

# INDEX MODELS AND THE ARBITRAGE PRICING THEORY

The exploitation of security mispricing in such a way that risk-free economic profits may be earned is called **arbitrage**. It typically involves the simultaneous purchase and sale of equivalent securities (usually in different markets) in order to profit from discrepancies in their price relationship. The concept of arbitrage is central to the theory of capital markets. This chapter discusses the nature, and illustrates the use, of arbitrage. We show how to identify arbitrage opportunities and why investors will take as large a position as they can in arbitrage portfolios.

Perhaps the most basic principle of capital market theory is that equilibrium market prices are rational in that they rule out risk-free arbitrage opportunities. Pricing relationships that guarantee the absence of risk-free arbitrage possibilities are extremely powerful. If actual security prices allow for risk-free arbitrage, the result will be strong pressure on security prices to restore equilibrium. Only a few investors need be aware of arbitrage opportunities to

bring about a large volume of trades, and these trades will bring prices back into balance.

The CAPM of the previous chapter gave us the security market line, a relationship between expected return and risk as measured by beta. The model discussed in this chapter, called the arbitrage pricing theory, or APT, also stipulates a relationship between expected return and risk, but it uses different assumptions and techniques. We explore



this relationship using well-diversified portfolios, showing that these portfolios are priced to satisfy the CAPM expected return–beta relationship. Because all well-diversified portfolios have to satisfy that relationship, we show that all individual securities almost certainly satisfy this same relationship. This reasoning allows the derivation of an SML relationship that avoids reliance on the unobservable, theoretical market portfolio that is central to the CAPM. We also show how the simple single-factor APT easily can be generalized to a richer multifactor version.

The single-factor APT is based on the assumption that only one systematic common factor affects the returns of all securities. This assumption, however, was also at the origin of another class of models, known as *index* or *market models*, that predate the APT by several years. These models are initially introduced in order to simplify the computations of the Markowitz portfolio selection model. Since they also offer significant new insights into the nature of systematic risk versus firm-specific risk and constitute a good introduction to the concept of factor models of security returns, they will be examined in the first sections of this chapter.



## 8.1 A SINGLE-INDEX SECURITY MARKET

### Systematic Risk Versus Firm-Specific Risk

The success of a portfolio selection rule depends on the quality of the input list, that is, the estimates of expected security returns and the covariance matrix. In the long run, efficient portfolios will beat portfolios with less reliable input lists and consequently inferior reward-to-risk tradeoffs.

Suppose your security analysts can thoroughly analyze 50 stocks. This means that your input list will include the following:

$$\begin{aligned} n &= 50 \text{ estimates of expected returns} \\ n &= 50 \text{ estimates of variances} \\ (n^2 - n)/2 &= 1,225 \text{ estimates of covariances} \\ &1,325 \text{ estimates} \end{aligned}$$

This is a formidable task, particularly in the light of the fact that a 50-security portfolio is relatively small. Doubling  $n$  to 100 will nearly quadruple the number of estimates to 5,150. If  $n = 1,600$ , roughly the number of TSE-listed stocks in 1990, we need nearly 1.3 million estimates.

Another difficulty in applying the Markowitz model to portfolio optimization is that errors in the assessment or estimation of correlation coefficients can lead to nonsensical results. This can happen because some sets of correlation coefficients are mutually inconsistent, as the following example demonstrates:<sup>1</sup>

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<sup>1</sup>We are grateful to Andy Kaplin for this example.