve the system of two equations.

$$\begin{aligned} 13 &= 2t_1 + d && ① \\ 23 &= 2t_1 + 3d && ② \\ \hline -10 &= -2d && ① - ② \\ 5 &= d \end{aligned}$$

Substitute $d = 5$ into one of the equations.

$$\begin{aligned} 13 &= 2t_1 + d \\ 13 &= 2t_1 + 5 \\ 8 &= 2t_1 \\ 4 &= t_1 \end{aligned}$$

With $t_1 = 4$ and $d = 5$, the first six terms of the series are $4 + 9 + 14 + 19 + 24 + 29$.

The sum of the first six terms is

$$\begin{aligned} S_n &= \frac{n}{2}[2t_1 + (n - 1)d] \\ S_6 &= \frac{6}{2}[2(4) + (6 - 1)5] \\ S_6 &= 3(8 + 25) \\ S_6 &= 99 \end{aligned}$$

or

$$\begin{aligned} S_n &= \frac{n}{2}(t_1 + t_n) \\ S_6 &= \frac{6}{2}(4 + 29) \\ S_6 &= 3(33) \\ S_6 &= 99 \end{aligned}$$

Which formula do you prefer to use? Why?

Your Turn

The sum of the first two terms of an arithmetic series is 19 and the sum of the first four terms is 50. What are the first six terms of the series and the sum to 20 terms?

Key Ideas

- Given the sequence $t_1, t_2, t_3, t_4, \ldots, t_n$ the associated series is $S_n = t_1 + t_2 + t_3 + t_4 + \cdots + t_n$.
- For the general arithmetic series,
  $t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + (t_1 + [n - 1]d)$ or
  $t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + (t_n - d) + t_n$,
  the sum of the first n terms is
  $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(t_1 + t_n)$,
  where t_1 is the first term
  n is number of terms
  d is the common difference
  t_n is the nth term
  S_n is the sum to n terms

Check Your Understanding

Practise

1. Determine the sum of each arithmetic series.

a) $5 + 8 + 11 + \cdots + 53$

b) $7 + 14 + 21 + \cdots + 98$

c) $8 + 3 + (-2) + \cdots + (-102)$

d) $\frac{2}{3} + \frac{5}{3} + \frac{8}{3} + \cdots + \frac{41}{3}$

2. For each of the following arithmetic series, determine the values of t_1 and d, and the value of S_n to the indicated sum.

a) $1 + 3 + 5 + \cdots (S_8)$

b) $40 + 35 + 30 + \cdots (S_{11})$

c) $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots (S_7)$

d) $(-3.5) + (-1.25) + 1 + \cdots (S_6)$

3. Determine the sum, S_n, for each arithmetic sequence described.

a) $t_1 = 7$, $t_n = 79$, $n = 8$

b) $t_1 = 58$, $t_n = -7$, $n = 26$

c) $t_1 = -12$, $t_n = 51$, $n = 10$

d) $t_1 = 12$, $d = 8$, $n = 9$

e) $t_1 = 42$, $d = -5$, $n = 14$

4. Determine the value of the first term, t_1, for each arithmetic series described.

a) $d = 6$, $S_n = 574$, $n = 14$

b) $d = -6$, $S_n = 32$, $n = 13$

c) $d = 0.5$, $S_n = 218.5$, $n = 23$

d) $d = -3$, $S_n = 279$, $n = 18$

5. For the arithmetic series, determine the value of n.

a) $t_1 = 8$, $t_n = 68$, $S_n = 608$

b) $t_1 = -6$, $t_n = 21$, $S_n = 75$

6. For each series find t_{10} and S_{10}.

a) $5 + 10 + 15 + \cdots$

b) $10 + 7 + 4 + \cdots$

c) $(-10) + (-14) + (-18) + \cdots$

d) $2.5 + 3 + 3.5 + \cdots$

Apply

7. a) Determine the sum of all the multiples of 4 between 1 and 999.

b) What is the sum of the multiples of 6 between 6 and 999?

8. It's About Time, in Langley, British Columbia, is Canada's largest custom clock manufacturer. They have a grandfather clock that, on the hours, chimes the number of times that corresponds to the time of day. For example, at 4:00 p.m., it chimes 4 times. How many times does the clock chime in a 24-h period?

9. A training program requires a pilot to fly circuits of an airfield. Each day, the pilot flies three more circuits than the previous day. On the fifth day, the pilot flew 14 circuits. How many circuits did the pilot fly

a) on the first day?

b) in total by the end of the fifth day?

c) in total by the end of the nth day?

10. The second and fifth terms of an arithmetic series are 40 and 121, respectively. Determine the sum of the first 25 terms of the series.

11. The sum of the first five terms of an arithmetic series is 85. The sum of the first six terms is 123. What are the first four terms of the series?

12. Galileo noticed a relationship between the distance travelled by a falling object and time. Suppose data show that when an object is dropped from a particular height it moves approximately 5 m during the first second of its fall, 15 m during the second second, 25 m during the third second, 35 during the fourth second, and so on. The formula describing the approximate distance, d, the object is from its starting position n seconds after it has been dropped is $d(n) = 5n^2$.

a) Using the general formula for the sum of a series, derive the formula $d(n) = 5n^2$.

b) Demonstrate algebraically, using $n = 100$, that the sum of the series $5 + 15 + 25 + \cdots$ is equivalent to $d(n) = 5n^2$.

13. At the sixth annual Vancouver Canstruction® Competition, architects and engineers competed to see whose team could build the most spectacular structure using little more than cans of food.

A Breach in Hunger

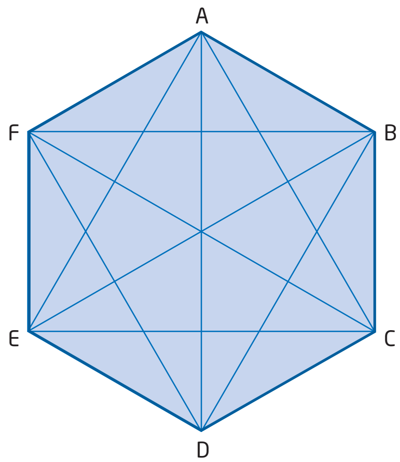
The UnBEARable Truth

Stores often stack cans for display purposes, although their designs are not usually as elaborate as the ones shown above. To calculate the number of cans in a display, an arithmetic series may be used. Suppose a store wishes to stack the cans in a pattern similar to the one shown. This display has one can at the top and each row thereafter adds one can. If there are 18 rows, how many cans in total are there in the display?

Did You Know?

The Vancouver Canstruction® Competition aids in the fight against hunger. At the end of the competition, all canned food is donated to food banks.

14. The number of handshakes between 6 people where everyone shakes hands with everyone else only once may be modelled using a hexagon. If you join each of the 6 vertices in the hexagon to every other point in the hexagon, there are 1 + 2 + 3 + 4 + 5 lines. Therefore, there are 15 lines.



a) What does the series 1 + 2 + 3 + 4 + 5 represent?

b) Write the series if there are 10 people in the room and everyone shakes hands with everyone else in the room once.

c) How many handshakes occur in a room of 30 people?

d) Describe a similar situation in which this method of determining the number of handshakes may apply.

15. The first three terms of an arithmetic sequence are given by x, $(2x - 5)$, 8.6.

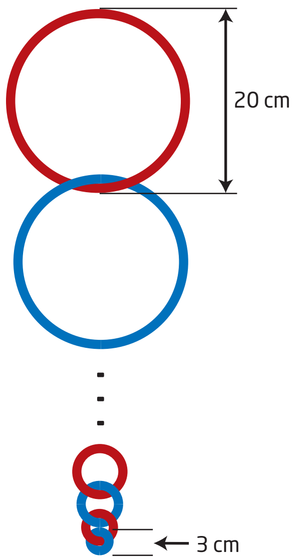
a) Determine the first term and the common difference for the sequence.

b) Determine the 20th term of the sequence.

c) Determine the sum of the first 20 terms of the series.

Extend

16. A number of interlocking rings each 1 cm thick are hanging from a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance from the top of the top ring to the bottom of the bottom ring?



17. Answer the following as either true or false. Justify your answers.

a) Doubling each term in an arithmetic series will double the sum of the series.

b) Keeping the first term constant and doubling the number of terms will double the sum of the series.

c) If each term of an arithmetic sequence is multiplied by a fixed number, the resulting sequence will always be an arithmetic sequence.

18. The sum of the first n terms of an arithmetic series is $S_n = 2n^2 + 5n$.

a) Determine the first three terms of this series.

b) Determine the sum of the first 10 terms of the series using the arithmetic sum formula.

c) Determine the sum of the first 10 terms of the series using the given formula.

d) Using the general formula for the sum of an arithmetic series, show how the formulas in parts b) and c) are equal.

19. Nathan Gerelus is a Manitoba farmer preparing to harvest his field of wheat. Nathan begins harvesting the crop at 11:00 a.m., after the morning dew has evaporated. By the end of the first hour he harvests 240 bushels of wheat. Nathan challenges himself to increase the number of bushels harvested by the end of each hour. Suppose that this increase produces an arithmetic series where Nathan harvests 250 bushels in the second hour, 260 bushels in the third hour, and so on.

a) Write the series that would illustrate the amount of wheat that Nathan has harvested by the end of the seventh hour.

b) Write the general sum formula that represents the number of bushels of wheat that Nathan took off the field by the end of the nth hour.

c) Determine the total number of bushels harvested by the end of the seventh hour.

d) State any assumptions that you made.

20. The 15th term in an arithmetic sequence is 43 and the sum of the first 15 terms of the series is 120. Determine the first three terms of the series.

Create Connections

21. An arithmetic series was defined where $t_1 = 12$, $n = 16$, $d = 6$, and $t_n = 102$. Two students were asked to determine the sum of the series. Their solutions are shown below.

Pierre's solution:

$$S_n = \frac{n}{2}(t_1 + t_n)$$
$$S_{16} = \frac{16}{2}(12 + 102)$$
$$S_{16} = 8(114)$$
$$S_{16} = 912$$

Jeanette's solution:

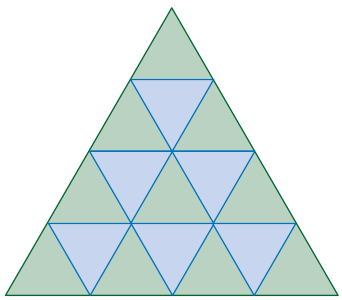
$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$
$$S_{16} = \frac{16}{2}[2(12) + (16 - 1)6]$$
$$S_{16} = 8(24 + 90)$$
$$S_{16} = 912$$

Both students arrived at a correct answer. Explain how both formulas lead to the correct answer.

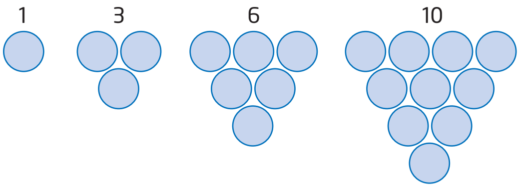
22. The triangular arrangement shown consists of a number of unit triangles. A unit triangle has side lengths equal to 1. The series for the total number of unit triangles in the diagram is 1 + 3 + 5 + 7.

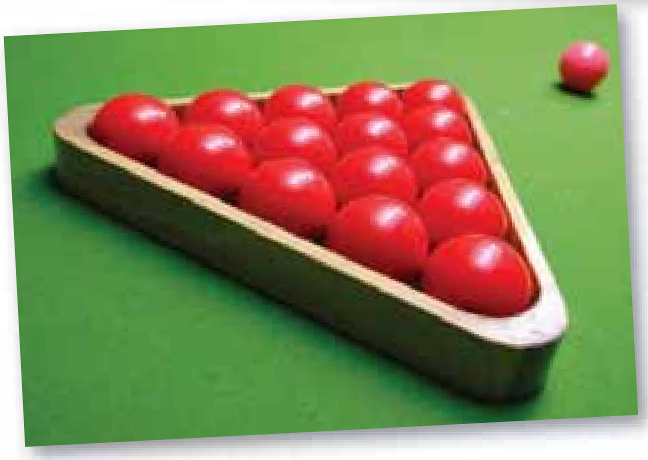
a) How many unit triangles are there if there are 10 rows in the triangular arrangement?

b) Using the sum of a series, show how the sum of the blue unit triangles plus the sum of the green unit triangles results in your answer from part a).



23. Bowling pins and snooker balls are often arranged in a triangular formation. A triangular number is a number that can be represented by a triangular array of dots. Each triangular number is an arithmetic series. The sequence 1, (1 + 2), (1 + 2 + 3), (1 + 2 + 3 + 4), ... gives the first four triangular numbers as 1, 3, 6, and 10.





a) What is the tenth triangular number?

b) Use the general formula for the sum of an arithmetic series to show that the nth triangular number is $\frac{n}{2}(n + 1)$.

Project Corner **Diamond Mining**

- In 1991, the first economic diamond deposit was discovered in the Lac de Gras area of the Northwest Territories. In October 1998, Ekati diamond mine opened about 300 km northeast of Yellowknife.
- By April 1999, the mine had produced one million carats. Ekati's average production over its projected 20-year life is expected to be 3 to 5 million carats per year.
- Diavik, Canada's second diamond mine, began production in January 2003. During its projected 20-year life, average diamond production from this mine is expected to be about 8 million carats per year, which represents about 6% of the world's total supply.

A polar bear diamond is a certified Canadian diamond mined, cut, and polished in Yellowknife.

1.3

Geometric Sequences

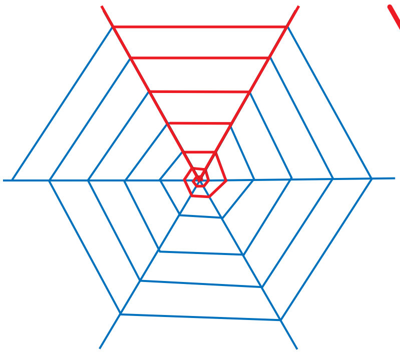
Focus on...

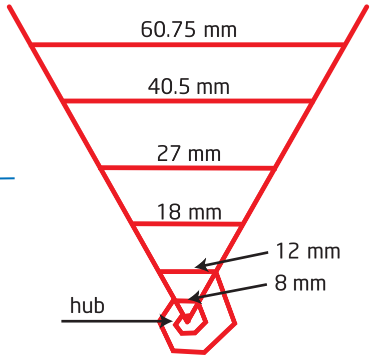
- providing and justifying an example of a geometric sequence
- deriving a rule for determining the general term of a geometric sequence
- solving a problem that involves a geometric sequence

Many types of sequences can be found in nature. The Fibonacci sequence, frequently found in flowers, seeds, and trees, is one example. A **geometric sequence** can be approximated by the orb web of the common garden spider. A spider's orb web is an impressive architectural feat. The web can capture the beauty of the morning dew, as well as the insects that the spider may feed upon. The following graphic was created to represent an approximation of the geometric sequence formed by the orb web.

geometric sequence

- a sequence in which the ratio of consecutive terms is constant





The capture spiral is constructed by the spider starting on the outside edge of the web frame, and winding inward toward the hub. The lengths of the sections of the silk between the radii for this section of the spiral produce a geometric sequence. What makes this sequence geometric?

Did You Know?

An orb web is a round spider web with a pattern of lines in a spiral formation.

Investigate a Geometric Sequence

Coin Toss Outcomes

Materials
- 3 coins

Work with a partner for the following activity.

1. a) Toss a single coin. How many possible outcomes are there?

b) Toss two coins. How many possible outcomes are there?

c) Create a tree diagram to show the possible outcomes for three coins.

2. Copy the table. Continue the pattern to complete the table.

Number of Coins, n	Number of Outcomes, t_n	Expanded Form	Using Exponents
1	2	(2)	2^1
2	4	(2)(2)	2^2
3			
4			
⋮	⋮	⋮	⋮
n			

3. a) As the number of coins increases, a sequence is formed by the number of outcomes. What are the first four terms of this sequence?

b) Describe how the terms of the sequence are related. Is this relationship different from an arithmetic sequence? Explain.

c) Predict the next two terms of the sequence. Describe the method you used to make your prediction.

d) Describe a method you could use to generate one term from the previous term.

4. a) For several pairs of consecutive terms in the sequence, divide the second term by the preceding term.

b) What observation can you make about your predictions in step 3c)?

Reflect and Respond

5. a) Is the sequence generated a geometric sequence? How do you know?

b) Write a general term that relates the number of outcomes to the number of coins tossed.

c) Show how to use your formula to determine the value of the 20th term of the sequence.

Link the Ideas

common ratio

- the ratio of successive terms in a geometric sequence, $r = \frac{t_n}{t_{n-1}}$
- the ratio may be positive or negative
- for example, in the sequence 2, 4, 8, 16, …, the common ratio is 2

In a geometric sequence, the ratio of consecutive terms is constant. The **common ratio**, r, can be found by taking any term, except the first, and dividing that term by the preceding term.

The general geometric sequence is $t_1, t_1r, t_1r^2, t_1r^3, \ldots$, where t_1 is the first term and r is the common ratio.

$$
\begin{aligned}
t_1 &= t_1\\
t_2 &= t_1r\\
t_3 &= t_1r^2\\
t_4 &= t_1r^3\\
&\vdots\\
t_n &= t_1r^{n-1}
\end{aligned}
$$

The *general term* of a geometric sequence where n is a positive integer is

$$\boldsymbol{t_n = t_1r^{n-1}}$$

where t_1 is the first term of the sequence
n is the number of terms
r is the common ratio
t_n is the general term or nth term

Example 1

Determine t_1, r, and t_n

In nature, many single-celled organisms, such as bacteria, reproduce by splitting in two so that one cell gives rise to 2, then 4, then 8 cells, and so on, producing a geometric sequence. Suppose there were 10 bacteria originally present in a bacteria sample. Determine the general term that relates the number of bacteria to the doubling period of the bacteria. State the values for t_1 and r in the geometric sequence produced.

Did You Know?

One of the most common bacteria on Earth, Shewanella oneidensis MR-1, uses oxygen as an energy source for respiration. This bacterium is generally associated with the removal of metal pollutants in aquatic and marine environments.

Solution

State the sequence generated by the doubling of the bacteria.

$$
\begin{aligned}
t_1 &= 10\\
t_2 &= 20\\
t_3 &= 40\\
t_4 &= 80\\
t_5 &= 160\\
&\vdots
\end{aligned}
$$

The common ratio, r, may be found by dividing any two consecutive terms, $r = \frac{t_n}{t_{n-1}}$.

$$\frac{20}{10} = 2 \quad \frac{40}{20} = 2 \quad \frac{80}{40} = 2 \quad \frac{160}{80} = 2$$

The common ratio is 2.

For the given sequence, $t_1 = 10$ and $r = 2$. Use the general term of a geometric sequence.

$t_n = t_1 r^{n-1}$

$t_n = (10)(2)^{n-1}$ Substitute known values.

The general term of the sequence is $t_n = 10(2)^{n-1}$.

Your Turn

Suppose there were three bacteria originally present in a sample. Determine the general term that relates the number of bacteria to the doubling period of the bacteria. State the values of t_1 and r in the geometric sequence formed.

Example 2

Determine a Particular Term

Sometimes you use a photocopier to create enlargements or reductions. Suppose the actual length of a photograph is 25 cm and the smallest size that a copier can make is 67% of the original. What is the shortest possible length of the photograph after 5 reductions? Express your answer to the nearest tenth of a centimetre.

Solution

This situation can be modelled by a geometric sequence.

For this sequence,

First term	$t_1 = 25$
Common ratio	$r = 0.67$
Number of terms	$n = 6$

Why is the number of terms 6 in this case?

You need to find the sixth term of the sequence.

Use the general term, $t_n = t_1 r^{n-1}$.

$t_n = t_1 r^{n-1}$

$t_6 = 25(0.67)^{6-1}$ Substitute known values.

$t_6 = 25(0.67)^5$

$t_6 = 3.375\ldots$

After five reductions, the shortest possible length of the photograph is approximately 3.4 cm.

Your Turn

Suppose the smallest reduction a photocopier could make is 60% of the original. What is the shortest possible length after 8 reductions of a photograph that is originally 42 cm long?

Example 3

Determine t_1 and r

In a geometric sequence, the third term is 54 and the sixth term is -1458. Determine the values of t_1 and r, and list the first three terms of the sequence.

Solution

Method 1: Use Logical Reasoning

The third term of the sequence is 54 and the sixth term is -1458.

$t_3 = 54$
$t_6 = -1458$

Since the sequence is geometric,

$$\begin{aligned} t_4 &= t_3(r) \\ t_5 &= t_3(r)(r) \\ t_6 &= t_3(r)(r)(r) \\ -1458 &= 54r^3 \\ \frac{-1458}{54} &= r^3 \\ -27 &= r^3 \\ \sqrt[3]{-27} &= r \\ -3 &= r \end{aligned}$$

Substitute known values.

You can use the general term of a geometric sequence to determine the value for t_1.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ t_3 &= t_1r^{3-1} \\ t_3 &= t_1r^2 \\ 54 &= t_1(-3)^2 \\ 54 &= 9t_1 \\ 6 &= t_1 \end{aligned}$$

Substitute known values.

The first term of the sequence is 6 and the common ratio is -3.
The first three terms of the sequence are 6, -18, 54.

Method 2: Use the General Term

You can write an equation for t_3 and an equation for t_6 using the general term of a geometric sequence.

$t_n = t_1r^{n-1}$

For the third term, $n = 3$.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ 54 &= t_1r^{3-1} \\ 54 &= t_1r^2 \end{aligned}$$

For the sixth term, $n = 6$.

$$\begin{aligned} t_n &= t_1r^{n-1} \\ -1458 &= t_1r^{6-1} \\ -1458 &= t_1r^5 \end{aligned}$$

Solve one of the equations for the variable t_1.

$$54 = t_1r^2$$
$$\frac{54}{r^2} = t_1$$

Substitute this expression for t_1 in the other equation. Solve for the variable r.

$$-1458 = t_1r^5$$
$$-1458 = \left(\frac{54}{r^2}\right)r^5$$
$$-1458 = 54r^3$$
$$\frac{-1458}{54} = \frac{54r^3}{54}$$
$$-27 = r^3$$
$$\sqrt[3]{-27} = r$$
$$-3 = r$$

Substitute the common ratio of -3 in one of the equations to solve for the first term, t_1.

Substitute $r = -3$

$$54 = t_1r^2$$
$$54 = t_1(-3)^2$$
$$54 = 9t_1$$
$$6 = t_1$$

The first term of the sequence is 6 and the common ratio is -3.

The first three terms of the sequence are 6, -18, 54.

Your Turn

In a geometric sequence, the second term is 28 and the fifth term is 1792. Determine the values of t_1 and r, and list the first three terms of the sequence.

Example 4

Apply Geometric Sequences

The modern piano has 88 keys. The frequency of the notes ranges from A_0, the lowest note, at 27.5 Hz, to C_7, the highest note on the piano, at 4186.009 Hz. The frequencies of these notes approximate a geometric sequence as you move up the keyboard.

a) Determine the common ratio of the geometric sequence produced from the lowest key, A_0, to the fourth key, C_0, at 32.7 Hz.

b) Use the lowest and highest frequencies to verify the common ratio found in part a).

Did You Know?

A sound has two characteristics, pitch and volume. The pitch corresponds to the frequency of the sound wave. High notes have high frequencies. Low notes have low frequencies. Frequency is measured in Hertz (Hz), which is the number of waves per second.

Solution

a) The situation may be modelled by a geometric sequence.
For this sequence,

First term $t_1 = 27.5$
Number of terms $n = 4$
nth term $t_n = 32.7$

Use the general term of a geometric sequence.

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ 32.7 &= (27.5)(r^{4-1}) && \text{Substitute known values.} \\ \frac{32.7}{27.5} &= \frac{27.5r^3}{27.5} \\ \frac{32.7}{27.5} &= r^3 \\ \sqrt[3]{\frac{32.7}{27.5}} &= r && \text{Take the cube root of both sides.} \\ 1.0594\ldots &= r \end{aligned}$$

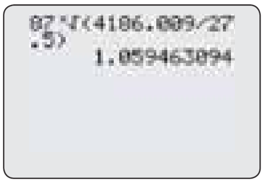
The common ratio for this sequence is approximately 1.06.

b) For this sequence,

First term $t_1 = 27.5$
Number of terms $n = 88$
nth term $t_n = 4186.009$

Use the general term of a geometric sequence.

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ 4186.009 &= (27.5)(r^{88-1}) && \text{Substitute known values.} \\ \frac{4186.009}{27.5} &= \frac{27.5r^{87}}{27.5} \\ \frac{4186.009}{27.5} &= r^{87} \\ \sqrt[87]{\frac{4186.009}{27.5}} &= r && \text{Take the 87th root of both sides.} \\ 1.0594\ldots &= r \end{aligned}$$



The common ratio of this sequence is approximately 1.06.

Your Turn

In 1990 the population of Canada was approximately 26.6 million. The population projection for 2025 is approximately 38.4 million. If this projection were based on a geometric sequence, what would be the annual growth rate? Given that this is a geometric sequence what assumptions would you have to make?

Key Ideas

- A geometric sequence is a sequence in which each term, after the first term, is found by multiplying the previous term by a non-zero constant, r, called the common ratio.
- The common ratio of successive terms of a geometric sequence can be found by dividing any two consecutive terms, $r = \frac{t_n}{t_{n-1}}$.
- The general term of a geometric sequence is
  $t_n = t_1 r^{n-1}$
  where t_1 is the first term
  n is the number of terms
  r is the common ratio
  t_n is the general term or nth term

Check Your Understanding

Practise

1. Determine if the sequence is geometric. If it is, state the common ratio and the general term in the form $t_n = t_1 r^{n-1}$.

a) 1, 2, 4, 8, ...

b) 2, 4, 6, 8, ...

c) 3, −9, 27, −81, ...

d) 1, 1, 2, 4, 8, ...

e) 10, 15, 22.5, 33.75, ...

f) −1, −5, −25, −125, ...

2. Copy and complete the following table for the given geometric sequences.

	Geometric Sequence	Common Ratio	6th Term	10th Term
a)	6, 18, 54, ...			
b)	1.28, 0.64, 0.32, ...			
c)	$\frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \ldots$			

3. Determine the first four terms of each geometric sequence.

a) $t_1 = 2, r = 3$

b) $t_1 = -3, r = -4$

c) $t_1 = 4, r = -3$

d) $t_1 = 2, r = 0.5$

4. Determine the missing terms, t_2, t_3, and t_4, in the geometric sequence in which $t_1 = 8.1$ and $t_5 = 240.1$.

5. Determine a formula for the nth term of each geometric sequence.

a) $r = 2, t_1 = 3$

b) 192, −48, 12, −3, ...

c) $t_3 = 5, t_6 = 135$

d) $t_1 = 4, t_{13} = 16\ 384$

Apply

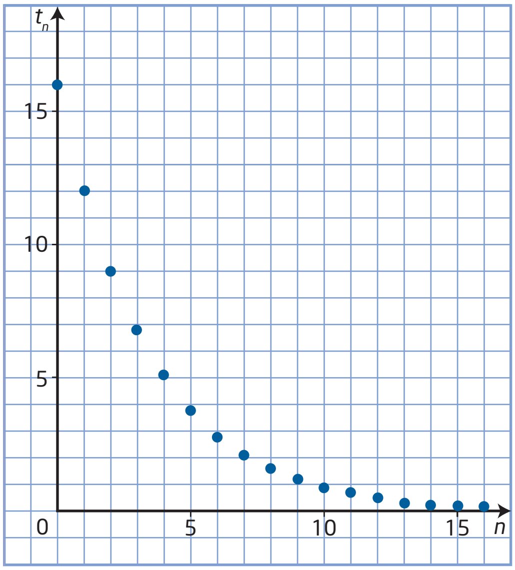
6. Given the following geometric sequences, determine the number of terms, n.

	Table A			
	First Term, t_1	Common Ratio, r	nth Term, t_n	Number of Terms, n
a)	5	3	135	
b)	−2	−3	−1458	
c)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{48}$	
d)	4	4	4096	
e)	$-\frac{1}{6}$	2	$-\frac{128}{3}$	
f)	$\frac{p^2}{2}$	$\frac{p}{2}$	$\frac{p^9}{256}$	

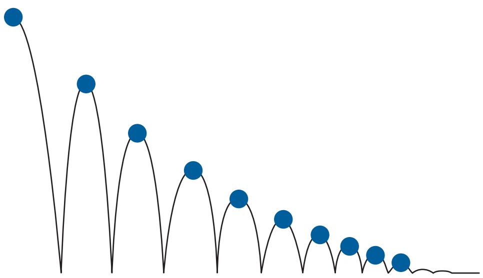
7. The following sequence is geometric. What is the value of y?

$3, 12, 48, 5y + 7, \ldots$

8. The following graph illustrates a geometric sequence. List the first three terms for the sequence and state the general term that describes the sequence.



9. A ball is dropped from a height of 3.0 m. After each bounce it rises to 75% of its previous height.



a) Write the first term and the common ratio of the geometric sequence.

b) Write the general term of the sequence that relates the height of the bounce to the number of bounces.

c) What height does the ball reach after the 6th bounce?

d) After how many bounces will the ball reach a height of approximately 40 cm?

10. The colour of some clothing fades over time when washed. Suppose a pair of jeans fades by 5% with each washing.

a) What percent of the colour remains after one washing?

b) If $t_1 = 100$, what are the first four terms of the sequence?

c) What is the value of r for your geometric sequence?

d) What percent of the colour remains after 10 washings?

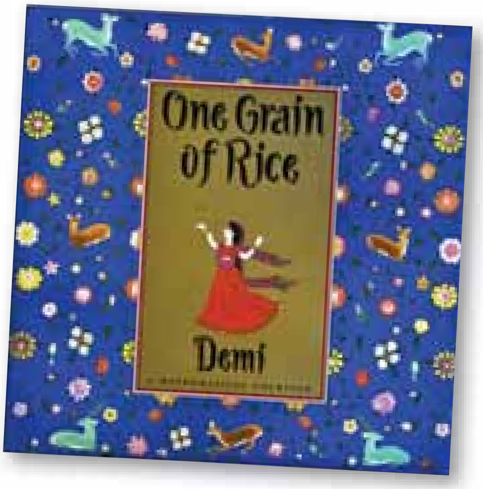
e) How many washings would it take so that only 25% of the original colour remains in the jeans? What assumptions did you make?

11. Pincher Creek, in the foothills of the Rocky Mountains in southern Alberta, is an ideal location to harness the wind power of the chinook winds that blow through the mountain passes. Kinetic energy from the moving air is converted to electricity by wind turbines. In 2004, the turbines generated 326 MW of wind energy, and it is projected that the amount will be 10 000 MW per year by 2010. If this growth were modelled by a geometric sequence, determine the value of the growth rate from 2004 to 2010.

Did You Know?

In an average year, a single 660-kW wind turbine produces 2000 MW of electricity, enough power for over 250 Canadian homes. Using wind to produce electricity rather than burning coal will leave 900 000 kg of coal in the ground and emit 2000 tonnes fewer greenhouse gases annually. This has the same positive impact as taking 417 cars off the road or planting 10 000 trees.

12. The following excerpt is taken from the book *One Grain of Rice* by Demi.



Long ago in India, there lived a raja who believed that he was wise and fair. But every year he kept nearly all of the people's rice for himself. Then when famine came, the raja refused to share the rice, and the people went hungry. Then a village girl named Rani devises a clever plan. She does a good deed for the raja, and in return, the raja lets her choose her reward. Rani asks for just one grain of rice, doubled every day for thirty days.

a) Write the sequence of terms for the first five days that Rani would receive the rice.

b) Write the general term that relates the number of grains of rice to the number of days.

c) Use the general term to determine the number of grains of rice that Rani would receive on the 30th day.

13. The Franco-Manitoban community of St-Pierre-Jolys celebrates Les Folies Grenouilles annually in August. Some of the featured activities include a slow pitch tournament, a parade, fireworks, and the Canadian National Frog Jumping Championships. During the competition, competitor's frogs have five chances to reach their maximum jump. One year, a frog by the name of Georges, achieved the winning jump in his 5th try. Georges' first jump was 191.41 cm, his second jump was 197.34 cm, and his third was 203.46 cm. The pattern of Georges' jumps approximated a geometric sequence.

a) By what ratio did Georges improve his performance with each jump? Express your answer to three decimal places.

b) How far was Georges' winning jump? Express your answer to the nearest tenth of a centimetre.

c) The world record frog jump is held by a frog named Santjie of South Africa. Santjie jumped approximately 10.2 m. If Georges, from St-Pierre-Jolys, had continued to increase his jumps following this same geometric sequence, how many jumps would Georges have needed to complete to beat Santjie's world record jump?

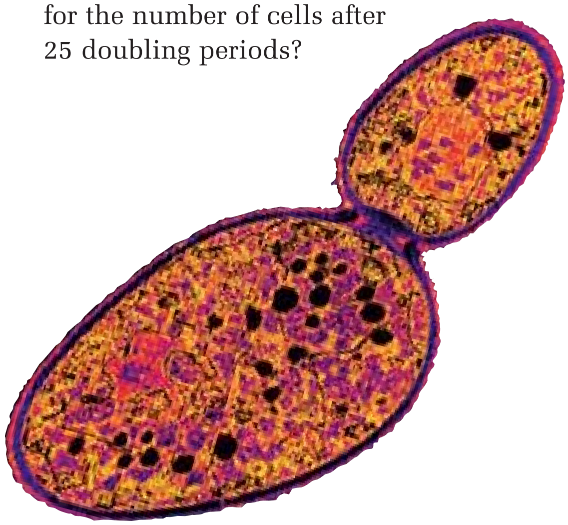
14. Bread and bread products have been part of our diet for centuries. To help bread rise, yeast is added to the dough. Yeast is a living unicellular micro-organism about one hundredth of a millimetre in size. Yeast multiplies by a biochemical process called budding. After mitosis and cell division, one cell results in two cells with exactly the same characteristics.

a) Write a sequence for the first six terms that describes the cell growth of yeast, beginning with a single cell.

b) Write the general term for the growth of yeast.

c) How many cells would there be after 25 doublings?

d) What assumptions would you make for the number of cells after 25 doubling periods?



15. The Arctic Winter Games is a high profile sports competition for northern and arctic athletes. The premier sports are the Dene and Inuit games, which include the arm pull, the one foot high kick, the two foot high kick, and the Dene hand games. The first Arctic Winter Games, held in 1970, drew 700 competitors. In 2008, the games were held in Yellowknife and drew 2000 competitors. If the number of competitors grew geometrically from 1970 to 2008, determine the rate of growth in the number of competitors for these games. Express your answer to the nearest tenth of a percent.

16. Jason Annahatak entered the Russian sledge jump competition at the Arctic Winter Games, held in Yellowknife. Suppose that to prepare for this event, Jason started training by jumping 2 sledges each day for the first week, 4 sledges each day for the second week, 8 sledges each day for the third week, and so on. During the competition, Jason jumped 142 sledges. Assuming he continued his training pattern, how many weeks did it take him to reach his competition number of 142 sledges?

Did You Know?

Sledge jump starts from a standing position. The athlete jumps consecutively over 10 sledges placed in a row, turns around using one jumping movement, and then jumps back over the 10 sledges. This process is repeated until the athlete misses a jump or touches a sledge.

17. At Galaxyland in the West Edmonton Mall, a boat swing ride has been modelled after a basic pendulum design. When the boat first reaches the top of the swing, this is considered to be the beginning of the first swing. A swing is completed when the boat changes direction. On each successive completed swing, the boat travels 96% as far as on the previous swing. The ride finishes when the arc length through which the boat travels is 30 m. If it takes 20 swings for the boat to reach this arc length, determine the arc length through which the boat travels on the first swing. Express your answer to the nearest tenth of a metre.

18. The Russian nesting doll or Matryoshka had its beginnings in 1890. The dolls are made so that the smallest doll fits inside a larger one, which fits inside a larger one, and so on, until all the dolls are hidden inside the largest doll. In a set of 50 dolls, the tallest doll is 60 cm and the smallest is 1 cm. If the decrease in doll size approximates a geometric sequence, determine the common ratio. Express your answer to three decimal places.

19. The primary function for our kidneys is to filter our blood to remove any impurities. Doctors take this into account when prescribing the dosage and frequency of medicine. A person's kidneys filter out 18% of a particular medicine every two hours.

a) How much of the medicine remains after 12 h if the initial dosage was 250 mL? Express your answer to the nearest tenth of a millimetre.

b) When there is less than 20 mL left in the body, the medicine becomes ineffective and another dosage is needed. After how many hours would this happen?

Did You Know?

Every day, a person's kidneys process about 190 L of blood to remove about 1.9 L of waste products and extra water.

20. The charge in a car battery, when the car is left to sit, decreases by about 2% per day and can be modelled by the formula $C = 100(0.98)^d$, where d is the time, in days, and C is the approximate level of charge, as a percent.

a) Copy and complete the chart to show the percent of charge remaining in relation to the time passed.

Time, *d* (days)	Charge Level, *C* (%)
0	100
1	
2	
3	

b) Write the general term of this geometric sequence.

c) Explain how this formula is different from the formula $C = 100(0.98)^d$.

d) How much charge is left after 10 days?

21. A coiled basket is made using dried pine needles and sinew. The basket is started from the centre using a small twist and spirals outward and upward to shape the basket. The circular coiling of the basket approximates a geometric sequence, where the radius of the first coil is 6 mm.

a) If the ratio of consecutive coils is 1.22, calculate the radius for the 8th coil.

b) If there are 18 coils, what is the circumference of the top coil of the basket?

Extend

22. Demonstrate that $6^a, 6^b, 6^c, \ldots$ forms a geometric sequence when a, b, c, ... forms an arithmetic sequence.

23. If $x + 2$, $2x + 1$, and $4x - 3$ are three consecutive terms of a geometric sequence, determine the value of the common ratio and the three given terms.

24. On a six-string guitar, the distance from the nut to the bridge is 38 cm. The distance from the first fret to the bridge is 35.87 cm, and the distance from the second fret to the bridge is 33.86 cm. This pattern approximates a geometric sequence.

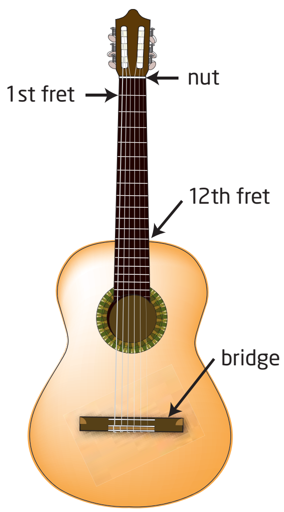
a) What is the distance from the 8th fret to the bridge?

b) What is the distance from the 12th fret to the bridge?

c) Determine the distance from the nut to the first fret.

d) Determine the distance from the first fret to the second fret.

e) Write the sequence for the first three terms of the distances between the frets. Is this sequence geometric or arithmetic? What is the common ratio or common difference?



Create Connections

25. Alex, Mala, and Paul were given the following problem to solve in class.

An aquarium that originally contains 40 L of water loses 8% of its water to evaporation every day. Determine how much water will be in the aquarium at the beginning of the 7th day.

The three students' solutions are shown below. Which approach to the solution is correct? Justify your reasoning.

Alex's solution:

Alex believed that the sequence was geometric, where $t_1 = 40$, $r = 0.08$, and $n = 7$. He used the general formula $t_n = t_1r^{n-1}$.

$t_n = t_1r^{n-1}$
$t_n = 40(0.08)^{n-1}$
$t_7 = 40(0.08)^{7-1}$
$t_7 = 40(0.08)^6$
$t_7 = 0.000\ 01$

There will be 0.000 01 L of water in the tank at the beginning of the 7th day.

Mala's solution:

Mala believed that the sequence was geometric, where $t_1 = 40$, $r = 0.92$, and $n = 7$. She used the general formula $t_n = t_1r^{n-1}$.

$t_n = t_1r^{n-1}$
$t_n = 40(0.92)^{n-1}$
$t_7 = 40(0.92)^{7-1}$
$t_7 = 40(0.92)^6$
$t_7 = 24.25$

There will be 24.25 L of water in the tank at the beginning of the 7th day.

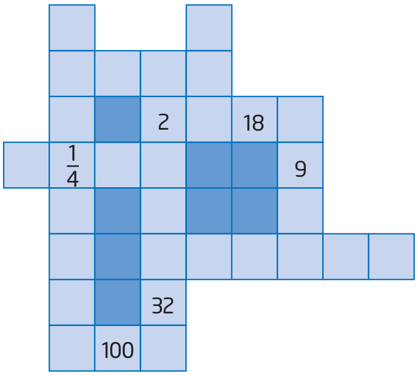
Paul's solution:

Paul believed that the sequence was arithmetic, where $t_1 = 40$ and $n = 7$. To calculate the value of d, Paul took 8% of 40 = 3.2. He reasoned that this would be a negative constant since the water was gradually disappearing. He used the general formula $t_n = t_1 + (n - 1)d$.

$t_n = t_1 + (n - 1)d$
$t_n = 40 + (n - 1)(-3.2)$
$t_7 = 40 + (7 - 1)(-3.2)$
$t_7 = 40 + (6)(-3.2)$
$t_7 = 20.8$

There will be 20.8 L of water in the tank at the beginning of the 7th day.

26. Copy the puzzle. Fill in the empty boxes with positive numbers so that each row and column forms a geometric sequence.



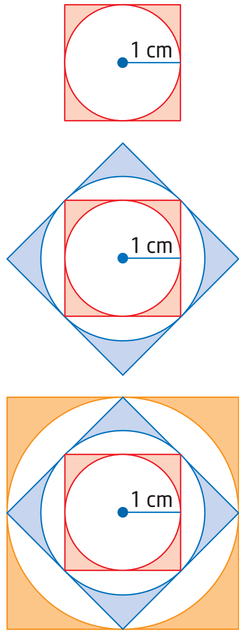
27. A square has an inscribed circle of radius 1 cm.

a) What is the area of the red portion of the square, to the nearest hundredth of a square centimetre?

b) If another square with an inscribed circle is drawn around the original, what is the area of the blue region, to the nearest hundredth of a square centimetre?

c) If another square with an inscribed circle is drawn around the squares, what is the area of the orange region, to the nearest hundredth of a square centimetre?

d) If this pattern were to continue, what would be the area of the newly coloured region for the 8th square, to the nearest hundredth of a square centimetre?



Project Corner — Forestry

- Canada has 402.1 million hectares (ha) of forest and other wooded lands. This value represents 41.1% of Canada's total surface area of 979.1 million hectares.
- Annually, Canada harvests 0.3% of its commercial forest area. In 2007, 0.9 million hectares were harvested.
- In 2008, British Columbia planted its 6 billionth tree seedling since the 1930s, as part of its reforestation programs.

1.4

Geometric Series

Focus on…

- deriving a rule for determining the sum of n terms of a geometric series
- determining t_1, r, n, or S_n involving a geometric sequence
- solving a problem that involves a geometric series
- identifying any assumptions made when identifying a geometric series

If you take the time to look closely at nature, chances are you have seen a fractal. Fractal geometry is the *geometry of nature*. The study of fractals is, mathematically, relatively new. A fractal is a geometric figure that is generated by starting with a very simple pattern and repeating that pattern over and over an infinite number of times. The basic concept of a fractal is that it contains a large degree of self-similarity. This means that a fractal usually contains small copies of itself buried within the original. Where do you see fractals in the images shown?

Investigate Fractals

Materials

- paper
- ruler

Fractal Tree

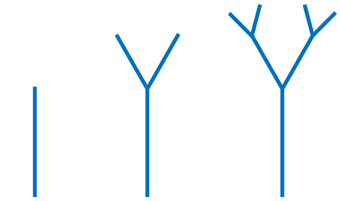
A *fractal tree* is a fractal pattern that results in a realistic looking tree.

You can build your own fractal tree:

1. a) Begin with a sheet of paper. Near the bottom of the paper and centred on the page, draw a vertical line segment approximately 3 cm to 4 cm in length.

b) At the top of the segment, draw two line segments, splitting away from each other as shown in Stage 2. These segments form the branches of the tree. Each new branch formed is a smaller version of the main trunk of the tree.

c) At the top of each new line segment, draw another two branches, as shown in Stage 3.



d) Continue this process to complete five stages of the fractal tree.

2. Copy and complete the following table.

Stage	1	2	3	4	5
Number of New Branches	1	2			

3. Decide whether a geometric sequence has been generated for the number of new branches formed at each stage. If a geometric sequence has been generated, state the first term, the common ratio, and the general term.

Reflect and Respond

4. a) Would a geometric sequence be generated if there were three new branches formed from the end of each previous branch?

b) Would a geometric sequence be generated if there were four new branches formed?

5. Describe a strategy you could use to determine the total number of branches that would be formed by the end of stage 5.

6. Would this be a suitable strategy to use if you wanted to determine the total number of branches up to stage 100? Explain.

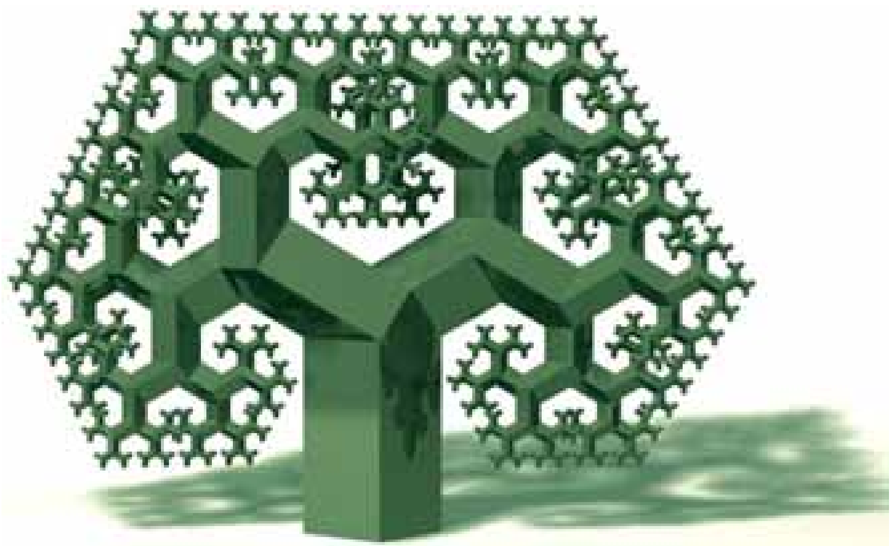


Image rendered by Anton Bakker based on a fractal tree design by Koos Verhoeff. Used with permission of the Foundation MathArt Koos Verhoeff.

Web Link

The complex mathematical equations of fractals are used in the creation of many works of art and computer generated fractals. To learn more about art and fractals, go to www.mhrprecalc11.ca and follow the links.

Link the Ideas

geometric series

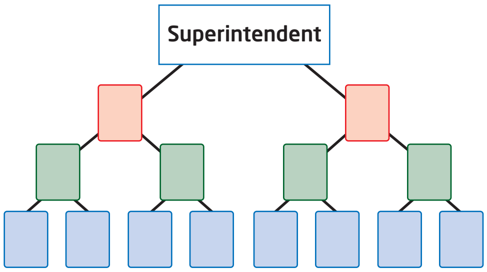
- the terms of a geometric sequence expressed as a sum
- for example, $3 + 6 + 12 + 24$ is a geometric series

A **geometric series** is the expression for the sum of the terms of a geometric sequence.

A school district emergency *fan-out* system is designed to enable important information to reach the entire staff of the district very quickly. At the first level, the superintendent calls two assistant superintendents. The two assistant superintendents each call two area superintendents. They in turn, each call two principals. The pattern continues with each person calling two other people.

At every level, the total number of people contacted is twice the number of people contacted in the previous level. The pattern can be modelled by a geometric series where the first term is 1 and the common ratio is 2. The series for the fan-out system would be $1 + 2 + 4 + 8$, which gives a sum of 15 people contacted after 4 levels.

To extend this series to 15 or 20 or 100 levels, you need to determine a way to calculate the sum of the series other than just adding the terms.



One way to calculate the sum of the series is to use a formula.

To develop a formula for the sum of a series,

List the original series.
$S_4 = 1 + 2 + 4 + 8$ ①

Multiply each term in the series by the common ratio.
$2(S_4 = 1 + 2 + 4 + 8)$
$2S_4 = 2 + 4 + 8 + 16$ ②

The number of staff contacted in the 5th level is 16.

Subtract equation ① from equation ②.

$$\begin{array}{rcl} 2S_4 &=& \phantom{1 + {}} 2 + 4 + 8 + 16 \\ -\ S_4 &=& 1 + 2 + 4 + 8 \\ \hline (2-1)S_4 &=& -1 + 0 + 0 + 0 + 16 \end{array}$$

Why are the two equations aligned as shown?

Isolate S_4 by dividing by $(2 - 1)$.

$$S_4 = \frac{16 - 1}{2 - 1}$$
$$S_4 = 15$$

You can use the above method to derive a general formula for the sum of a geometric series.

The general geometric series may be represented by the following series.
$S_n = t_1 + t_1r + t_1r^2 + t_1r^3 + \cdots + t_1r^{n-1}$

Multiply every term in the series by the common ratio, r.
$rS_n = t_1r + t_1r(r) + t_1r^2(r) + t_1r^3(r) + \cdots + t_1r^{n-1}(r)$
$rS_n = t_1r + t_1r^2 + t_1r^3 + t_1r^3 + \cdots + t_1r^n$

Subtract the two equations.

$$
\begin{array}{rl}
rS_n = & \quad\;\; t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \cdots + t_1r^{n-1} + t_1r^n \\
S_n = & t_1 + t_1r + t_1r^2 + t_1r^3 + \cdots \qquad\quad + t_1r^{n-1} \\
\hline
(r-1)S_n = & -t_1 + 0 + 0 + 0 + \cdots + 0 + 0 + t_1r^n
\end{array}
$$

Isolate S_n by dividing by $r - 1$.

$$S_n = \frac{t_1r^n - t_1}{r - 1} \text{ or } S_n = \frac{t_1(r^n - 1)}{r - 1},\ r \neq 1$$

Why can r not be equal to 1?

The *sum* of a geometric series is can be determined using the formula

$$\boldsymbol{S_n = \frac{t_1(r^n - 1)}{r - 1},\ r \neq 1}$$

where t_1 is the first term of the series
n is the number of terms
r is the common ratio
S_n is the sum of the first n terms

Example 1

Determine the Sum of a Geometric Series

Determine the sum of the first 10 terms of each geometric series.

a) $4 + 12 + 36 + \cdots$

b) $t_1 = 5,\ r = \frac{1}{2}$

Solution

a) In the series, $t_1 = 4$, $r = 3$, and $n = 10$.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{4(3^{10} - 1)}{3 - 1}$$
$$S_{10} = \frac{4(59\ 048)}{2}$$
$$S_{10} = 118\ 096$$

The sum of the first 10 terms of the geometric series is 118 096.

b) In the series, $t_1 = 5$, $r = \frac{1}{2}$, $n = 10$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{5\left[\left(\frac{1}{2}\right)^{10} - 1\right]}{\frac{1}{2} - 1}$$

$$S_{10} = \frac{5\left(\frac{1}{1024} - 1\right)}{-\frac{1}{2}}$$

$$S_{10} = -10\left(\frac{-1023}{1024}\right)$$

$$S_{10} = \frac{5115}{512}$$

The sum of the first 10 terms of the geometric series is $\frac{5115}{512}$ or $9\frac{507}{512}$.

Your Turn

Determine the sum of the first 8 terms of the following geometric series.

a) $5 + 15 + 45 + \cdots$

b) $t_1 = 64$, $r = \frac{1}{4}$

Example 2

Determine the Sum of a Geometric Series for an Unspecified Number of Terms

Determine the sum of each geometric series.

a) $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \cdots + 729$

b) $4 - 16 + 64 - \cdots - 65\,536$

Solution

a) Method 1: Determine the Number of Terms

$$t_n = t_1 r^{n-1}$$ Use the general term.

$$729 = \frac{1}{27}(3)^{n-1}$$ Substitute known values.

$$(27)(729) = \left[\frac{1}{27}(3)^{n-1}\right](27)$$ Multiply both sides by 27.

$$(27)(729) = (3)^{n-1}$$

$$(3^3)(3^6) = (3)^{n-1}$$ Write as powers with a base of 3.

$$(3)^9 = (3)^{n-1}$$

$$9 = n - 1$$

$$10 = n$$ Since the bases are the same, the exponents must be equal.

There are 10 terms in the series.

Use the general formula for the sum of a geometric series where $n = 10$, $t_1 = \frac{1}{27}$, and $r = \frac{1}{3}$.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{\left(\frac{1}{27}\right)[(3)^{10} - 1]}{3 - 1}$$

$$S_{10} = \frac{29\,524}{27}$$

The sum of the series is $\frac{29\,524}{27}$ or $1093\frac{13}{27}$.

Method 2: Use an Alternate Formula

Begin with the formula for the general term of a geometric sequence, $t_n = t_1r^{n-1}$.

Multiply both sides by r.

$rt_n = (t_1r^{n-1})(r)$

Simplify the right-hand side of the equation.

$rt_n = t_1r^n$

From the previous work, you know that the general formula for the sum of a geometric series may be written as

$$S_n = \frac{t_1r^n - t_1}{r - 1}$$

Substitute rt_n for t_1r^n.

$S_n = \frac{rt_n - t_1}{r - 1}$ where $r \neq 1$.

This results in a general formula for the sum of a geometric series when the first term, the nth term, and the common ratio are known.

Determine the sum where $r = 3$, $t_n = 729$, and $t_1 = \frac{1}{27}$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{(3)(729) - \frac{1}{27}}{3 - 1}$$

$$S_n = \frac{29\,524}{27}$$

The sum of the series is $\frac{29\,524}{27}$ or approximately 1093.48.

b) Use the alternate formula $S_n = \frac{rt_n - t_1}{r - 1}$, where $t_1 = 4$, $r = -4$, and $t_n = -65\,536$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{(-4)(-65\,536) - 4}{-4 - 1}$$

$$S_n = -52\,428$$

The sum of the series is $-52\,428$.

Your Turn

Determine the sum of the following geometric series.

a) $\frac{1}{64} + \frac{1}{16} + \frac{1}{4} + \cdots + 1024$ **b)** $-2 + 4 - 8 + \cdots - 8192$

Example 3

Apply Geometric Series

The Western Scrabble™ Network is an organization whose goal is to promote the game of Scrabble™. It offers Internet tournaments throughout the year that WSN members participate in. The format of these tournaments is such that the losers of each round are eliminated from the next round. The winners continue to play until a final match determines the champion. If there are 256 entries in an Internet Scrabble™ tournament, what is the total number of matches that will be played in the tournament?

Solution

The number of matches played at each stage of the tournament models the terms of a geometric sequence. There are two players per match, so the first term, t_1, is $\frac{256}{2} = 128$ matches. After the first round, half of the players are eliminated due to a loss. The common ratio, r, is $\frac{1}{2}$.

A single match is played at the end of the tournament to decide the winner. The nth term of the series, t_n, is 1 final match.

Use the formula $S_n = \frac{rt_n - t_1}{r - 1}$ for the sum of a geometric series where $t_1 = 128$, $r = \frac{1}{2}$, and $t_n = 1$.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{\left(\frac{1}{2}\right)(1) - 128}{\left(\frac{1}{2}\right) - 1}$$

$$S_n = \frac{\frac{-255}{2}}{-\frac{1}{2}}$$

$$S_n = \left(\frac{-255}{2}\right)\left(-\frac{2}{1}\right)$$

$$S_n = 255$$

There will be 255 matches played in the tournament

Your Turn

If a tournament has 512 participants, how many matches will be played?

Key Ideas

- A geometric series is the expression for the sum of the terms of a geometric sequence.
  For example, $5 + 10 + 20 + 40 + \cdots$ is a geometric series.
- The general formula for the sum of the first n terms of a geometric series with the first term, t_1, and the common ratio, r, is
  $$S_n = \frac{t_1(r^n - 1)}{r - 1}, r \neq 1$$
- A variation of this formula may be used when the first term, t_1, the common ratio, r, and the nth term, t_n, are known, but the number of terms, n, is not known.
  $$S_n = \frac{rt_n - t_1}{r - 1}, r \neq 1$$

Check Your Understanding

Practise

1. Determine whether each series is geometric. Justify your answer.

a) $4 + 24 + 144 + 864 + \cdots$

b) $-40 + 20 - 10 + 5 - \cdots$

c) $3 + 9 + 18 + 54 + \cdots$

d) $10 + 11 + 12.1 + 13.31 + \cdots$

2. For each geometric series, state the values of t_1 and r. Then determine each indicated sum. Express your answers as exact values in fraction form and to the nearest hundredth.

a) $6 + 9 + 13.5 + \cdots (S_{10})$

b) $18 - 9 + 4.5 + \cdots (S_{12})$

c) $2.1 + 4.2 + 8.4 + \cdots (S_9)$

d) $0.3 + 0.003 + 0.000\,03 + \cdots (S_{12})$

3. What is S_n for each geometric series described? Express your answers as exact values in fraction form.

a) $t_1 = 12, r = 2, n = 10$

b) $t_1 = 27, r = \frac{1}{3}, n = 8$

c) $t_1 = \frac{1}{256}, r = -4, n = 10$

d) $t_1 = 72, r = \frac{1}{2}, n = 12$

4. Determine S_n for each geometric series. Express your answers to the nearest hundredth, if necessary.

a) $27 + 9 + 3 + \cdots + \frac{1}{243}$

b) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{128}{6561}$

c) $t_1 = 5, t_n = 81\,920, r = 4$

d) $t_1 = 3, t_n = 46\,875, r = -5$

5. What is the value of the first term for each geometric series described? Express your answers to the nearest tenth, if necessary.

a) $S_n = 33$, $t_n = 48$, $r = -2$

b) $S_n = 443$, $n = 6$, $r = \frac{1}{3}$

6. The sum of $4 + 12 + 36 + 108 + \cdots + t_n$ is 4372. How many terms are in the series?

7. The common ratio of a geometric series is $\frac{1}{3}$ and the sum of the first 5 terms is 121.

a) What is the value of the first term?

b) Write the first 5 terms of the series.

8. What is the second term of a geometric series in which the third term is $\frac{9}{4}$ and the sixth term is $-\frac{16}{81}$? Determine the sum of the first 6 terms. Express your answer to the nearest tenth.

Apply

9. A fan-out system is used to contact a large group of people. The person in charge of the contact committee relays the information to four people. Each of these four people notifies four more people, who in turn each notify four more people, and so on.

a) Write the corresponding series for the number of people contacted.

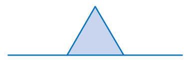
b) How many people are notified after 10 levels of this system?

10. A tennis ball dropped from a height of 20 m bounces to 40% of its previous height on each bounce. The total vertical distance travelled is made up of upward bounces and downward drops. Draw a diagram to represent this situation. What is the total vertical distance the ball has travelled when it hits the floor for the sixth time? Express your answer to the nearest tenth of a metre.

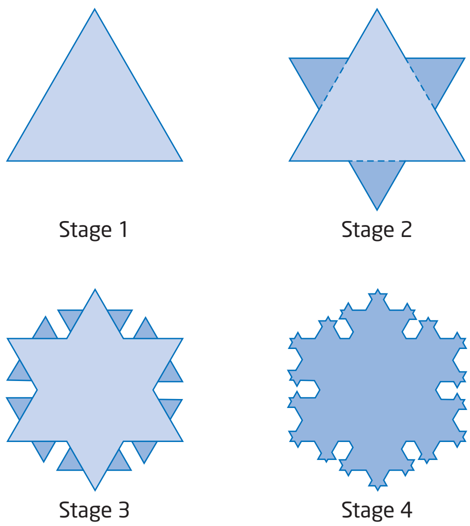
11. Celia is training to run a marathon. In the first week she runs 25 km and increases this distance by 10% each week. This situation may be modelled by the series $25 + 25(1.1) + 25(1.1)^2 + \cdots$. She wishes to continue this pattern for 15 weeks. How far will she have run in total when she completes the 15th week? Express your answer to the nearest tenth of a kilometre.

12. **MINI LAB** Building the Koch snowflake is a step-by-step process.

- Start with an equilateral triangle. (Stage 1)
- In the middle of each line segment forming the sides of the triangle, construct an equilateral triangle with side length equal to $\frac{1}{3}$ of the length of the line segment.



- Delete the base of this new triangle. (Stage 2)
- For each line segment in Stage 2, construct an equilateral triangle, deleting its base. (Stage 3)
- Repeat this process for each line segment, as you move from one stage to the next.



a) Work with a partner. Use dot paper to draw three stages of the Koch snowflake.

b) Copy and complete the following table.

Stage Number	Length of Each Line Segment	Number of Line Segments	Perimeter of Snowflake
1	1	3	3
2	$\frac{1}{3}$	12	4
3	$\frac{1}{9}$		
4			
5			

c) Determine the general term for the length of each line segment, the number of line segments, and the perimeter of the snowflake.

d) What is the total perimeter of the snowflake up to Stage 6?

13. An advertising company designs a campaign to introduce a new product to a metropolitan area. The company determines that 1000 people are aware of the product at the beginning of the campaign. The number of new people aware increases by 40% every 10 days during the advertising campaign. Determine the total number of people who will be aware of the product after 100 days.

14. Bead working has a long history among Canada's Indigenous peoples. Floral designs are the predominate patterns found among people of the boreal forests and northern plains. Geometric patterns are found predominately in the Great Plains. As bead work continues to be popular, traditional patterns are being exchanged among people in all regions. Suppose a set of 10 beads were laid in a line where each successive bead had a diameter that was $\frac{3}{4}$ of the diameter of the previous bead. If the first bead had a diameter of 24 mm, determine the total length of the line of beads. Express your answer to the nearest millimetre.

Did You Know?

Wampum belts consist of rows of beads woven together. Weaving traditionally involves stringing the beads onto twisted plant fibres, and then securing them to animal sinew.

Wanuskewan Native Heritage Park, Cree Nation, Saskatchewan

15. When doctors prescribe medicine at equally spaced time intervals, they are aware that the body metabolizes the drug gradually. After some period of time, only a certain percent of the original amount remains. After each dose, the amount of the drug in the body is equal to the amount of the given dose plus the amount remaining from the previous doses. The amount of the drug present in the body after the nth dose is modelled by a geometric series where t_1 is the prescribed dosage and r is the previous dose remaining in the body.

Suppose a person with an ear infection takes a 200-mg ampicillin tablet every 4 h. About 12% of the drug in the body at the start of a four-hour period is still present at the end of that period. What amount of ampicillin is in the body, to the nearest tenth of a milligram,

a) after taking the third tablet?

b) after taking the sixth tablet?

Extend

16. Determine the number of terms, n, if $3 + 3^2 + 3^3 + \cdots + 3^n = 9840$.

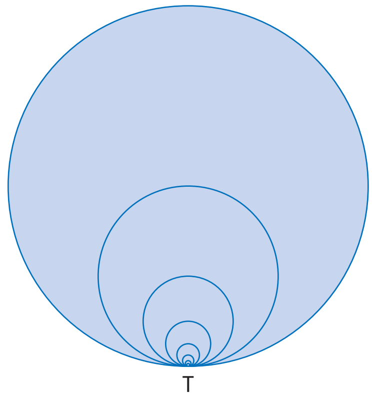
17. The third term of a geometric series is 24 and the fourth term is 36. Determine the sum of the first 10 terms. Express your answer as an exact fraction.

18. Three numbers, a, b, and c, form a geometric series so that $a + b + c = 35$ and $abc = 1000$. What are the values of a, b, and c?

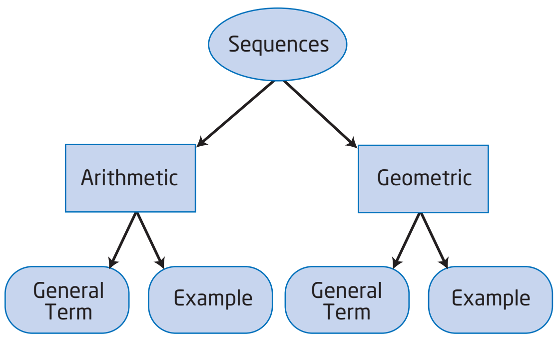
19. The sum of the first 7 terms of a geometric series is 89, and the sum of the first 8 terms is 104. What is the value of the eighth term?

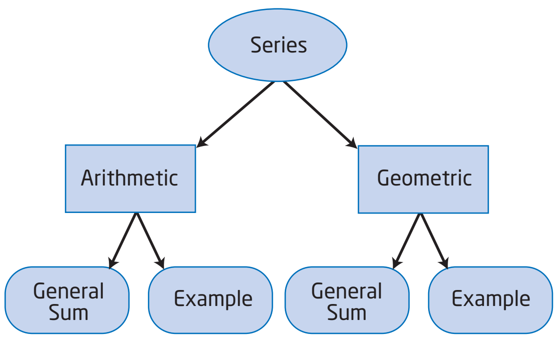
Create Connections

20. A fractal is created as follows: A circle is drawn with radius 8 cm. Another circle is drawn with half the radius of the previous circle. The new circle is tangent to the previous circle at point T as shown. Suppose this pattern continues through five steps. What is the sum of the areas of the circles? Express your answer as an exact fraction.



21. Copy the following flowcharts. In the appropriate segment of each chart, give a definition, a general term or sum, or an example, as required.





22. Tom learned that the monarch butterfly lays an average of 400 eggs. He decided to calculate the growth of the butterfly population from a single butterfly by using the logic that the first butterfly produced 400 butterflies. Each of those butterflies would produce 400 butterflies, and this pattern would continue. Tom wanted to estimate how many butterflies there would be in total in the fifth generation following this pattern. His calculation is shown below.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{1(400^5 - 1)}{400 - 1}$$

$$S_5 \approx 2.566 \times 10^{10}$$

Tom calculated that there would be approximately 2.566×10^{10} monarch butterflies in the fifth generation.

a) What assumptions did Tom make in his calculations?

b) Do you agree with the method Tom used to arrive at the number of butterflies? Explain.

c) Would this be a reasonable estimate of the total number of butterflies in the fifth generation? Explain.

d) Explain a method you would use to calculate the number of butterflies in the fifth generation.

Did You Know?

According to the American Indian Butterfly Legend:

If anyone desires a wish to come true they must first capture a butterfly and whisper that wish to it.

Since a butterfly can make no sound, the butterfly cannot reveal the wish to anyone but the Great Spirit who hears and sees all.

In gratitude for giving the beautiful butterfly its freedom, the Great Spirit always grants the wish.

So, according to legend, by making a wish and giving the butterfly its freedom, the wish will be taken to the heavens and be granted.

Project Corner — Oil Discovery

- The first oil well in Canada was discovered by James Miller Williams in 1858 near Oil Springs, Ontario. The oil was taken to Hamilton, Ontario, where it was refined into lamp oil. This well produced 37 barrels a day. By 1861 there were 400 wells in the area.
- In 1941, Alberta's population was approximately 800 000. By 1961, it was about 1.3 million.
- In February 1947, oil was struck in Leduc, Alberta. Leduc was the largest discovery in Canada in 33 years. By the end of 1947, 147 more wells were drilled in the Leduc-Woodbend oilfield.
- With these oil discoveries came accelerated population growth. In 1941, Leduc was inhabited by 871 people. By 1951, its population had grown to 1842.
- Leduc #1 was capped in 1974, after producing 300 000 barrels of oil and 9 million cubic metres of natural gas.

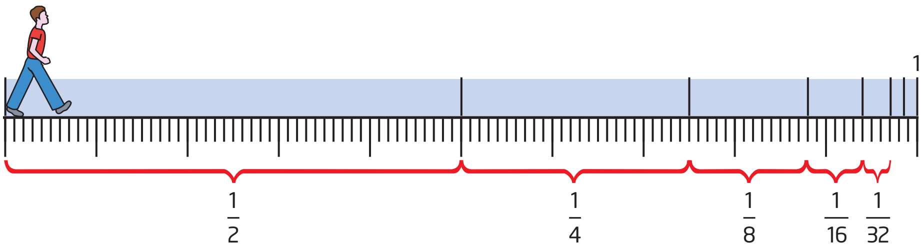
1.5

Infinite Geometric Series

Focus on…

- generalizing a rule for determining the sum of an infinite geometric series
- explaining why a geometric series is convergent or divergent
- solving a problem that involves a geometric sequence or series

In the fifth century B.C.E., the Greek philosopher Zeno of Elea posed four problems, now known as Zeno's paradoxes. These problems were intended to challenge some of the ideas that were held in his day. His paradox of motion states that a person standing in a room cannot walk to the wall. In order to do so, the person would first have to go half the distance, then half the remaining distance, and then half of what still remains. This process can always be continued and can never end.



Zeno's argument is that there is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on to infinity.

Where does the argument break down? Why?

Did You Know?

The word *paradox* comes from the Greek *para doxa*, meaning something contrary to opinion.

Investigate an Infinite Series

Materials

- square piece of paper
- ruler

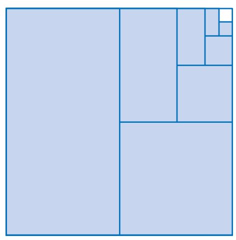
1. Start with a square piece of paper.

a) Draw a line dividing it in half.

b) Shade one of the halves.

c) In the unshaded half of the square, draw a line to divide it in half. Shade one of the halves.

d) Repeat part c) at least six more times.



2. Write a sequence of terms indicating the area of each newly shaded region as a fraction of the entire page. List the first five terms.

3. Predict the next two terms for the sequence.

4. Is the sequence arithmetic, geometric, or neither? Justify your answer.
5. Write the rule for the nth term of the sequence.
6. Ignoring physical limitations, could this sequence continue indefinitely? In other words, would this be an infinite sequence? Explain your answer.
7. What conclusion can you make about the area of the square that would remain unshaded as the number of terms in the sequence approaches infinity?

Reflect and Respond

8. Using a graphing calculator, input the function $y = \left(\frac{1}{2}\right)^x$.
   a) Using the table of values from the calculator, what happens to the value of $y = \left(\frac{1}{2}\right)^x$ as x gets larger and larger?
   b) Can the value of $\left(\frac{1}{2}\right)^x$ ever equal zero?
9. The geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ can be written as $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^x$.
   You can use the general formula to determine the sum of the series.

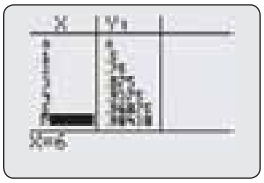
$$S_x = \frac{t_1(1 - r^x)}{(1 - r)}$$

$$S_x = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^x\right)}{1 - \frac{1}{2}}$$

$$S_x = 1 - \left(\frac{1}{2}\right)^x$$

For values of $r < 1$, the general formula $S_x = \frac{t_1(r^x - 1)}{(r - 1)}$ can be written for convenience as $S_x = \frac{t_1(1 - r^x)}{(1 - r)}$. Why do you think this is true?

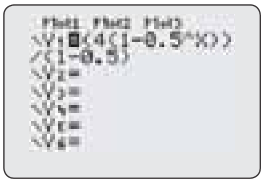
   Enter the function into your calculator and use the table feature to find the sum, S_x, as x gets larger.

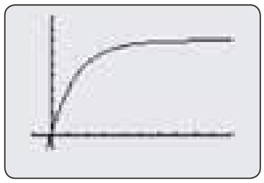


   a) What happens to the sum, S_x, as x gets larger?
   b) Will the sum increase without limit? Explain your reasoning.
10. a) As the value of x gets very large, what value can you assume that r^x becomes close to?
   b) Use your answer from part a) to modify the formula for the sum of a geometric series to determine the sum of an infinite geometric series.
   c) Use your formula from part b) to determine the sum of the infinite geometric series $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$.

Convergent Series

Consider the series $4 + 2 + 1 + 0.5 + 0.25 + \cdots$





$S_5 = 7.75$
$S_7 = 7.9375$
$S_9 = 7.9844$
$S_{11} = 7.9961$
$S_{13} = 7.999$
$S_{15} = 7.9998$
$S_{17} = 7.9999$

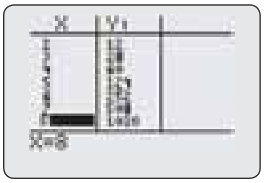
As the number of terms increases, the sequence of partial sums approaches a fixed value of 8. Therefore, the sum of this series is 8. This series is said to be a **convergent series**.

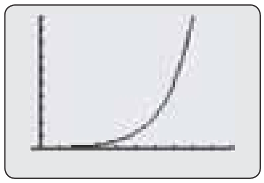
convergent series
- a series with an infinite number of terms, in which the sequence of partial sums approaches a fixed value
- a series for which r is between -1 and 1
- for example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Divergent Series

Consider the series $4 + 8 + 16 + 32 + \cdots$

$S_1 = 4$
$S_2 = 12$
$S_3 = 28$
$S_4 = 60$
$S_5 = 124$





As the number of terms increases, the sum of the series continues to grow. The sequence of partial sums does not approach a fixed value. Therefore, the sum of this series cannot be calculated. This series is said to be a **divergent series**.

divergent series
- a series with an infinite number of terms, in which the sequence of partial sums does not approach a fixed value
- a series for which $r > 1$ or $r < -1$
- for example, $2 + 4 + 8 + 16 + \cdots$

Infinite Geometric Series

The formula for the sum of a geometric series is

$$S_n = \frac{t_1(1 - r^n)}{1 - r}.$$

As n gets very large, the value of the r^n approaches 0, for values of r between -1 and 1.

So, as n gets large, the partial sum S_n approaches $\frac{t_1}{1 - r}$.

Therefore, the sum of an infinite geometric series is

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1.$$

The *sum* of an infinite geometric series, where $-1 < r < 1$, can be determined using the formula

$$\boldsymbol{S_\infty = \frac{t_1}{1 - r}}$$

where t_1 is the first term of the series
r is the common ratio
S_∞ represents the sum of an infinite number of terms

Applying the formula to the series $4 + 2 + 1 + 0.5 + 0.25 + \cdots$

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$
$$S_\infty = \frac{4}{1 - 0.5}$$
$$S_\infty = \frac{4}{0.5}$$
$$S_\infty = 8$$

Example 1

Sum of an Infinite Geometric Series

Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

a) $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

b) $2 - 4 + 8 - \cdots$

Solution

a) $t_1 = 1$, $r = -\frac{1}{3}$

Since $-1 < r < 1$, the series is convergent.
Use the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$
$$S_\infty = \frac{1}{1 - \left(-\frac{1}{3}\right)}$$
$$S_\infty = \frac{1}{\frac{4}{3}}$$
$$S_\infty = (1)\left(\frac{3}{4}\right)$$
$$S_\infty = \frac{3}{4}$$

b) $t_1 = 2$, $r = -2$

Since $r < -1$, the series is divergent and has no sum.

Your Turn

Determine whether each infinite geometric series converges or diverges. Calculate the sum, if it exists.

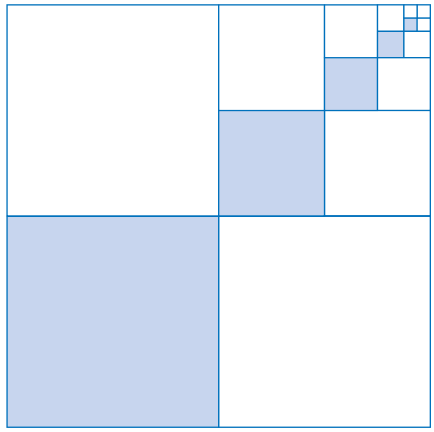
a) $1 + \frac{1}{5} + \frac{1}{25} + \cdots$

b) $4 + 8 + 16 + \cdots$

Example 2

Apply the Sum of an Infinite Geometric Series

Assume that each shaded square represents $\frac{1}{4}$ of the area of the larger square bordering two of its adjacent sides and that the shading continues indefinitely in the indicated manner.



a) Write the series of terms that would represent this situation.

b) How much of the total area of the largest square is shaded?

Solution

a) The sequence of shaded regions generates an infinite geometric sequence. The series of terms that represents this situation is $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$

b) To determine the total area shaded, you need to determine the sum of all the shaded regions within the largest square.

For this series,

First term $t_1 = \frac{1}{4}$

Common ratio $r = \frac{1}{4}$

Use the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}, \text{ where } -1 < r < 1,$$

$$S_\infty = \frac{\frac{1}{4}}{1 - \frac{1}{4}}$$

$$S_\infty = \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$S_\infty = \left(\frac{1}{4}\right)\left(\frac{4}{3}\right)$$

$$S_\infty = \frac{1}{3}$$

A total area of $\frac{1}{3}$ of the largest square is shaded.

Your Turn

You can express $0.\overline{584}$ as an infinite geometric series.

$0.\overline{584} = 0.584\ 584\ 584\ldots$

$\quad = 0.584 + 0.000\ 584 + 0.000\ 000\ 584 + \cdots$

Determine the sum of the series.

Key Ideas

- An infinite geometric series is a geometric series that has an infinite number of terms; that is, the series has no last term.
- An infinite series is said to be convergent if its sequence of partial sums approaches a finite number. This number is the sum of the infinite series. An infinite series that is not convergent is said to be divergent.
- An infinite geometric series has a sum when $-1 < r < 1$ and the sum is given by $S_\infty = \dfrac{t_1}{1 - r}$.

Check Your Understanding

Practise

1. State whether each infinite geometric series is convergent or divergent.

a) $t_1 = -3$, $r = 4$

b) $t_1 = 4$, $r = -\dfrac{1}{4}$

c) $125 + 25 + 5 + \cdots$

d) $(-2) + (-4) + (-8) + \cdots$

e) $\dfrac{243}{3125} - \dfrac{81}{625} + \dfrac{27}{25} - \dfrac{9}{5} + \cdots$

2. Determine the sum of each infinite geometric series, if it exists.

a) $t_1 = 8$, $r = -\dfrac{1}{4}$

b) $t_1 = 3$, $r = \dfrac{4}{3}$

c) $t_1 = 5$, $r = 1$

d) $1 + 0.5 + 0.25 + \cdots$

e) $4 - \dfrac{12}{5} + \dfrac{36}{25} - \dfrac{108}{125} + \cdots$

3. Express each of the following as an infinite geometric series. Determine the sum of the series.

a) $0.\overline{87}$

b) $0.\overline{437}$

4. Does $0.999\ldots = 1$? Support your answer.

5. What is the sum of each infinite geometric series?

a) $5 + 5\left(\dfrac{2}{3}\right) + 5\left(\dfrac{2}{3}\right)^2 + 5\left(\dfrac{2}{3}\right)^3 + \cdots$

b) $1 + \left(-\dfrac{1}{4}\right) + \left(-\dfrac{1}{4}\right)^2 + \left(-\dfrac{1}{4}\right)^3 + \cdots$

c) $7 + 7\left(\dfrac{1}{2}\right) + 7\left(\dfrac{1}{2}\right)^2 + 7\left(\dfrac{1}{2}\right)^3 + \cdots$

Apply

6. The sum of an infinite geometric series is 81, and its common ratio is $\dfrac{2}{3}$. What is the value of the first term? Write the first three terms of the series.

7. The first term of an infinite geometric series is -8, and its sum is $-\dfrac{40}{3}$. What is the common ratio? Write the first four terms of the series.

8. In its first month, an oil well near Virden, Manitoba produced 24 000 barrels of crude. Every month after that, it produced 94% of the previous month's production.

a) If this trend continued, what would be the lifetime production of this well?

b) What assumption are you making? Is your assumption reasonable?

9. The infinite series given by $1 + 3x + 9x^2 + 27x^3 + \cdots$ has a sum of 4. What is the value of x? List the first four terms of the series.

10. The sum of an infinite series is twice its first term. Determine the value of the common ratio.

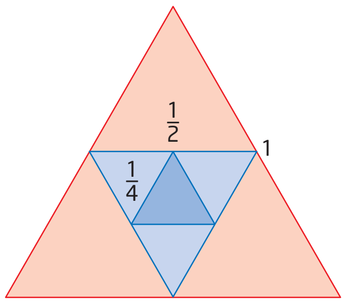
11. Each of the following represents an infinite geometric series. For what values of x will each series be convergent?

a) $5 + 5x + 5x^2 + 5x^3 + \cdots$

b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots$

c) $2 + 4x + 8x^2 + 16x^3 + \cdots$

12. Each side of an equilateral triangle has length of 1 cm. The midpoints of the sides are joined to form an inscribed equilateral triangle. Then, the midpoints of the sides of that triangle are joined to form another triangle. If this process continues forever, what is the sum of the perimeters of the triangles?



13. The length of the initial swing of a pendulum is 50 cm. Each successive swing is 0.8 times the length of the previous swing. If this process continues forever, how far will the pendulum swing?

14. Andrew uses the formula for the sum of an infinite geometric series to evaluate $1 + 1.1 + 1.21 + 1.331 + \cdots$. He calculates the sum of the series to be 10. Is Andrew's answer reasonable? Explain.

15. A ball is dropped from a height of 16 m. The ball rebounds to one half of its previous height each time it bounces. If the ball keeps bouncing, what is the total vertical distance the ball travels?

16. A pile driver pounds a metal post into the ground. With the first impact, the post moves 30 cm; with the second impact it moves 27 cm. Predict the total distance that the post will be driven into the ground if

a) the distances form a geometric sequence and the post is pounded 8 times

b) the distances form a geometric sequence and the post is pounded indefinitely

17. Dominique and Rita are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \cdots$. Dominique says that the sum of the series is $-\frac{1}{7}$. Rita says that the series is divergent and has no sum.

a) Who is correct?

b) Explain your reasoning.

18. A hot air balloon rises 25 m in its first minute of flight. Suppose that in each succeeding minute the balloon rises only 80% as high as in the previous minute. What would be the balloon's maximum altitude?

Hot air balloon rising over Calgary.

Extend

19. A square piece of paper with a side length of 24 cm is cut into four small squares, each with side lengths of 12 cm. Three of these squares are placed side by side. The remaining square is cut into four smaller squares, each with side lengths of 6 cm. Three of these squares are placed side by side with the bigger squares. The fourth square is cut into four smaller squares and three of these squares are placed side by side with the bigger squares. Suppose this process continues indefinitely. What is the length of the arrangement of squares?

20. The sum of the series
$0.98 + 0.98^2 + 0.98^3 + \cdots + 0.98^n = 49$.
The sum of the series
$0.02 + 0.0004 + 0.000\,008 + \cdots = \frac{1}{49}$.
The common ratio in the first series is 0.98 and the common ratio in the second series is 0.02. The sum of these ratios is equal to 1. Suppose that $\frac{1}{z} = x + x^2 + x^3 + \cdots$, where z is an integer and $x = \frac{1}{z + 1}$.

a) Create another pair of series that would follow this pattern, where the sum of the common ratios of the two series is 1.

b) Determine the sum of each series using the formula for the sum of an infinite series.

Create Connections

21. Under what circumstances will an infinite geometric series converge?

22. The first two terms of a series are 1 and $\frac{1}{4}$. Determine a formula for the sum of n terms if the series is

a) an arithmetic series

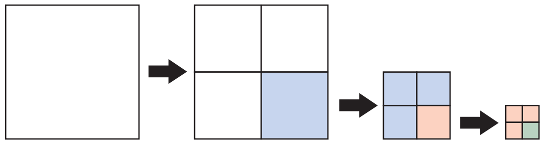
b) a geometric series

c) an infinite geometric series

23. **MINI LAB** Work in a group of three.

Step 1 Begin with a large sheet of graph paper and draw a square. Assume that the area of this square is 1.

Step 2 Cut the square into 4 equal parts. Distribute one part to each member of your group. Cut the remaining part into 4 equal parts. Again distribute one part to each group member. Subdivide the remaining part into 4 equal parts. Suppose you could continue this pattern indefinitely.



Step 3 Write a sequence for the fraction of the original square that each student received at each stage.

n	0	1	2	3	4
Fraction of Paper	0				

Step 4 Write the total area of paper each student has as a series of partial sums. What do you expect the sum to be?

Project Corner — Petroleum

- The Athabasca Oil Sands have estimated oil reserves in excess of that of the rest of the world. These reserves are estimated to be 1.6 trillion barrels.
- Canada is the seventh largest oil producing country in the world. In 2008, Canada produced an average of 438 000 m^3 per day of crude oil, crude bitumen, and natural gas.
- As Alberta's reserves of light crude oil began to deplete, so did production. By 1997, Alberta's light crude oil production totalled 37.3 million cubic metres. This production has continued to decline each year since, falling to just over half of its 1990 total at 21.7 million cubic metres in 2005.

Chapter 1 Review

1.1 Arithmetic Sequences, pages 6–21

1. Determine whether each of the following sequences is arithmetic. If it is arithmetic, state the common difference.

a) 36, 40, 44, 48, ...

b) −35, −40, −45, −50, ...

c) 1, 2, 4, 8, ...

d) 8.3, 4.3, 0.3, −3, −3.7, ...

2. Match the equation for the nth term of an arithmetic sequence to the correct sequence.

a) 18, 30, 42, 54, 66, ...	**A** $t_n = 3n + 1$
b) 7, 12, 17, 22, ...	**B** $t_n = -4(n + 1)$
c) 2, 4, 6, 8, ...	**C** $t_n = 12n + 6$
d) −8, −12, −16, −20, ...	**D** $t_n = 5n + 2$
e) 4, 7, 10, 13, ...	**E** $t_n = 2n$

3. Consider the sequence 7, 14, 21, 28, ... Determine whether each of the following numbers is a term of this sequence. Justify your answer. If the number is a term of the sequence, determine the value of n for that term.

a) 98 **b)** 110

c) 378 **d)** 575

4. Two sequences are given:
Sequence 1 is 2, 9, 16, 23, ...
Sequence 2 is 4, 10, 16, 22, ...

a) Which of following statements is correct?

A t_{17} is greater in sequence 1.

B t_{17} is greater in sequence 2.

C t_{17} is equal in both sequences.

b) On a grid, sketch a graph of each sequence. Does the graph support your answers in part a)? Explain.

5. Determine the tenth term of the arithmetic sequence in which the first term is 5 and the fourth term is 17.

6. The Gardiner Dam, located 100 km south of Saskatoon, Saskatchewan, is the largest earth-filled dam in the world. Upon its opening in 1967, engineers discovered that the pressure from Lake Diefenbaker had moved the clay-based structure 200 cm downstream. Since then, the dam has been moving at a rate of 2 cm per year. Determine the distance the dam will have moved downstream by the year 2020.

1.2 Arithmetic Series, pages 22–31

7. Determine the indicated sum for each of the following arithmetic series

a) $6 + 9 + 12 + \cdots (S_{10})$

b) $4.5 + 8 + 11.5 + \cdots (S_{12})$

c) $6 + 3 + 0 + \cdots (S_{10})$

d) $60 + 70 + 80 + \cdots (S_{20})$

8. The sum of the first 12 terms of an arithmetic series is 186, and the 20th term is 83. What is the sum of the first 40 terms?

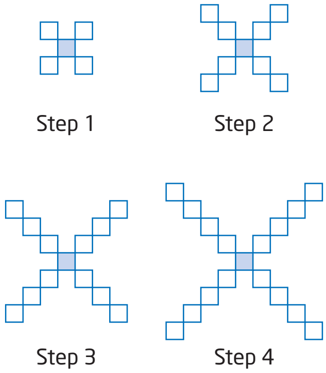
9. You have taken a job that requires being in contact with all the people in your neighbourhood. On the first day, you are able to contact only one person. On the second day, you contact two more people than you did on the first day. On day three, you contact two more people than you did on the previous day. Assume that the pattern continues.

a) How many people would you contact on the 15th day?

b) Determine the total number of people you would have been in contact with by the end of the 15th day.

c) How many days would you need to contact the 625 people in your neighbourhood?

10. A new set of designs is created by the addition of squares to the previous pattern.



a) Determine the total number of squares in the 15th step of this design.

b) Determine the total number of squares required to build all 15 steps.

11. A concert hall has 10 seats in the first row. The second row has 12 seats. If each row has 2 seats more than the row before it and there are 30 rows of seats, how many seats are in the entire concert hall?

1.3 Geometric Sequences, pages 32–45

12. Determine whether each of the following sequences is geometric. If it is geometric, determine the common ratio, r, the first term, t_1, and the general term of the sequence.

a) 3, 6, 10, 15, ...

b) 1, −2, 4, −8, ...

c) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

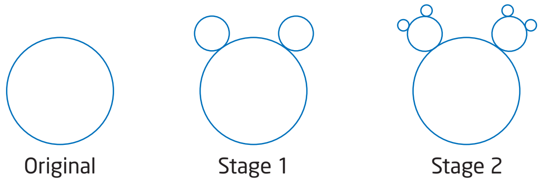
d) $\frac{16}{9}, -\frac{3}{4}, 1, \ldots$

13. A culture initially has 5000 bacteria, and the number increases by 8% every hour.

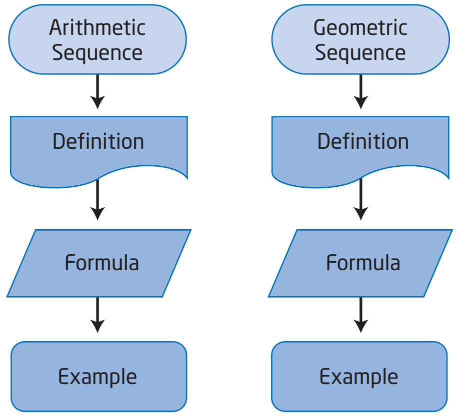
a) How many bacteria are present at the end of 5 h?

b) Determine a formula for the number of bacteria present after n hours.

14. In the Mickey Mouse fractal shown below, the original diagram has a radius of 81 cm. Each successive circle has a radius $\frac{1}{3}$ of the previous radius. What is the circumference of the smallest circle in the 4th stage?



15. Use the following flowcharts to describe what you know about arithmetic and geometric sequences.



1.4 Geometric Series, pages 46–57

16. Decide whether each of the following statements relates to an arithmetic series or a geometric series.

a) A sum of terms in which the difference between consecutive terms is constant.

b) A sum of terms in which the ratio of consecutive terms is constant.

c) $S_n = \frac{t_1(r^n - 1)}{r - 1}, r \neq 1$

d) $S_n = \frac{n[2t_1 + (n - 1)d]}{2}$

e) $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \cdots$

f) $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{2}{27} + \cdots$

17. Determine the sum indicated for each of the following geometric series.

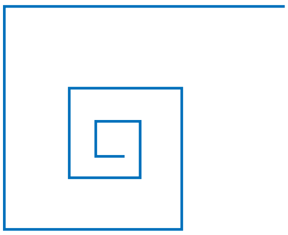
a) $6 + 9 + 13.5 + \cdots$ (S_{10})

b) $18 + 9 + 4.5 + \cdots$ (S_{12})

c) $6000 + 600 + 60 + \cdots$ (S_{20})

d) $80 + 20 + 5 + \cdots$ (S_9)

18. A student programs a computer to draw a series of straight lines with each line beginning at the end of the previous line and at right angles to it. The first line is 4 mm long. Each subsequent line is 25% longer than the previous one, so that a spiral shape is formed as shown.



a) What is the length, in millimetres, of the eighth straight line drawn by the program? Express your answer to the nearest tenth of a millimetre.

b) Determine the total length of the spiral, in metres, when 20 straight lines have been drawn. Express your answer to the nearest hundredth of a metre.

1.5 Infinite Geometric Series, pages 58–65

19. Determine the sum of the each of the following infinite geometric series.

a) $5 + 5\left(\frac{2}{3}\right) + 5\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right)^3 + \cdots$

b) $1 + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^3 + \cdots$

20. For each of the following series, state whether it is convergent or divergent. For those that are convergent, determine the sum.

a) $8 + 4 + 2 + 1 + \cdots$

b) $8 + 12 + 27 + 40.5 + \cdots$

c) $-42 + 21 - 10.5 + 5.25 - \cdots$

d) $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots$

21. Given the infinite geometric series: $7 - 2.8 + 1.12 - 0.448 + \cdots$

a) What is the common ratio, r?

b) Determine S_1, S_2, S_3, S_4, and S_5.

c) What is the particular value that the sums are approaching?

d) What is the sum of the series?

22. Draw four squares adjacent to each other. The first square has a side length of 1 unit, the second has a side length of $\frac{1}{2}$ unit, the third has a side length of $\frac{1}{4}$ unit, and the fourth has a side length of $\frac{1}{8}$ unit.

a) Calculate the area of each square. Do the areas form a geometric sequence? Justify your answer.

b) What is the total area of the four squares?

c) If the process of adding squares with half the side length of the previous square continued indefinitely, what would the total area of all the squares be?

23. a) Copy and complete each of the following statements.

- A series is geometric if there is a common ratio r such that ____.
- An infinite geometric series converges if ____.
- An infinite geometric series diverges if ____.

b) Give two examples of convergent infinite geometric series one with positive common ratio and one with negative common ratio. Determine the sum of each of your series.

Chapter 1 Practice Test

Multiple Choice

For # 1 to #5, choose the best answer.

1. What are the missing terms of the arithmetic sequence ■, 3, 9, ■, ■?

A 1, 27, 81 **B** 9, 3, 9

C −6, 12, 17 **D** −3, 15, 21

2. Marc has set up in his father's grocery store a display of cans as shown in the diagram. The top row (Row 1) has 1 can and each successive row has 3 more cans than the previous row. Which expression would represent the number of cans in row n?

A $S_n = 3n + 1$ **B** $t_n = 3n - 2$

C $t_n = 3n + 2$ **D** $S_n = 3n - 3$

3. What is the sum of the first five terms of the geometric series 16 807 − 2401 + 343 − ⋯?

A 19 607 **B** 14 707

C 16 807.29 **D** 14 706.25

4. The numbers represented by a, b and c are the first three terms of an arithmetic sequence. The number c, when expressed in terms of a and b, would be represented by

A $a + b$ **B** $2b - a$

C $a + (n - 1)b$ **D** $2a + b$

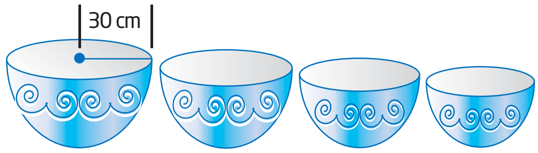
5. The 20th term of a geometric sequence is 524 288 and the 14th term is 8192. The value of the third term could be

A 4 only **B** 8 only

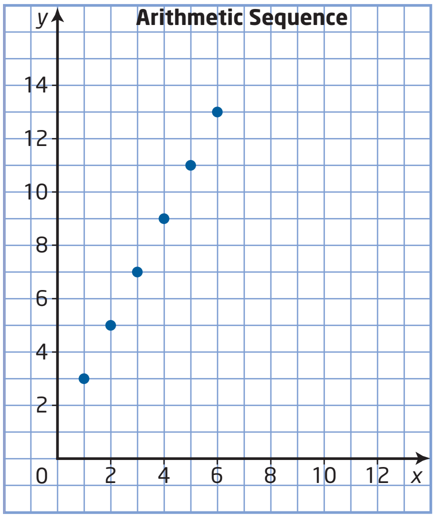
C +4 or −4 **D** +8 or −8

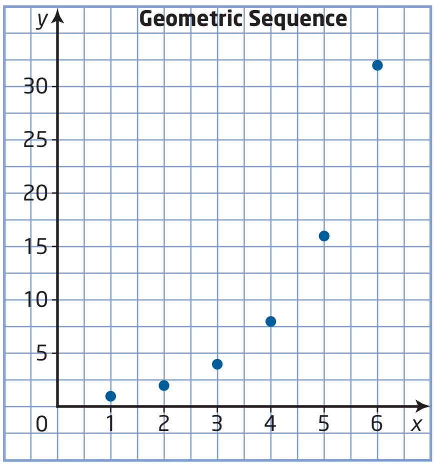
Short Answer

6. A set of hemispherical bowls are made so they can be nested for easy storage. The largest bowl has a radius of 30 cm and each successive bowl has a radius 90% of the preceding one. What is the radius of the tenth bowl?



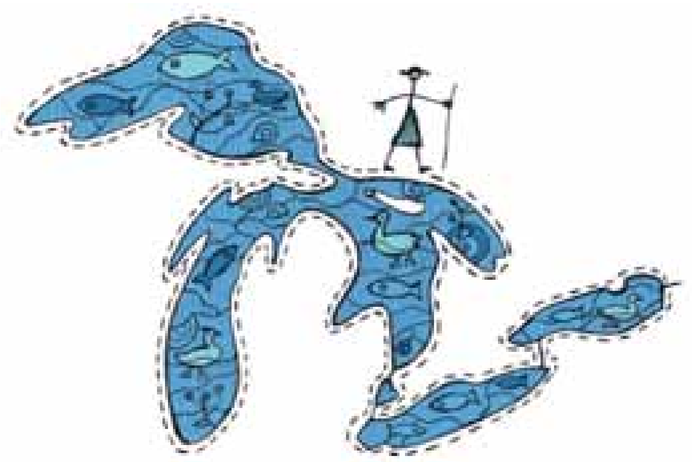
7. Use the following graphs to compare and contrast an arithmetic and a geometric sequence.





8. If 3, A, 27 is an arithmetic sequence and 3, B, 27 is a geometric sequence where $B > 0$, then what are the values of A and B?

9. Josephine Mandamin, an Anishinabe elder from Thunder Bay, Ontario, set out to walk around the Great Lakes to raise awareness about the quality of water in the lakes. In six years, she walked 17 000 km. If Josephine increased the number of kilometres walked per week by 2% every week, how many kilometres did she walk in the first week?



10. Consider the sequence 5, ■, ■, ■, ■, 160.

a) Assume the sequence is arithmetic. Determine the unknown terms of the sequence.

b) What is the general term of the arithmetic sequence?

c) Assume the sequence is geometric. Determine the unknown terms of the sequence.

d) What is the general term of the geometric sequence?

Extended Response

11. Scientists have been measuring the continental drift between Europe and North America for about 25 years. The data collected show that the continents are moving apart at a steady rate of about 17 mm per year.

a) According to the Pangaea theory, Europe and North America were connected at one time. Assuming this theory is correct, write an arithmetic sequence that describes how far apart the continents were at the end of each of the first five years after separation.

b) Determine the general term that describes the arithmetic sequence.

c) Approximately how many years did it take to separate to the current distance of 6000 km?

d) What assumptions did you make in part c)?

12. Photodynamic therapy is used in patients with certain types of disease. A doctor injects a patient with a drug that is attracted to the diseased cells. The diseased cells are then exposed to red light from a laser. This procedure targets and destroys diseased cells while limiting damage to surrounding healthy tissue. The drug remains in the normal cells of the body and must be bleached out by exposure to the sun. A patient must be exposed to the sun for 30 s on the first day, and then increase the exposure by 30 s every day until a total of 30 min is reached.

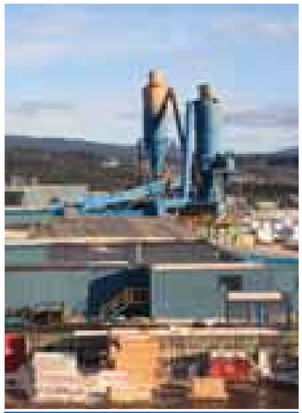
a) Write the first five terms of the sequence of sun exposure times.

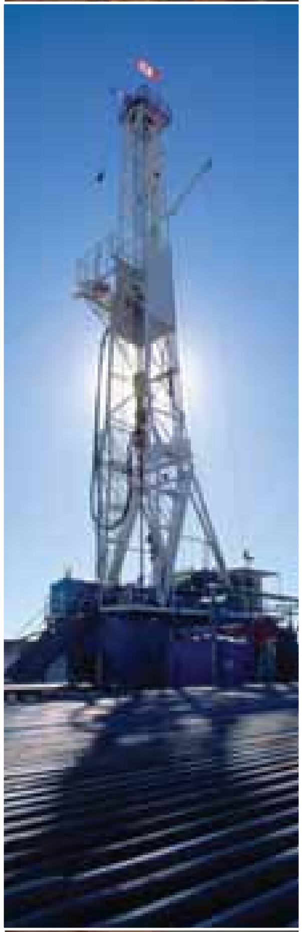
b) Is the sequence arithmetic or geometric?

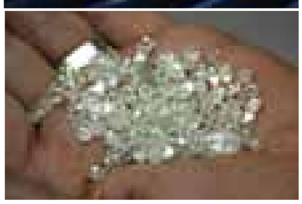
c) How many days are required to reach the goal of 30 min of exposure to the sun?

d) What is the total number of minutes of sun exposure when a patient reaches the 30 min goal?

Canada's Natural Resources







Canada is the source of more than 60 mineral commodities, including metals, non-metals, structural materials, and mineral fuels.

Quarrying and mining are among the oldest industries in Canada. In 1672, coal was discovered on Cape Breton Island.

In the 1850s, gold discoveries in British Columbia, oil finds in Ontario, and increased production of Cape Breton coal marked a turning point in Canadian mineral history.

In 1896, gold was found in the Klondike District of what became Yukon Territory, giving rise to one of the world's most spectacular gold rushes.

In the late 1800s, large deposits of coal and oil sands were evident in part of the North-West Territories that later became Alberta.

In the post-war era there were many major mineral discoveries: deposits of nickel in Manitoba; zinc-lead, copper, and molybdenum in British Columbia; and base metals and asbestos in Québec, Ontario, Manitoba, Newfoundland, Yukon Territory, and British Columbia.

The discovery of the famous Leduc oil field in Alberta in 1947 was followed by a great expansion of Canada's petroleum industry.

In the late 1940s and early 1950s, uranium was discovered in Saskatchewan and Ontario. In fact, Canada is now the world's largest uranium producer.

Canada's first diamond-mining operation began production in October 1998 at the Ekati mine in Lac de Gras, Northwest Territories, followed by the Diavik mine in 2002.

Chapter 1 Task

Choose a natural resource that you would like to research. You may wish to look at some of the information presented in the Project Corner boxes throughout Chapter 1 for ideas. Research your chosen resource.

- List interesting facts about your chosen resource, including what it is, how it is produced, where it is exported, how much is exported, and so on.
- Look for data that would support using a sequence or series in discussing or describing your resource. List the terms for the sequence or series you include.
- Use the information you have gathered in a sequence or series to predict possible trends in the use or production of the resource over a ten-year period.
- Describe any effects the production of the natural resource has on the community.

Unit 1 End Matter

Unit 1 consists of Chapters 1–3.

To preview chapters 2 and 3 please visit
www.westernmath.ca

Unit 1 Project

Canada's Natural Resources

The emphasis of the Chapter 2 Task is the location of your resource. You will describe the route of discovery of the resource and the planned area of the resource.

Chapter 2 Task

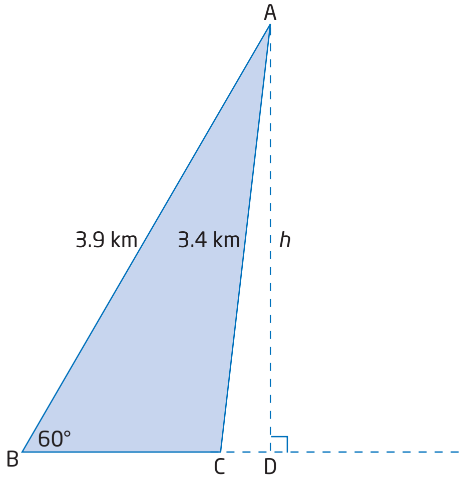
The Journey to Locate the Resource

- Use the map provided. Include a brief log of the journey leading to your discovery. The exploration map is the route that you followed to discover your chosen resource.
- With your exploration map, determine the total distance of your route, to the nearest tenth of a kilometre. Begin your journey at point A and conclude at point J. Include the height of the Sawback Ridge and the width of Crow River in your calculations.

Developing the Area of Your Planned Resource

- Your job as a resource development officer for the company is to present a possible area of development. You are restricted by land boundaries to the triangular shape shown, with side AB of 3.9 km, side AC of 3.4 km, and $\angle B = 60°$.
- Determine all measures of the triangular region that your company could develop.

Possible Proposed Development



Web Link

To obtain a copy of an exploration map, go to www.mhrprecalc11.ca and follow the links.

Unit 1 Project Wrap-Up

Canada's Natural Resources

You need investment capital to develop your resource. Prepare a presentation to make to your investors to encourage them to invest in your project. You can use a written or visual presentation, a brochure, a video production, a computer slide show presentation, or an interactive whiteboard presentation.

Your presentation should include the following:

- Actual data taken from Canadian sources on the production of your chosen resource. Use sequences and series to show how production has increased or decreased over time, and to predict future production and sales.
- A fictitious account of a recent discovery of your resource, including a map of the area showing the accompanying distances.
- A proposal for how the resource area will be developed over the next few years

Cumulative Review, Chapters 1–2

Chapter 1 Sequences and Series

1. Match each term to the correct expression.

a) arithmetic sequence

b) geometric sequence

c) arithmetic series

d) geometric series

e) convergent series

A 3, 7, 11, 15, 19, ...

B $5 + 1 + \frac{1}{5} + \frac{1}{25} + \cdots$

C $1 + 2 + 4 + 8 + 16 + \cdots$

D 1, 3, 9, 27, 81, ...

E $2 + 5 + 8 + 11 + 14 + \cdots$

2. Classify each sequence as arithmetic or geometric. State the value of the common difference or common ratio. Then, write the next three terms in each sequence.

a) 27, 18, 12, 8, ...

b) 17, 14, 11, 8, ...

c) $-21, -16, -11, -6, \ldots$

d) $3, -6, 12, -24, \ldots$

3. For each arithmetic sequence, determine the general term. Express your answer in simplified form.

a) 18, 15, 12, 9, ...

b) $1, \frac{5}{2}, 4, \frac{11}{2}, \ldots$

4. Use the general term to determine t_{20} in the geometric sequence $2, -4, 8, -16, \ldots$.

5. **a)** What is S_{12} for the arithmetic series with a common difference of 3 and $t_{12} = 31$?

b) What is S_5 for a geometric series where $t_1 = 4$ and $t_{10} = 78\ 732$?

6. Phytoplankton, or algae, is a nutritional supplement used in natural health programs. Canadian Pacific Phytoplankton Ltd. is located in Nanaimo, British Columbia. The company can grow 10 t of marine phytoplankton on a regular 11-day cycle. Assume this cycle continues.

a) Create a graph showing the amount of phytoplankton produced for the first five cycles of production.

b) Write the general term for the sequence produced.

c) How does the general term relate to the characteristics of the linear function described by the graph?

7. The Living Shangri-La is the tallest building in Metro Vancouver. The ground floor of the building is 5.8 m high, and each floor above the ground floor is 3.2 m high. There are 62 floors altogether, including the ground floor. How tall is the building?

8. Tristan and Julie are preparing a math display for the school open house. Both students create posters to debate the following question:

Does 0.999... = 1?

Julie's Poster

0.999... ≠ 1
0.999... = 0.999 999 999 999 9...

The decimal will continue to infinity and will never reach exactly one.

Tristan's Poster

0.999... = 1
Rewrite 0.999... in expanded form.

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

This can be written as a geometric series where $t_1 = \frac{9}{10}$ and $r =$

a) Finish Tristan's poster by determining the value of the common ratio and then finding the sum of the infinite geometric series.

b) Which student do you think correctly answered the question?

Chapter 2 Trigonometry

9. Determine the exact distance, in simplified form, from the origin to a point P(−2, 4) on the terminal arm of an angle.

10. Point P(15, 8) is on the terminal arm of angle θ. Determine the exact values for sin θ, cos θ, and tan θ.

11. Sketch each angle in standard position and determine the measure of the reference angle.

a) 40°

b) 120°

c) 225°

d) 300°

12. The clock tower on the post office in Battleford, Saskatchewan, demonstrates the distinctive Romanesque Revival style of public buildings designed in the early decades of the twentieth century. The Battleford Post Office is similar in design to the post offices in Humboldt and Melfort. These three buildings are the only surviving post offices of this type in the Prairie Provinces.

a) What is the measure of the angle, θ, created between the hands of the clock when it reads 3 o'clock?

b) If the length of the minute hand is 2 ft, sketch a diagram to represent the clock face on a coordinate grid with the centre at the origin. Label the coordinates represented by the tip of the minute hand.

c) What are the exact values for sin θ, cos θ, and tan θ at 3 o'clock?

Battleford post office

13. Determine the exact value of each trigonometric ratio.

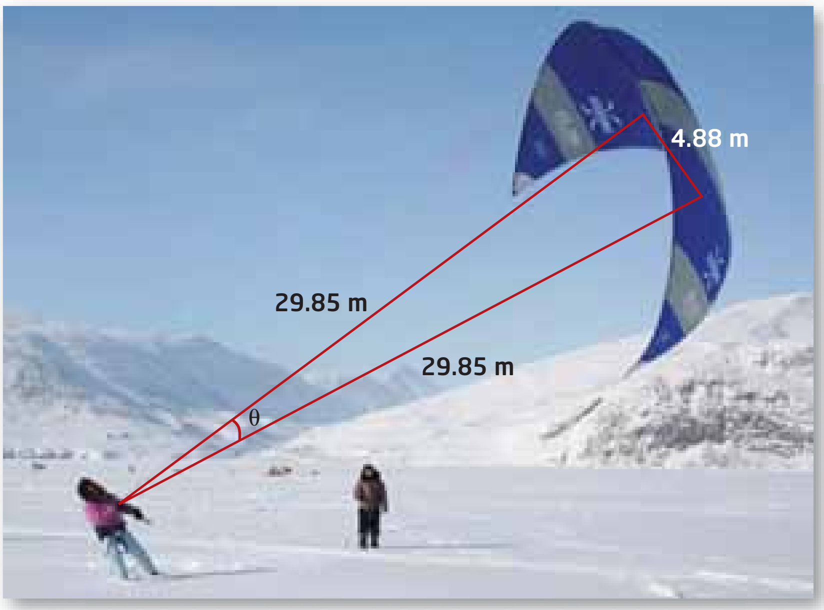
a) sin 405° **b)** cos 330°

c) tan 225° **d)** cos 180°

e) tan 150° **f)** sin 270°

14. Radio collars are used to track polar bears by sending signals via GPS to receiving stations. Two receiving stations are 9 km apart along a straight road. At station A, the signal from one of the collars comes from a direction of 49° from the road. At station B, the signal from the same collar comes from a direction of 65° from the road. Determine the distance the polar bear is from each of the stations.

15. The Arctic Wind Riders is a program developed to introduce youth in the communities of Northern Canada to the unique sport of Paraski. This sport allows participants to sail over frozen bays, rivers, and snowy tundra using wind power. The program has been offered to close to 700 students and young adults. A typical Paraski is shown below. Determine the measure of angle θ, to the nearest tenth of a degree.



Clara Akulukjuk, Pangnirtung, Nunavut, learning to Paraski.

16. Waterton Lakes National Park in Alberta is a popular site for birdwatching, with over 250 species of birds recorded. Chelsea spots a rare pileated woodpecker in a tree at an angle of elevation of 52°. After walking 16 m closer to the tree, she determines the new angle of elevation to be 70°.

a) Sketch and label a diagram to represent the situation.

b) What is the closest distance that Chelsea is from the bird, to the nearest tenth of a metre?

17. In △RST, RT = 2 m, ST = 1.4 m, and ∠R = 30°. Determine the measure of obtuse ∠S to the nearest tenth of a degree.

Unit 1 Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Which of the following expressions could represent the general term of the sequence 8, 4, 0, …?

A $t_n = 8 + (n - 1)4$

B $t_n = 8 - (n - 1)4$

C $t_n = 4n + 4$

D $t_n = 8(-2)^{n-1}$

2. The expression for the 14th term of the geometric sequence x, x^3, x^5, … is

A x^{13}

B x^{14}

C x^{27}

D x^{29}

3. The sum of the series $6 + 18 + 54 + \cdots$ to n terms is 2184. How many terms are in the series?

A 5

B 7

C 8

D 6

4. Which angle has a reference angle of 55°?

A 35°

B 135°

C 235°

D 255°

5. Given the point $P(x, \sqrt{5})$ on the terminal arm of angle θ, where $\sin \theta = \frac{\sqrt{5}}{5}$ and $90° \le \theta \le 180°$, what is the exact value of $\cos \theta$?

A $\frac{3}{5}$

B $-\frac{3}{\sqrt{5}}$

C $\frac{2}{\sqrt{5}}$

D $-\frac{2\sqrt{5}}{5}$

Numerical Response

Complete the statements in #6 to #8.

6. A coffee shop is holding its annual fundraiser to help send a local child to summer camp. The coffee shop plans to donate a portion of the profit for every cup of coffee served. At the beginning of the day, the owner buys the first cup of coffee and donates \$20 to the fundraiser. If the coffee shop regularly serves another 2200 cups of coffee in one day, they must collect \$■ per cup to raise \$350.

7. An angle of 315° drawn in standard position has a reference angle of ■°.

8. The terminal arm of an angle, θ, in standard position lies in quadrant IV, and it is known that $\sin \theta = -\frac{\sqrt{3}}{2}$. The measure of θ is ■.

Written Response

9. Jacques Chenier is one of Manitoba's premier children's entertainers. Jacques was a Juno Award Nominee for his album *Walking in the Sun*. He has performed in over 600 school fairs and festivals across the country. Suppose there were 150 people in the audience for his first performance. If this number increased by 5 for each of the next 14 performances, what total number of people attended the first 15 of Jacques Chenier's performances?

10. The third term in an arithmetic sequence is 4 and the seventh term in the sequence is 24.

a) Determine the value of the common difference.

b) What is the value of t_1?

c) Write the general term of the sequence.

d) What is the sum of the first 10 terms of the sequence?

11. A new car that is worth $35 000 depreciates 20% in the first year and 10% every year after that. About how much will the car be worth 7 years after it is purchased?

12. The Multiple Sclerosis Walk is a significant contributor to the Multiple Sclerosis Society's fundraising efforts to support research. One walker was sponsored $100 plus $5 for the first kilometre, $10 for the second kilometre, $15 for the third kilometre, and so on. How far would this walker need to walk to earn $150?

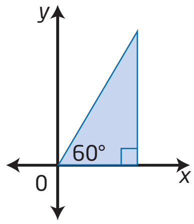
13. Les Jeux de la Francophonie Canadienne are held each summer to celebrate sport, leadership, and French culture. Badminton is one of the popular events at the games. There are 64 entrants in the boys' singles tournament. There are two players per game, and only the winner advances to the next round. The number of players in each round models a geometric sequence.

a) Write the first four terms of the geometric sequence.

b) Write the general term that could be used to determine the number of players in any round of the tournament.

c) How many games must be played altogether to determine the winner of the boys' singles tournament?

14. A right triangle with a reference angle of 60° is drawn in standard position on a coordinate grid.

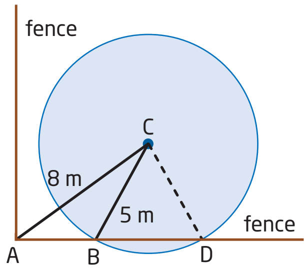


a) Apply consecutive rotations of 60° counterclockwise to complete one 360° revolution about the origin.

b) Write the sequence that represents the measures of the angles in standard position formed by the rotations.

c) Write the general term for the sequence.

15. A circular water sprinkler in a backyard sprays a radius of 5 m. The sprinkler is placed 8 m from the corner of the lot.



a) If the measure of ∠CAB is 32°, what is the measure of ∠CDA?

b) What length of fence, to the nearest tenth of a metre, would get wet from the sprinkler?

16. A triangle has sides that measure 61 cm, 38 cm, and 43 cm. Determine the measure of the smallest angle in the triangle.

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