SUPPLEMENT TO CHAPTER SIX

Linear Programming

SUPPLEMENT OUTLINE

LEARNING OBJECTIVES

After completing this supplement, you should be able to:

1. Describe the type of problem that would lend itself to solution using linear programming.
2. Formulate a linear programming model from a description of a problem.
3. Solve simple linear programming problems using the graphical method.
4. Interpret computer solutions of linear programming problems.
5. Do sensitivity analysis on the solution of a linear programming problem.

Linear programming is a powerful quantitative tool used by operations managers and other managers to obtain optimal solutions to problems that involve restrictions or limitations, such as the available materials, budgets, and labour and machine time. These problems are referred to as *constrained optimization* problems. There are numerous examples of linear programming applications to such problems, including:

- Establishing locations for emergency equipment and personnel that will minimize response time
- Determining optimal schedules for airlines for planes, pilots, and ground personnel
- Developing financial plans
- Determining optimal blends of animal feed mixes
- Determining optimal diet plans
- Identifying the best set of worker–job assignments
- Developing optimal production schedules
- Developing shipping plans that will minimize shipping costs
- Identifying the optimal mix of products in a factory

Introduction

Linear programming (LP) techniques consist of a sequence of steps that will lead to an optimal solution to problems, in cases where an optimum exists. There are a number of different linear programming techniques; some are special-purpose (i.e., used to find solutions for specific types of problems) and others are more general in scope. This supplement covers the two general-purpose solution techniques: graphical linear programming and computer solutions. Graphical linear programming provides a visual portrayal of many of the important concepts of linear programming. However, it is limited to problems with only two variables. In practice, computers are used to obtain solutions for problems, some of which involve a large number of variables.

Linear Programming Models

Linear programming models are mathematical representations of constrained optimization problems. These models have certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that can be solved using linear programming. In addition, it also can help us formulate LP models. The characteristics can be grouped into two categories: components and assumptions. First, let's consider the components.

Four components provide the structure of a linear programming model:

1. Objective.
2. Decision variables.
3. Constraints.
4. Parameters.

Linear programming algorithms require that a single goal or *objective*, such as the maximization of profits, be specified. The two general types of objectives are maximization and minimization. A maximization objective might involve profits, revenues, efficiency, or rate of return. Conversely, a minimization objective might involve cost, time, distance travelled, or scrap. The **objective function** is a mathematical expression that can be used to determine the total profit (or cost, etc., depending on the objective) for a given solution.

objective function Mathematical statement of profit (or cost, etc.) for a given solution.

Decision variables represent choices available to the decision maker in terms of amounts of either inputs or outputs. For example, some problems require choosing a combination of inputs to minimize total costs, while others require selecting a combination of outputs to maximize profits or revenues.

decision variables Amounts of either inputs or outputs.

Constraints are limitations that restrict the alternatives available to decision makers. The three types of constraints are less than or equal to ($\leq$), greater than or equal to ($\geq$), and simply equal to ($=$). A $\leq$ constraint implies an upper limit on the amount of some scarce resource (e.g., machine hours, labour hours, materials) available for use. A $\geq$ constraint specifies a minimum that must be achieved in the final solution (e.g., must contain at least 10 percent real fruit juice, must get at least 30 km/L on the highway). The $=$ constraint is more restrictive in the sense that it specifies *exactly* what a decision variable should equal (e.g., make 200 units of product A). A linear programming model can consist of one or more constraints. The constraints of a given problem define the set of all feasible combinations of decision variables; this set is referred to as the **feasible solution space**. Linear programming algorithms are designed to search the feasible solution space for the combination of decision variables that will yield an optimum in terms of the objective function.

constraints Limitations that restrict the available alternatives.

feasible solution space The set of all feasible combinations of decision variables as defined by the constraints.

An LP model consists of a mathematical statement of the objective and a mathematical statement of each constraint. These statements consist of symbols (e.g., x_1, x_2) that represent the decision variables and numerical values, called **parameters**. The parameters are fixed values; the model is solved *given* those values.

parameters Numerical constants.

Example S–1 illustrates the components of an LP model.

Example S-1

Decision variables
$$\begin{cases} x_1 = \text{Quantity of product 1 to produce} \\ x_2 = \text{Quantity of product 2 to produce} \\ x_3 = \text{Quantity of product 3 to produce} \end{cases}$$

Maximize	$5x_1 + 8x_2 + 4x_3$ (profit)	(Objective function)
Subject to		
Labour	$2x_1 + 4x_2 + 8x_3 \leq 250$ hours	
Material	$7x_1 + 6x_2 + 5x_3 \leq 100$ kg	(Constraints)
Product 1	$x_1 \geq 10$ units	
	$x_1, x_2, x_3 \geq 0$	(Nonnegativity constraints)

First, the model lists and defines the decision variables. These typically represent *quantities*. In this case, they are quantities of three different products that might be produced.

Next, the model states the objective function. It includes every decision variable in the model and the contribution (profit per unit) of each decision variable. Thus, product x_1 has a profit of \$5 per unit. The profit from product x_1 for a given solution will be 5 times the value of x_1 specified by the solution; the total profit from all products will be the sum of the individual product profits. Thus, if $x_1 = 10$, $x_2 = 0$, and $x_3 = 6$, the value of the objective function would be:

$$5(10) + 8(0) + 4(6) = 74$$

The objective function is followed by a list (in no particular order) of three constraints. Each constraint has a right-side numerical value (e.g., the labour constraint has a right-side value of 250) that indicates the amount of the constraint and a relation sign that indicates whether that amount is a maximum ($\leq$), a minimum ($\geq$), or an equality ($=$). The left side of each constraint consists of the variables subject to that particular constraint and a coefficient for each variable that indicates how much of the right-side quantity *one unit* of the decision variable represents. For instance, for the labour constraint, one unit of x_1 will require two hours of labour. The sum of the values on the left side of each constraint represents the amount of that constraint used by a solution. Thus, if $x_1 = 10$, $x_2 = 0$, and $x_3 = 6$, the amount of labour used would be:

$$2(10) + 4(0) + 8(6) = 68 \text{ hours}$$

Because this amount does not exceed the quantity on the right-hand side of the constraint, it is *feasible.*

Note that the third constraint refers to only a single variable; x_1 must be at least 10 units. Its coefficient is, in effect, 1, although that is not shown.

Finally, there are the nonnegativity constraints. These are listed on a single line; they reflect the condition that no decision variable is allowed to have a negative value.

In order for linear-programming models to be used effectively, certain *assumptions* must be satisfied. These are:

1. *Linearity:* the impact of decision variables is linear in constraints and the objective function.
2. *Divisibility:* noninteger values of decision variables are acceptable.
3. *Certainty:* values of parameters are known and constant.
4. *Nonnegativity:* negative values of decision variables are unacceptable.

Model Formulation

An understanding of the components of linear programming models is necessary for model formulation. This helps provide organization to the process of assembling information about a problem into a model.

Naturally, it is important to obtain valid information on what constraints are appropriate, as well as on what values of the parameters are appropriate. If this is not done, the usefulness of the model will be questionable. Consequently, in some instances, considerable effort must be expended to obtain that information.

In formulating a model, use the format illustrated in Example 1. Begin by identifying the decision variables. Very often, decision variables are "the quantity of" something, such as x_1 = the quantity of product 1. Generally, decision variables have profits, costs, times, or a similar measure of value associated with them. Knowing this can help you identify the decision variables in a problem.

Constraints are restrictions or requirements on one or more decision variables, and they refer to available amounts of resources such as labour, material, or machine time, or to minimal requirements, such as "make at least 10 units of product 1." It can be helpful to give a name to each constraint, such as "labour" or "material 1." Let's consider some of the different kinds of constraints you will encounter.

1. A constraint that refers to one or more decision variables. This is the most common kind of constraint. The constraints in Example 1 are of this type.

2. A constraint that specifies a ratio. For example, "the ratio of x_1 to x_2 must be at least 3 to 2." To formulate this, begin by setting up the ratio:

$$\frac{x_1}{x_2} \geq \frac{3}{2}$$

Then, cross multiply, obtaining

$$2x_1 \geq 3x_2$$

This is not yet in a suitable form because all variables in a constraint must be on the left side of the inequality (or equality) sign, leaving only a constant on the right side. To achieve this, we must subtract the variable amount that is on the right side from both sides. That yields:

$$2x_1 - 3x_2 \geq 0$$

[Note that the direction of the inequality remains the same.]

3. A constraint that specifies a percentage for one or more variables relative to one or more other variables. For example, "x_1 cannot be more than 20 percent of the mix." Suppose that the mix consists of variables x_1, x_2, and x_3. In mathematical terms, this would be:

$$x_1 \leq .20(x_1 + x_2 + x_3)$$

As always, all variables must appear on the left side of the relationship. To accomplish that, we can expand the right side, and then subtract the result from both sides. Thus,

$$x_1 \leq .20x_1 + .20x_2 + .20x_3$$

Subtracting yields

$$.80x_1 - .20x_2 - .20x_3 \leq 0$$

Once you have formulated a model, the next task is to solve it. The following sections describe two approaches to problem solution: graphical solutions and computer solutions.

Graphical Linear Programming

Graphical linear programming is a method for finding optimal solutions to two-variable problems. This section describes that approach.

graphical linear programming Graphical method for finding optimal solutions to two-variable problems.

Outline of Graphical Procedure

The graphical method of linear programming plots the constraints on a graph and identifies an area that satisfies all of the constraints. The area is referred to as the *feasible solution space.* Next, the objective function is plotted and used to identify the optimal point in the feasible solution space. The coordinates of the point can sometimes be read directly from the graph, although generally an algebraic determination of the coordinates of the point is necessary.

The general procedure followed in the graphical approach is:

1. Set up the objective function and the constraints in mathematical format.
2. Plot the constraints.
3. Identify the feasible solution space.
4. Plot the objective function.
5. Determine the optimum solution.

The technique can best be illustrated through solution of a typical problem. Consider the problem described in Example S–2.

Example S–2

General description: A firm that assembles computers and computer equipment is about to start production of two new types of microcomputers. Each type will require assembly time, inspection time, and storage space. The amounts of each of these resources that can be devoted to the production of the microcomputers is limited. The manager of the firm would like to determine the quantity of each microcomputer to produce in order to maximize the profit generated by sales of these microcomputers.

Additional information: In order to develop a suitable model of the problem, the manager has met with design and manufacturing personnel. As a result of those meetings, the manager has obtained the following information:

	Type 1	Type 2
Profit per unit	$60	$50
Assembly time per unit	4 hours	10 hours
Inspection time per unit	2 hours	1 hour
Storage space per unit	3 cubic feet	3 cubic feet

The manager also has acquired information on the availability of company resources. These (daily) amounts are:

Resource	Amount Available
Assembly time	100 hours
Inspection time	22 hours
Storage space	39 cubic feet

The manager met with the firm's marketing manager and learned that demand for the microcomputers was such that whatever combination of these two types of microcomputers is produced, all of the output can be sold.

In terms of meeting the assumptions, it would appear that the relationships are *linear*: The contribution to profit per unit of each type of computer and the time and storage space per unit of each type of computer is the same regardless of the quantity produced. Therefore, the total impact of each type of computer on the profit and each constraint is a linear function of the quantity of that variable. There may be a question of *divisibility* because, presumably, only whole units of computers will be sold. However, because this is a recurring process (i.e., the computers will be produced daily, a noninteger solution such as 3.5 computers per day will result in 7 computers every other day), this does not seem to pose a problem. The question of *certainty* cannot be explored here; in practice, the manager could be questioned to determine if there are any other possible constraints and whether the values shown for assembly times, and so forth, are known with certainty. For the purposes of discussion, we will assume certainty. Last, the assumption of *nonnegativity* seems justified; negative values for production quantities would not make sense.

Because we have concluded that linear programming is appropriate, let us now turn our attention to constructing a model of the microcomputer problem. First, we must define the decision variables. Based on the statement, "The manager … would like to determine the quantity of each microcomputer to produce," the decision variables are the quantities of each type of computer. Thus,

x_1 = quantity of type 1 to produce

x_2 = quantity of type 2 to produce

Next, we can formulate the objective function. The profit per unit of type 1 is listed as $60, and the profit per unit of type 2 is listed as $50, so the appropriate objective function is

$$\text{Maximize } Z = 60x_1 + 50x_2$$

where Z is the value of the objective function, given values of x_1 and x_2. Theoretically, a mathematical function requires such a variable for completeness. However, in practice, the objective function often is written without the Z, as sort of a shorthand version. That approach is underscored by the fact that computer input does not call for Z: it is understood. The output of a computerized model does include a Z, though.

Now for the constraints. There are three resources with limited availability: assembly time, inspection time, and storage space. The fact that availability is limited means that these constraints will all be $\leq$ constraints. Suppose we begin with the assembly constraint. The type 1 microcomputer requires 4 hours of assembly time per unit, whereas the type 2 microcomputer requires 10 hours of assembly time per unit. Therefore, with a limit of 100 hours available, the assembly constraint is

$$4x_1 + 10x_2 \leq 100 \text{ hours}$$

Similarly, each unit of type 1 requires 2 hours of inspection time, and each unit of type 2 requires 1 hour of inspection time. With 22 hours available, the inspection constraint is

$$2x_1 + 1x_2 \leq 22$$

(Note: The coefficient of 1 for x_2 need not be shown. Thus, an alternative form for this constraint is: $2x_1 + x_2 \leq 22$.) The storage constraint is determined in a similar manner:

$$3x_1 + 3x_2 \leq 39$$

There are no other system or individual constraints. The nonnegativity constraints are

$x_1, x_2 \geq 0$

In summary, the mathematical model of the microcomputer problem is

x_1 = quantity of type 1 to produce

x_2 = quantity of type 2 to produce

Maximize $60x_1 + 50x_2$

Subject to

Assembly	$4x_1 + 10x_2 \leq 100$ hours
Inspection	$2x_1 + 1x_2 \leq 22$ hours
Storage	$3x_1 + 3x_2 \leq 39$ cubic feet
	$x_1, x_2 \geq 0$

The next step is to plot the constraints.

Plotting Constraints

Begin by placing the nonnegativity constraints on a graph, as in Figure 6S–1. The procedure for plotting the other constraints is simple:

1. Replace the inequality sign with an equal sign. This transforms the constraint into an *equation of a straight line.*
2. Determine where the line intersects each axis.
   a. To find where it crosses the x_2 axis, set x_1 equal to zero and solve the equation for the value of x_2.
   b. To find where it crosses the x_1 axis, set x_2 equal to zero and solve the equation for the value of x_1.
3. Mark these intersections on the axes, and connect them with a straight line. (Note: If a constraint has only one variable, it will be a vertical line on a graph if the variable is x_1, or a horizontal line if the variable is x_2.)
4. Indicate by shading (or by arrows at the ends of the constraint line) whether the inequality is greater than or less than. (A general rule to determine which side of the line satisfies the inequality is to pick a point that is not on the line, such as 0,0, and see whether it is greater than or less than the constraint amount.)
5. Repeat steps 1–4 for each constraint.

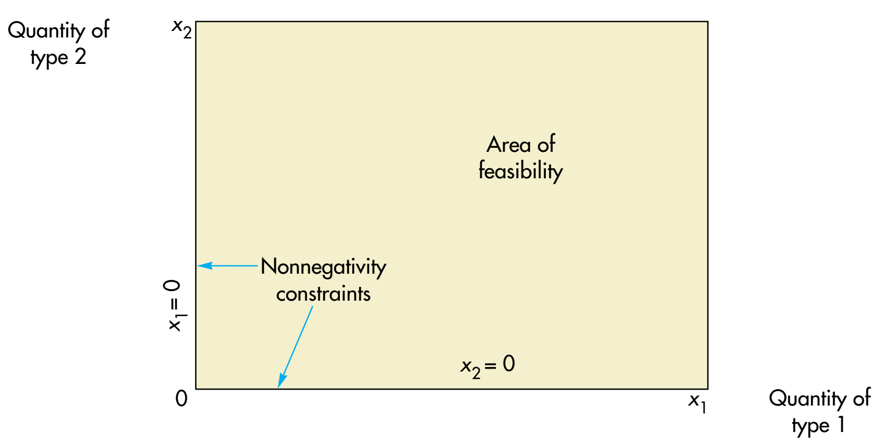
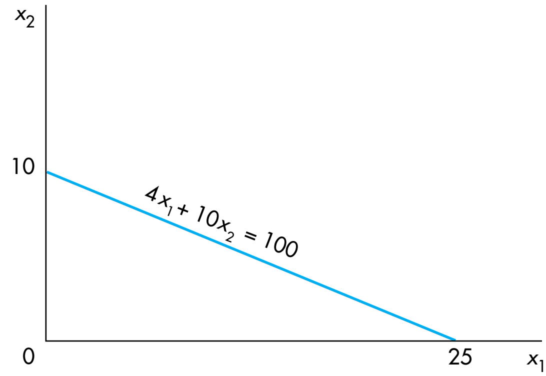


FIGURE 6S–1

Graph showing the nonnegativity constraints

FIGURE 6S–2

Plot of the first constraint (assembly time)



Consider the assembly time constraint:

$4x_1 + 10x_2 \leq 100$

Removing the inequality portion of the constraint produces this straight line:

$4x_1 + 10x_2 = 100$

Next, identify the points where the line intersects each axis, as step 2 describes. Thus with $x_2 = 0$, we find

$4x_1 + 10(0) = 100$

Solving, we find that $4x_1 = 100$, so $x_1 = 25$ when $x_2 = 0$. Similarly, we can solve the equation for x_2 when $x_1 = 0$:

$4(0) + 10x_2 = 100$

Solving for x_2, we find $x_2 = 10$ when $x_1 = 0$.

Thus, we have two points: $x_1 = 0$, $x_2 = 10$, and $x_1 = 25$, $x_2 = 0$. We can now add this line to our graph of the nonnegativity constraints by connecting these two points (see Figure 6S–2).

Next we must determine which side of the line represents points that are less than 100. To do this, we can select a test point that is not on the line, and we can substitute the x_1 and x_2 values of that point into the left side of the equation of the line. If the result is less than 100, this tells us that all points on that side of the line are less than the value of the line (e.g., 100). Conversely, if the result is greater than 100, this indicates that the other side of the line represents the set of points that will yield values that are less than 100. A relatively simple test point to use is the origin (i.e., $x_1 = 0$, $x_2 = 0$). Substituting these values into the equation yields

$4(0) + 10(0) = 0$

Obviously this is less than 100. Hence, the side of the line closest to the origin represents the "less than" area (i.e., the feasible region).

The feasible region for this constraint and the nonnegativity constraints then becomes the shaded portion shown in Figure 6S–3.

For the sake of illustration, suppose we try one other point, say $x_1 = 10$, $x_2 = 10$. Substituting these values into the assembly constraint yields

$4(10) + 10(10) = 140$

Clearly this is greater than 100. Therefore, all points on this side of the line are greater than 100 (see Figure 6S–4).

Continuing with the problem, we can add the two remaining constraints to the graph. For the inspection constraint:

1. Convert the constraint into the equation of a straight line by replacing the inequality sign with an equality sign:

   $2x_1 + 1x_2 \leq 22$ becomes $2x_1 + 1x_2 = 22$

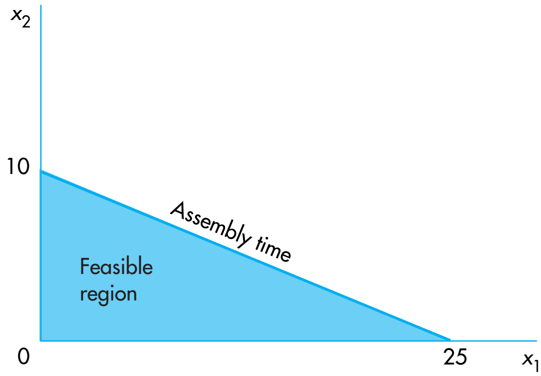


FIGURE 6S–3

The feasible region, given the first constraint and the nonnegativity constraints

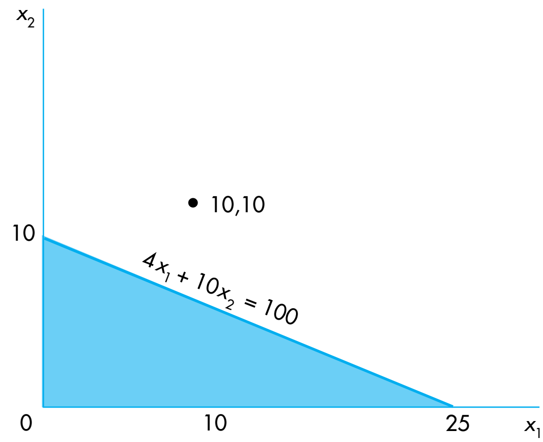


FIGURE 6S–4

The point 10, 10 is above the constraint line

2. Set x_1 equal to zero and solve for x_2:

   $2(0) + 1x_2 = 22$

   Solving, we find $x_2 = 22$. Thus, the line will intersect the x_2 axis at 22.
3. Next, set x_2 equal to zero and solve for x_1:

   $2x_1 + 1(0) = 22$

   Solving, we find $x_1 = 11$. Thus, the other end of the line will intersect the x_1 axis at 11.
4. Add the line to the graph (see Figure 6S–5).

Note that the area of feasibility for this constraint is below the line (Figure 6S–5). Again the area of feasibility at this point is shaded in for illustration, although when graphing problems it is more practical to refrain from shading in the feasible region until all constraint lines have been drawn. However, because constraints are plotted one at a time, using a small arrow at the end of each constraint to indicate the direction of feasibility can be helpful.

The storage constraint is handled in the same manner:

1. Convert it into an equality:

   $3x_1 + 3x_2 = 39$
2. Set x_1 equal to zero and solve for x_2:

   $3(0) + 3x_2 = 39$

   Solving, $x_2 = 13$. Thus, $x_2 = 13$ when $x_1 = 0$.

FIGURE 6S–5

Partially completed graph, showing the assembly, inspection, and nonnegativity constraints

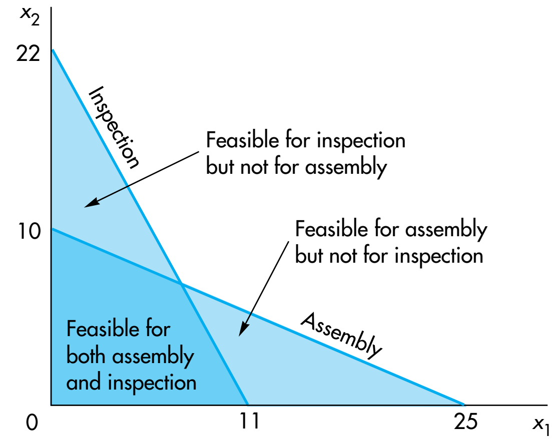
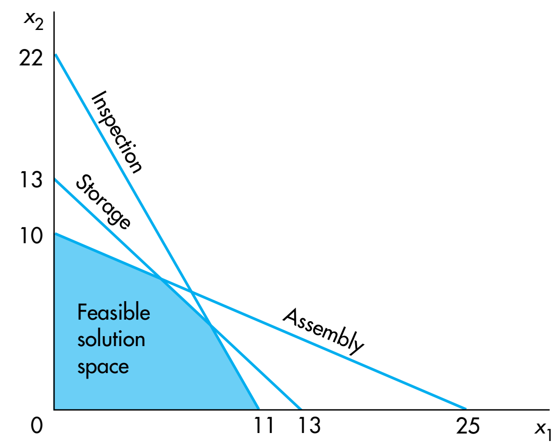


FIGURE 6S–6

Completed graph of the microcomputer problem showing all constraints and the feasible solution space



3. Set x_2 equal to zero and solve for x_1:

   $3x_1 + 3(0) = 39$

   Solving, $x_1 = 13$. Thus, $x_1 = 13$ when $x_2 = 0$.
4. Add the line to the graph (see Figure 6S–6).

Identifying the Feasible Solution Space

The feasible solution space is the set of all points that satisfies *all* constraints. (Recall that the x_1 and x_2 axes form nonnegativity constraints.) The shaded area shown in Figure 6S–6 is the feasible solution space for our problem.

The next step is to determine which point in the feasible solution space will produce the optimal value of the objective function. This determination is made using the objective function.

Plotting the Objective Function Line

Plotting an objective function line involves the same logic as plotting a constraint line: Determine where the line intersects each axis. Recall that the objective function for the microcomputer problem is

$$60x_1 + 50x_2$$

This is not an equation because it does not include an equal sign. We can get around this by simply setting it equal to some quantity. Any quantity will do, although one that is evenly divisible by both coefficients is desirable.

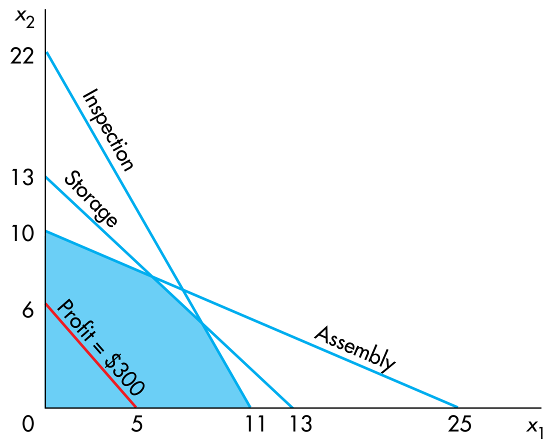


FIGURE 6S–7

Microcomputer problem with $300 profit line added

Suppose we decide to set the objective function equal to 300. That is,

$$60x_1 + 50x_2 = 300$$

We can now plot the line of our graph. As before, we can determine the x_1 and x_2 intercepts of the line by setting one of the two variables equal to zero, solving for the other, and then reversing the process. Thus, with $x_1 = 0$, we have

$$60(0) + 50x_2 = 300$$

Solving, we find $x_2 = 6$. Similarly, with $x_2 = 0$, we have

$$60x_1 + 50(0) = 300$$

Solving, we find $x_1 = 5$. This line is plotted in Figure 6S–7.

The profit line can be interpreted in the following way. It is an *isoprofit* line; every point on the line (i.e., every combination of x_1 and x_2 that lies on the line) will provide a profit of $300. We can see from the graph many combinations that are both on the $300 profit line and within the feasible solution space. In fact, considering noninteger as well as integer solutions, the possibilities are infinite.

Suppose we now consider another line, say the $600 line. To do this, we set the objective function equal to this amount. Thus,

$$60x_1 + 50x_2 = 600$$

Solving for the x_1 and x_2 intercepts yields these two points:

x_1 intercept	x_2 intercept
$x_1 = 10$	$x_1 = 0$
$x_2 = 0$	$x_2 = 12$

This line is plotted in Figure 6S–8, along with the previous $300 line for purposes of comparison.

Two things are evident in Figure 6S–8 regarding the profit lines. One is that the $600 line is *farther* from the origin than the $300 line; the other is that the two lines are *parallel.* The lines are parallel because they both have the same slope. The slope is not affected by the right side of the equation. Rather, it is determined solely by the coefficients 60 and 50. It would be correct to conclude that regardless of the quantity we select for the value of the objective function, the resulting line will be parallel to these two lines. Moreover, if the amount is greater than 600, the line will be even farther away from the origin than the $600 line. If the value is less than 300, the line will be closer to the origin than the $300 line. And if the value is between 300 and 600, the line will fall between the $300 and $600 lines. This knowledge will help in determining the optimal solution.

FIGURE 6S–8

Microcomputer problem with profit lines of $300 and $600

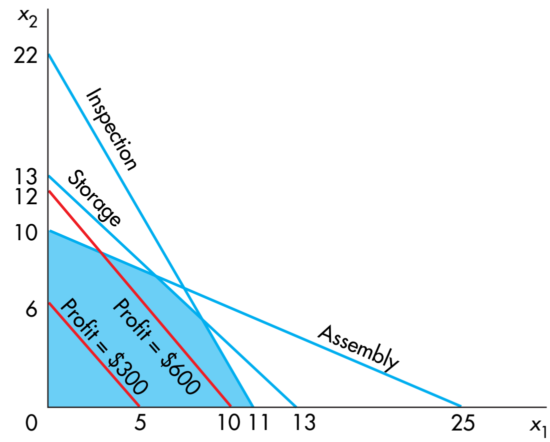
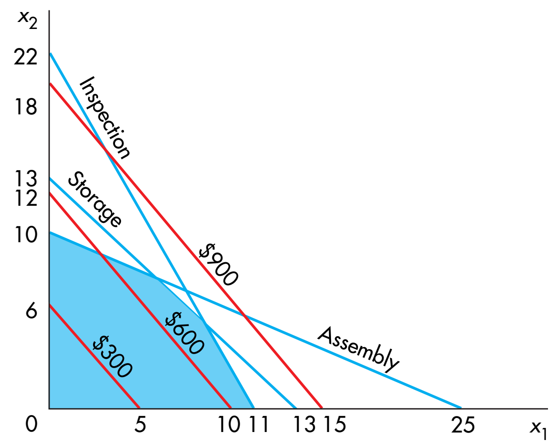


FIGURE 6S–9

Microcomputer problem with profit lines of $300, $600, and $900



Consider a third line, one with the profit equal to $900. Figure 6S–9 shows that line along with the previous two profit lines. As expected, it is parallel to the other two, and even farther away from the origin. However, the line does not touch the feasible solution space at all. Consequently, there is no feasible combination of x_1 and x_2 that will yield that amount of profit. Evidently, the maximum possible profit is an amount between $600 and $900, which we can see by referring to Figure 6S–9. We could continue to select profit lines in this manner, and eventually, we could determine an amount that would yield the greatest profit. However, there is a much simpler alternative. We can plot just one line, say the $300 line. We know that all other lines will be parallel to it. Consequently, by moving this one line parallel to itself we can represent other profit lines. We also know that as we move away from the origin, the profits get larger. What we want to know is how far the line can be moved out from the origin and still be touching the feasible solution space, and the values of the decision variables at that point of greatest profit (i.e., the optimal solution). Locate this point on the graph by placing a straight edge along the $300 line (or any other convenient line) and sliding it away from the origin, being careful to keep it parallel to the line. This approach is illustrated in Figure 6S–10.

Once we have determined where the optimal solution is in the feasible solution space, we must determine the values of the decision variables at that point. Then, we can use that information to compute the profit for that combination.

Note that the optimal solution is at the intersection of the inspection boundary and the storage boundary (see Figure 6S–10). In other words, the optimal combination of x_1 and x_2 must satisfy both boundary (equality) conditions. We can determine those values by

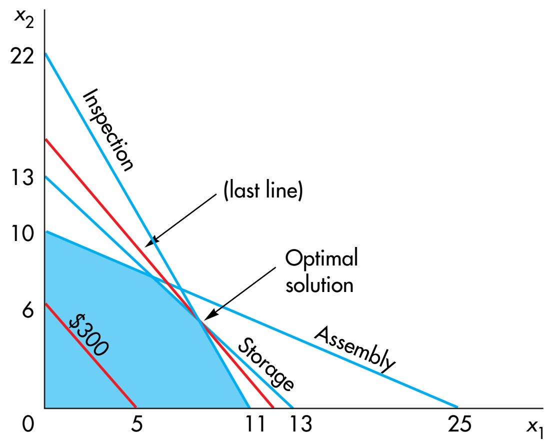


FIGURE 6S–10

Finding the optimal solution to the microcomputer problem

solving the two equations *simultaneously.* The equations are

Inspection $2x_1 + 1x_2 = 22$

Storage $3x_1 + 3x_2 = 39$

The idea behind solving two *simultaneous equations* is to algebraically eliminate one of the unknown variables (i.e., to obtain an equation with a single unknown). This can be accomplished by multiplying the constants of one of the equations by a fixed amount and then adding (or subtracting) the modified equation from the other. (Occasionally, it is easier to multiply each equation by a fixed quantity.) For example, we can eliminate x_2 by multiplying the inspection equation by 3 and then subtracting the storage equation from the modified inspection equation. Thus,

$3(2x_1 + 1x_2 = 22)$ becomes $6x_1 + 3x_2 = 66$

Subtracting the storage equation from this produces

$$\begin{array}{r} 6x_1 + 3x_2 = 66 \\ -\ \underline{(3x_1 + 3x_2 = 39)} \\ 3x_1 + 0x_2 = 27 \end{array}$$

Solving the resulting equation yields $x_1 = 9$. The value of x_2 can be found by substituting $x_1 = 9$ into either of the original equations or the modified inspection equation. Suppose we use the original inspection equation. We have

$2(9) + 1x_2 = 22$

Solving, we find $x_2 = 4$.

Hence, the optimal solution to the microcomputer problem is to produce nine type 1 computers and four type 2 computers per day. We can substitute these values into the objective function to find the optimal profit:

$\$60(9) + \$50(4) = \$740$

Hence, the last line—the one that would last touch the feasible solution space as we moved away from the origin parallel to the $300 profit line—would be the line where profit equalled $740.

In this problem, the optimal values for both decision variables are integers. This will not always be the case; one or both of the decision variables may turn out to be noninteger. In some situations noninteger values would be of little consequence. This would be true if the decision variables were measured on a continuous scale, such as the amount of water, sand, sugar, fuel oil, time, or distance needed for optimality, or if the contribution per unit (profit, cost, etc.) were small, as with the number of nails or ball bearings to make. In some cases, the answer would simply be rounded down (maximization problems) or up (minimization problems) with very little impact on the objective function. Here, we assume that noninteger answers are acceptable as such.

Let's review the procedure for finding the optimal solution using the objective function approach:

1. Graph the constraints.
2. Identify the feasible solution space.
3. Set the objective function equal to some amount that is divisible by each of the objective function coefficients. This will yield integer values for the x_1 and x_2 intercepts and simplify plotting the line. Often, the product of the two objective function coefficients provides a satisfactory line. Ideally, the line will cross the feasible solution space close to the optimal point, and it will not be necessary to slide a straight edge because the optimal solution can be readily identified visually.
4. After identifying the optimal point, determine which two constraints intersect there. Solve their equations simultaneously to obtain the values of the decision variables at the optimum.
5. Substitute the values obtained in the previous step into the objective function to determine the value of the objective function at the optimum.

Redundant Constraints

redundant constraint A constraint that does not form a unique boundary of the feasible solution space.

In some cases, a constraint does not form a unique boundary of the feasible solution space. Such a constraint is called a **redundant constraint**. Two such constraints are illustrated in Figure 6S–11. Note that a constraint is redundant if it meets the following test: Its removal would not alter the feasible solution space.

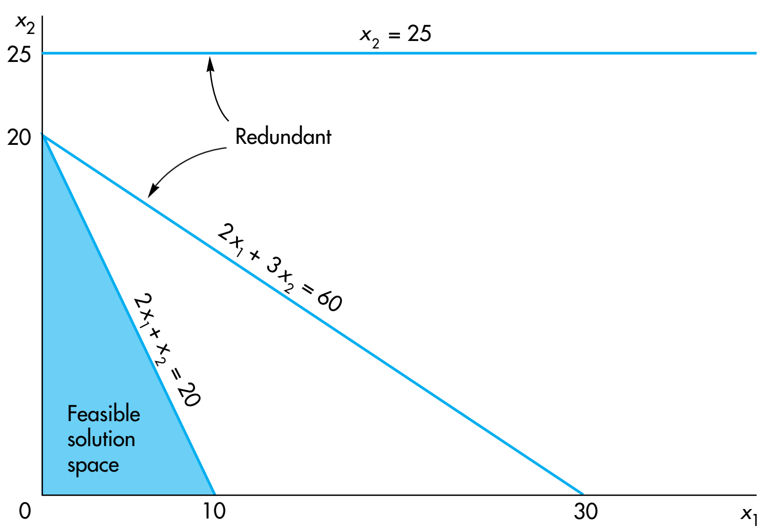
When a problem has a redundant constraint, at least one of the other constraints in the problem is more restrictive than the redundant constraint.

Solutions and Corner Points

The feasible solution space in graphical linear programming is a polygon. Moreover, the solution to any problem will be at one of the corner points (intersections of constraints) of the polygon. It is possible to determine the coordinates of each corner point of the feasible solution space, and use those values to compute the value of the objective function at those points. Because the solution is always at a corner point, comparing the values of the objective function at the corner points and identifying the best one (e.g., the maximum

FIGURE 6S–11

Examples of redundant constraints



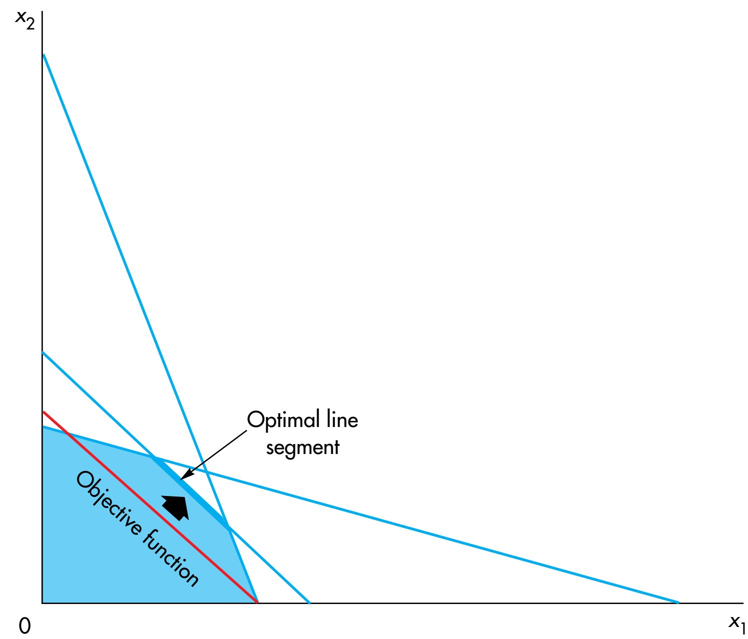


FIGURE 6S–12
Some LP problems have multiple optimal solutions

value) is another way to identify the optimal corner point. Using the graphical approach, it is much easier to plot the objective function and use that to identify the optimal corner point. However, for problems that have more than two decision variables, and the graphical method isn't appropriate, this alternate approach is used to find the optimal solution.

In some instances, the objective function will be *parallel* to one of the constraint lines that forms a *boundary of the feasible solution space.* When this happens, *every* combination of x_1 and x_2 on the segment of the constraint that touches the feasible solution space represents an optimal solution. Hence, there are multiple optimal solutions to the problem. Even in such a case, the solution will also be a corner point—in fact, the solution will be at *two* corner points: those at the ends of the segment that touches the feasible solution space. Figure 6S–12 illustrates an objective function line that is parallel to a constraint line.

Minimization

Graphical minimization problems are quite similar to maximization problems. There are, however, two important differences. One is that at least one of the constraints must be of the = or ≥ variety. This causes the feasible solution space to be away from the origin. The other difference is that the optimal point is the one closest to the origin. We find the optimal corner point by sliding the objective function (which is an *isocost* line) *toward* the origin instead of away from it.

Example S–3

Solve the following problem using graphical linear programming.

Minimize $Z = 8x_1 + 12x_2$

Subject to
$$5x_1 + 2x_2 \geq 20$$
$$4x_1 + 3x_2 \geq 24$$
$$x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Solution

1. Plot the constraints (shown in Figure 6S–13).
   a. Change constraints to equalities.
   b. For each constraint, set $x_1 = 0$ and solve for x_2, then set $x_2 = 0$ and solve for x_1.

FIGURE 6S–13

The constraints define the feasible solution space

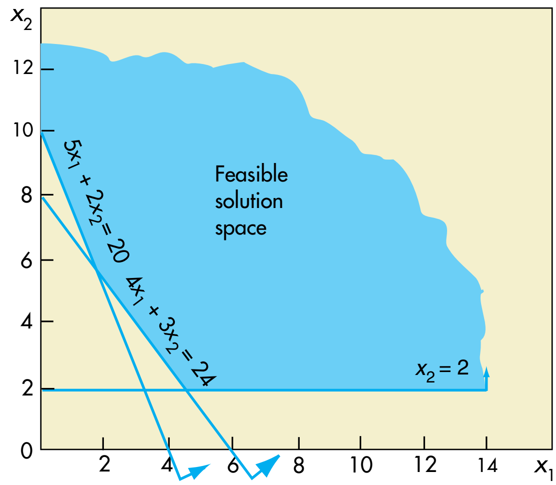
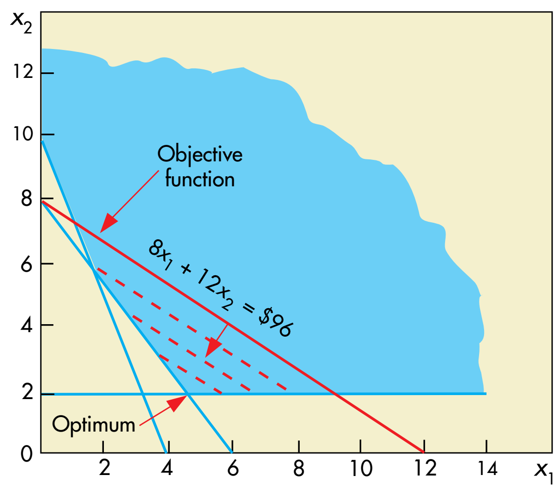


FIGURE 6S–14

The optimum is the last point the objective function touches as it is moved toward the origin



   c. Graph each constraint. Note that $x_2 = 2$ is a horizontal line parallel to the x_1 axis and 2 units above it.

2. Shade the feasible solution space (see Figure 6S–13).
3. Plot the objective function.

   a. Select a value for the objective function that causes it to cross the feasible solution space. Try $8 \times 12 = 96$; $8x_1 + 12x_2 = 96$ (acceptable).

   b. Graph the line (see Figure 6S–14).

4. Slide the objective function toward the origin, being careful to keep it parallel to the original line.
5. The optimum (last feasible point) is shown in Figure 6S–14. The x_2 coordinate ($x_2 = 2$) can be determined by inspection of the graph. Note that the optimum point is at the intersection of the line $x_2 = 2$ and the line $4x_1 + 3x_2 = 24$. Substituting the value of $x_2 = 2$ into the latter equation will yield the value of x_1 at the intersection:

   $$4x_1 + 3(2) = 24 \quad x_1 = 4.5$$

   Thus, the optimum is $x_1 = 4.5$ units and $x_2 = 2$.

6. Compute the minimum cost:

   $$8x_1 + 12x_2 = 8(4.5) + 12(2) = 60$$

Slack and Surplus

If a constraint forms the optimal corner point of the feasible solution space, it is called a **binding constraint**. In effect, it limits the value of the objective function; if the constraint could be relaxed (less restrictive), an improved solution would be possible. For constraints that are not binding, making them less restrictive will have no impact on the solution.

binding constraint A constraint that forms the optimal corner point of the feasible solution space.

If the optimal values of the decision variables are substituted into the left side of a binding constraint, the resulting value will exactly equal the right-hand value of the constraint. However, there will be a difference with a nonbinding constraint. If the left side is greater than the right side, we say that there is **surplus**; if the left side is less than the right side, we say that there is **slack**. Slack can only occur in a $\leq$ constraint; it is the amount by which the left side is less than the right side when the optimal values of the decision variables are substituted into the left side. And surplus can only occur in a $\geq$ constraint; it is the amount by which the left side exceeds the right side of the constraint when the optimal values of the decision variables are substituted into the left side.

surplus When the values of decision variables are substituted into a $\geq$ constraint and the resulting value exceeds the right-side value.

slack When the values of decision variables are substituted into a $\leq$ constraint and the resulting value is less than the right-side value.

For example, suppose the optimal values for a problem are $x_1 = 10$ and $x_2 = 20$. If one of the constraints is

$$3x_1 + 2x_2 \leq 100$$

substituting the optimal values into the left side yields

$$3(10) + 2(20) = 70$$

Because the constraint is $\leq$, the difference between the values of 100 and 70 (i.e., 30) is slack. Suppose the optimal values had been $x_1 = 20$ and $x_2 = 20$. Substituting these values into the left side of the constraint would yield $3(20) + 2(20) = 100$. Because the left side equals the right side, this is a binding constraint; slack is equal to zero.

Now consider this constraint:

$$4x_1 + x_2 \geq 50$$

Suppose the optimal values are $x_1 = 10$ and $x_2 = 15$; substituting into the left side yields

$$4(10) + 15 = 55$$

Because this is a $\geq$ constraint, the difference between the left- and right-side values is *surplus*. If the optimal values had been $x_1 = 12$ and $x_2 = 2$, substitution would result in the left side being equal to 50. Hence, the constraint would be a binding constraint, and there would be no surplus (i.e., surplus would be zero).

The Simplex Method

The **simplex** method is a general-purpose linear programming algorithm widely used to solve large-scale problems. Although it lacks the intuitive appeal of the graphical approach, its ability to handle problems with more than two decision variables makes it extremely valuable for solving problems often encountered in operations management.

simplex A linear programming algorithm that can solve problems having more than two decision variables.

Although manual solution of linear programming problems using simplex can yield a number of insights on how solutions are derived, space limitations preclude describing it here. However, it is available on the CD that accompanies this book. The discusion here will focus on computer solutions.

Computer Solutions

The microcomputer problem will be used to illustrate computer solutions. We repeat it here for ease of reference.

Maximize	$60x_1 + 50x_2$	where x_1 = the number of type 1 computers
		x_2 = the number of type 2 computers

Subject to

Assembly	$4x_1 + 10x_2 \leq 100$ hours	
Inspection	$2x_1 + 1x_2 \leq 22$ hours	
Storage	$3x_1 + 3x_2 \leq 39$ cubic feet	
	$x_1, x_2 \geq 0$	

Solving LP Models Using MS Excel

Solutions to linear programming models can be obtained from spreadsheet software such as Microsoft's Excel. Excel has a routine called *Solver* that performs the necessary calculations.

To use Solver:

1. First, enter the problem in a worksheet, as shown in Figure 6S–15. What is not obvious from the figure is the need to enter a formula for each cell where there is a zero (Solver automatically inserts the zero after you input the formula). The formulas are for the value of the objective function and the constraints, in the appropriate cells. Before you enter the formulas, designate the cells where you want the optimal values of x_1 and x_2. Here, cells D4 and E4 are used. To enter a formula, click on the cell that the formula will pertain to, and then enter the formula, starting with an equals sign. We want the optimal value of the objective function to appear in cell G4. For G4, enter the formula

= 60*D4 + 50*E4

The constraint formulas, in cells C7, C8, and C9, are

for C7: = 4*D4 + 10*E4

for C8: = 2*D4 + 1*E4

for C9: = 3*D4 + 3*E4

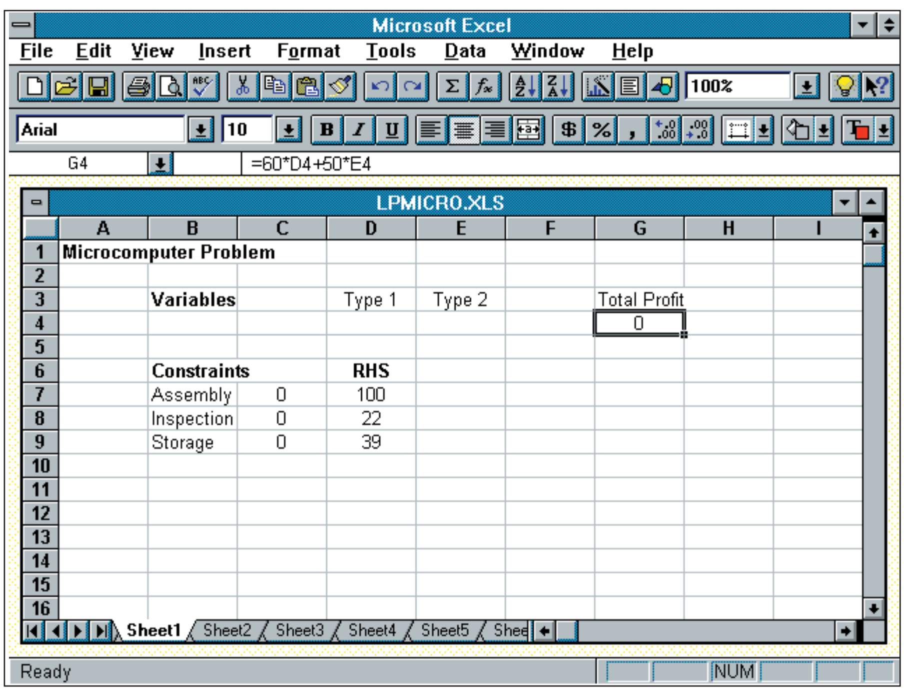


FIGURE 6S–15

MS Excel worksheet for microcomputer problem

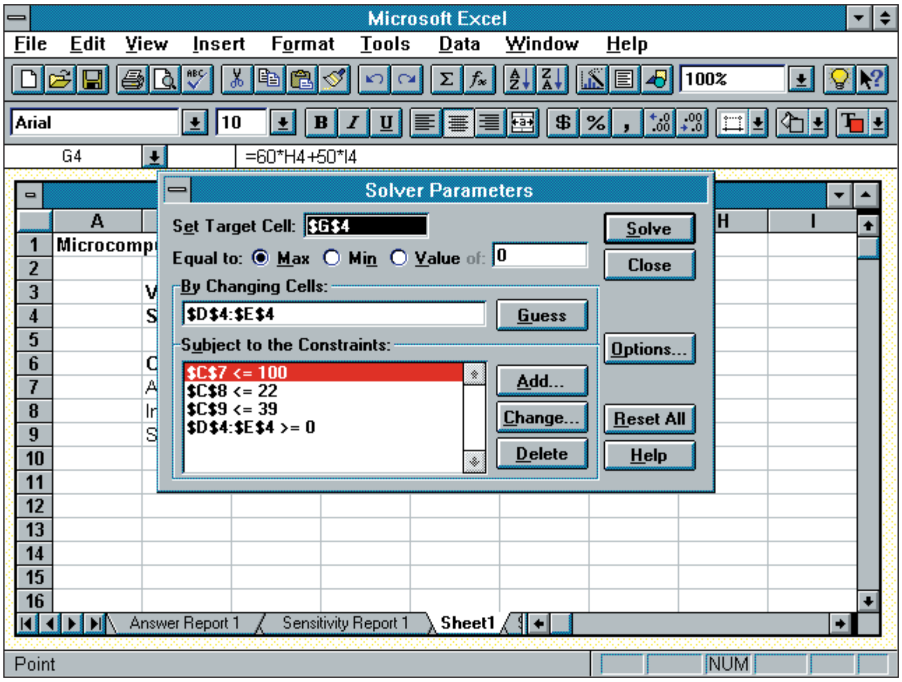


FIGURE 6S–16

MS Excel Solver parameters for microcomputer problem

2. Now, click on Tools on the top of the worksheet, and in that menu, click on Solver. The Solver menu will appear as illustrated in Figure 6S–16. Begin by setting the Target Cell (i.e., indicating the cell where you want the optimal value of the objective function to appear). Note, if the activated cell is the cell designated for the value of *Z* when you click on the Tools menu, Solver will automatically set that cell as the target cell.

Highlight Max if it isn't already highlighted. The Changing Cells are the cells where you want the optimal values of the decision variables to appear. Here, they are cells D4 and E4. We indicate this by the range D4:E4 (Solver will add the $ signs).

Finally, add the constraints by clicking on Add . . . When that menu appears, for each constraint, enter the cell that contains the formula for the left side of the constraint, then select the appropriate inequality sign, and then enter either the right-side amount or the cell that has the right-side amount. Here, the right-side amounts are used. After you have entered each constraint, click on Add, and then enter the next constraint. (Note, constraints can be entered in any order.) For the nonnegativity constraints, enter the range of cells designated for the optimal values of the decision variables, choose ≥ sign, and enter 0 for the right-hand side. Then, click on OK rather than Add, and you will return to the Solver menu. Click on Options . . . , and in the Options menu, click on Assume Linear Model, and then click on OK. This will return you to the Solver Parameters menu. Click on Solve.

3. The Solver Results menu will then appear, indicating that a solution has been found, or that an error has occurred. If there has been an error, go back to the Solver Parameters menu and check to see that your constraints refer to the correct changing cells, and that the inequality directions are correct. Make the corrections and click on Solve.

Assuming everything is correct, in the Solver Results menu, in the Reports box, highlight both Answer and Sensitivity, and then click on OK.

4. Solver will incorporate the optimal values of the decision variables and the objective function in your original layout on your worksheet (see Figure 6S–17). We can see that the optimal values are type 1 = 9 units and type 2 = 4 units, and the total profit is 740. The answer report will also show the optimal values of the decision variables (upper part of Figure 6S–18), and some information on the constraints (lower part of Figure 6S–18). Of particular interest here is the indication of which constraints have slack and how much slack. We can see that the constraint entered in cell C7 (assembly) has a slack of 24, and

FIGURE 6S–17

MS Excel worksheet solution to microcomputer problem

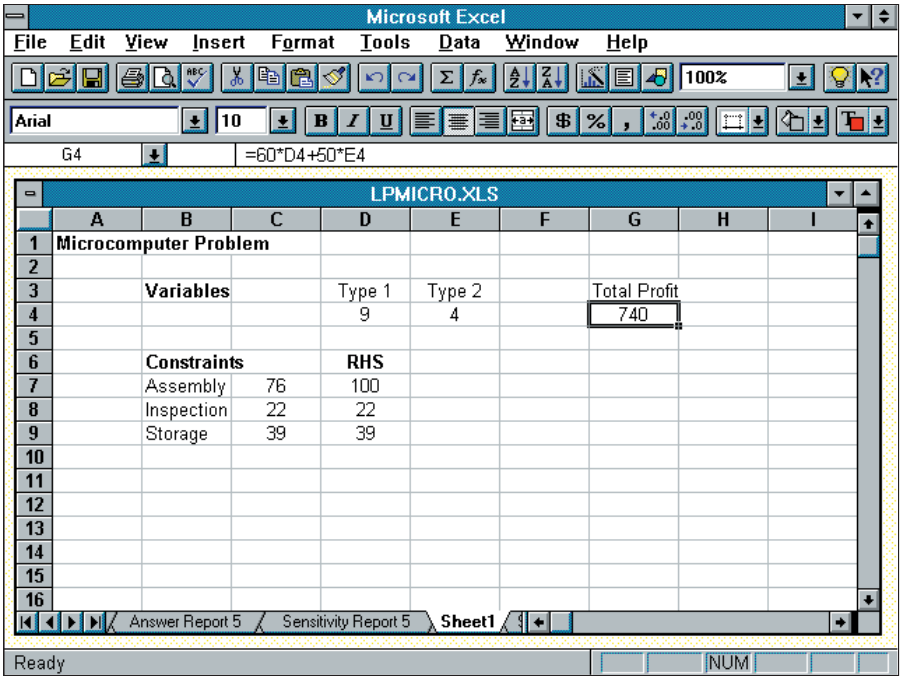
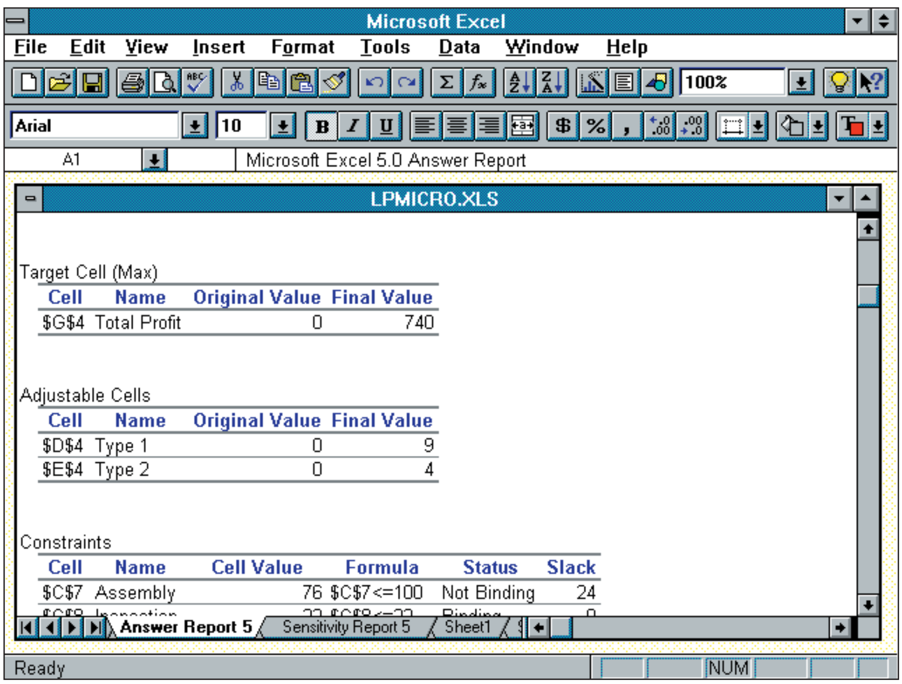


FIGURE 6S–18

MS Excel Answer Report for microcomputer problem



that the constraints entered in cells C8 (inspection) and C9 (storage) have slack equal to zero, indicating that they are binding constraints.

Sensitivity Analysis

sensitivity analysis Assessing the impact of potential changes to the numerical values of an LP model.

Sensitivity analysis is a means of assessing the impact of potential changes to the parameters (the numerical values) of an LP model. Such changes may occur due to forces beyond a manager's control; or a manager may be contemplating making the changes, say, to increase profits or reduce costs.

There are three types of potential changes:

1. Objective function coefficients.
2. Right-hand values of constraints.
3. Constraint coefficients.

We will consider the first two of these here. We begin with changes to objective function coefficients.

Objective Function Coefficient Changes

A change in the value of an objective function coefficient can cause a change in the optimal solution of a problem. In a graphical solution, this would mean a change to another corner point of the feasible solution space. However, not every change in the value of an objective function coefficient will lead to a changed solution; generally there is a *range of values for which the optimal values of the decision variables will not change.* For example, in the microcomputer problem, if the profit on type 1 computers increased from $60 per unit to, say, $65 per unit, the optimal solution would still be to produce nine units of type 1 and four units of type 2 computers. Similarly, if the profit per unit on type 1 computers decreased from $60 to, say, $58, producing nine of type 1 and four of type 2 would still be optimal. These sorts of changes are not uncommon; they may be the result of such things as price changes in raw materials, price discounts, cost reductions in production, and so on. Obviously, when a change does occur in the value of an objective function coefficient, it can be helpful for a manager to know if that change will affect the optimal values of the decision variables. The manager can quickly determine this by referring to that coefficient's **range of optimality**, which is the range in possible values of that objective function coefficient over which the optimal values of the decision variables will not change. Before we see how to determine the range, consider the implication of the range. The range of optimality for the type 1 coefficient in the microcomputer problem is 50 to 100. That means that as long as the coefficient's value is in that range, the optimal values will be 9 units of type 1 and 4 units of type 2. Conversely, *if a change extends beyond the range of optimality, the solution will change.*

range of optimality Range of values over which the solution quantities of all the decision variables remain the same.

Similarly suppose instead the coefficient of type 2 computers were to change. Its range of optimality is 30 to 60. As long as the value of the change doesn't take it outside of this range, nine and four will still be the optimal values. Note, however, even for changes that are *within* the range of optimality, the optimal value of the objective function *will* change. If the type 1 coefficient increased from $60 to $61, and nine units of type 1 is still optimum, profit would increase by $9: nine units times $1 per unit. Thus, for a change that is within the range of optimality, a revised value of the objective function must be determined.

Now let's see how we can determine the range of optimality using computer output.

Using MS Excel. There is a table for the Changing Cells (see Figure 6S–19). It shows the value of the objective function that was used in the problem for each type of computer (i.e., 60 and 50), and the allowable increase and allowable decrease for each coefficient. By subtracting the allowable decrease from the original value of the coefficient, and adding the allowable increase to the original value of the coefficient, we obtain the range of optimality for each coefficient. Thus, we find for type 1:

$$60 - 10 = 50 \quad \text{and} \quad 60 + 40 = 100$$

Hence, the range for the type 1 coefficient is 50 to 100. For type 2:

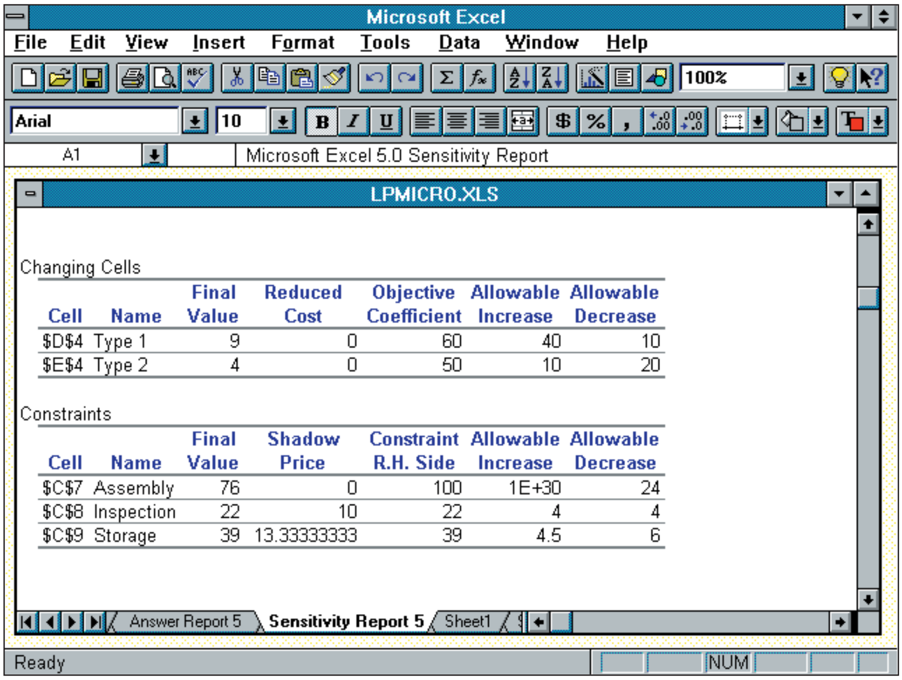
$$50 - 20 = 30 \quad \text{and} \quad 50 + 10 = 60$$

Hence the range for the type 2 coefficient is 30 to 60.

In this example, both of the decision variables are *basic* (i.e., nonzero). However, in other problems, one or more decision variables may be *nonbasic* (i.e., have an optimal

FIGURE 6S–19

MS Excel sensitivity report for microcomputer problem



Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D4	Type 1	9	0	60	40	10
E4	Type 2	4	0	50	10	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C7	Assembly	76	0	100	1E+30	24
C8	Inspection	22	10	22	4	4
C9	Storage	39	13.33333333	39	4.5	6

value of zero). In such instances, unless the value of that variable's objective function coefficient increases by more than a certain amount called its reduced cost, it won't come into solution (i.e., become a basic variable). Hence, the range of optimality (sometimes referred to as the *range of insignificance*) for a nonbasic variable is from negative infinity to the sum of its current value and its reduced cost.

Now let's see how we can handle multiple changes to objective function coefficients; that is, a change in more than one coefficient. To do this, divide each coefficient's change by the allowable change in the same direction. Thus, if the change is a decrease, divide that amount by the allowable decrease. Treat all resulting fractions as positive. Sum the fractions. If the sum does not exceed 1.00, then multiple changes are within the range of optimality and will not result in any change to the optimal values of the decision variables.

Changes in the Right-Hand-Side (RHS) Value of a Constraint

shadow price Amount by which the value of the objective function would change with a one-unit change in the RHS value of a constraint.

In considering right-hand-side changes, it is important to know if a particular constraint is binding on a solution. A constraint is binding if substituting the values of the decision variables of that solution into the left side of the constraint results in a value that is equal to the RHS value. In other words, that constraint stops the objective function from achieving a better value (e.g., a greater profit or a lower cost). Each constraint has a corresponding **shadow price**, which is a marginal value that indicates the amount by which the value of the objective function would change if there were a one-unit change in the RHS value of that constraint. If a constraint is nonbinding, its shadow price is zero, meaning that increasing or decreasing its RHS value by one unit will have no impact on the value of the objective function. Nonbinding constraints have either slack (if the constraint is $\leq$) or surplus (if the constraint is $\geq$). Suppose a constraint has 10 units of slack in the optimal solution, which means 10 units that are unused. If we were to increase or decrease the constraint's RHS value by one unit, the only effect would be to increase or decrease its slack by one unit. But there is no profit associated with slack, so the value of the objective function wouldn't change. On the other hand, if the change is to the RHS value of a binding constraint, then the optimal value of the objective function would change. Any change in a binding constraint will cause the optimal values of the decision variables to change, and hence cause the value of the objective function to change. For example, in the microcomputer problem, the inspection constraint is a binding constraint; it has a shadow price of 10. That means if there was one hour less of inspection time, total profit would decrease by $10, or if there were one more hour of inspection time available, total profit would increase by $10. In general, multiplying the amount of change in the RHS value of a constraint by the constraint's shadow price will indicate the change's impact on the optimal value of the objective function. However, this is true only over a limited range called the **range of feasibility**. In this range, the value of the shadow price remains constant. Hence, as long as a change in the RHS value of a constraint is within its range of feasibility, the shadow price will remain the same, and one can readily determine the impact on the objective function.

range of feasibility Range of values for the RHS of a constraint over which the shadow price remains the same.

Let's see how to determine the range of feasibility from computer output.

Using MS Excel. In the sensitivity report there is a table labelled "Constraints" (see Figure 6S–19). The table shows the shadow price for each constraint, its RHS value, and the allowable increase and allowable decrease. Adding the allowable increase to the RHS value and subtracting the allowable decrease will produce the range of feasibility for that constraint. For example, for the inspection constraint the range would be

$$22 + 4 = 26; \quad 22 - 4 = 18$$

Hence, the range of feasibility for Inspection is 18 to 26 hours. Similarly, for the storage constraint, the range is

$$39 - 6 = 33 \quad \text{to} \quad 39 + 4.5 = 43.5$$

The range for the assembly constraint is a little different; the assembly constraint is nonbinding (note the shadow price of 0) while the other two are binding (note their nonzero shadow prices). The assembly constraint has a slack of 24 (the difference between its RHS value of 100 and its final value of 76). With its slack of 24, its RHS value could be decreased by as much as 24 (to 76) before it would become binding. Conversely, increasing its right-hand side will only produce more slack. Thus, no amount of increase in the RHS value will make it binding, so there is no upper limit on the allowable increase. Excel indicates this by the large value (1E+30) shown for the allowable increase. So its range of feasibility has a lower limit of 76 and no upper limit.

If there are changes to more than one constraint's RHS value, analyze these in the same way as multiple changes to objective function coefficients. That is, if the change is an increase, divide that amount by that constraint's allowable increase; if the change is a decrease, divide the decrease by the allowable decrease. Treat all resulting fractions as positives. Sum the fractions. As long as the sum does not exceed 1.00, the changes are within the range of feasibility for multiple changes, and the shadow prices won't change.

Table 6S–1 summarizes the impacts of changes that fall within either the range of optimality or the range of feasibility.

Now let's consider what happens if a change goes beyond a particular range. In a situation involving the range of optimality, a change in an objective function that is beyond the range of optimality will result in a new solution. Hence, it will be necessary to recompute the solution. For a situation involving the range of feasibility, there are two cases to consider. The first case would be increasing the RHS value of a $\leq$ constraint to beyond the upper limit of its range of feasibility. This would produce slack equal to the amount by which the upper limit is exceeded. Hence, if the upper limit is 200, and the increase is 220, the result is that the constraint has a slack of 20. Similarly, for a $\geq$ constraint, going below its lower bound creates a surplus for that constraint. The second case for each of these would be exceeding the opposite limit (the lower bound for a $\leq$ constraint, or the upper bound for a $\geq$ constraint). In either instance, a new solution would have to be generated.

TABLE 6S–1

Summary of the impact of changes within their respective ranges

CHANGES TO OBJECTIVE FUNCTION COEFFICIENTS THAT ARE WITHIN THE RANGE OF OPTIMALITY	
Component	**Result**
Values of decision variables	No change
Value of objective function	Will change
CHANGES TO RHS VALUES OF CONSTRAINTS THAT ARE WITHIN THE RANGE OF FEASIBILITY	
Component	**Result**
Value of shadow price	No change
List of basic variables	No change
Values of basic variables	Will change
Value of objective function	Will change

Key Terms

Solved Problems

Problem 1

A small construction firm specializes in building and selling single-family homes. The firm offers two basic types of houses, model A and model B. Model A houses require 4,000 labour hours, 2 tons of stone, and 2,000 board feet of lumber. Model B houses require 10,000 labour hours, 3 tons of stone, and 2,000 board feet of lumber. Due to long lead times for ordering supplies and the scarcity of skilled and semiskilled workers in the area, the firm will be forced to rely on its present resources for the upcoming building season. It has 400,000 hours of labour, 150 tons of stone, and 200,000 board feet of lumber. What mix of model A and B houses should the firm construct if model As yield a profit of $1,000 per unit and model Bs yield $2,000 per unit? Assume that the firm will be able to sell all the units it builds.

Solution

a. Formulate the objective function and constraints:[1]

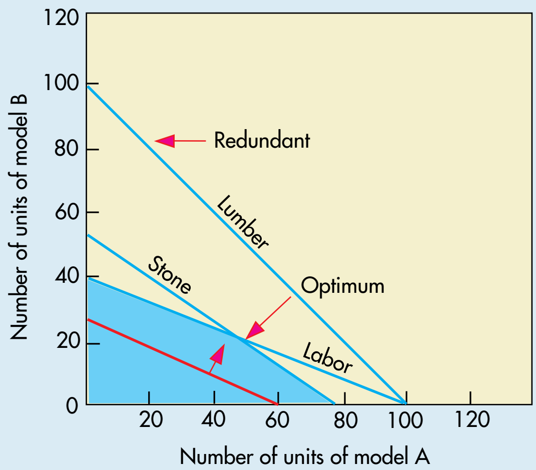
Maximize $Z = 1{,}000A + 2{,}000B$

Subject to

Labour	$4{,}000A + 10{,}000B$	$\leq$	400,000 labour hours
Stone	$2A + 3B$	$\leq$	150 tons
Lumber	$2{,}000A + 2{,}000B$	$\leq$	200,000 board feet
	A, B	$\geq$	0

b. Graph the constraints and objective function, and identify the optimum corner point (see graph). Note that the lumber constraint is *redundant*: It does not form a boundary of the feasible solution space.

[1]For the sake of consistency, we will assign to the horizontal axis the first decision variable mentioned in the problem. In this case, variable *A* will be represented on the horizontal axis and variable *B* on the vertical axis.



c. Determine the optimal quantities of models A and B, and compute the resulting profit. Because the optimum point is at the intersection of the stone and labour constraints, solve those two equations for their common point:

$$\begin{array}{rrcl} \text{Labour} & 4{,}000A + 10{,}000B & = & 400{,}000 \\ -2{,}000 \times (\text{Stone } 2A + & 3B & = & 150) \\ \hline & 4{,}000B & = & 100{,}000 \\ & B & = & 25 \end{array}$$

Substitute $B = 25$ in one of the equations, and solve for A:

$2A + 3(25) = 150 \qquad A = 37.5$

$Z = 1{,}000(37.5) + 2{,}000(25) = 87{,}500$

Problem 2

This LP model was solved by computer:

Maximize $15x_1 + 20x_2 + 14x_3$ where x_1 = quantity of product 1

x_2 = quantity of product 2

x_3 = quantity of product 3

Subject to

Labour	$5x_1 + 6x_2 + 4x_3 \le 210$	hours
Material	$10x_1 + 8x_2 + 5x_3 \le 200$	pounds
Machine	$4x_1 + 2x_2 + 5x_3 \le 170$	minutes
	$x_1, x_2, x_3 \ge 0$	

The following information was obtained from the output. The ranges were also computed based on the output, and they are shown as well.

```
Total profit = 548.00

Variable     Value   Reduced Cost   Range of Optimality

Product 1      0         10.6          0.00 to 25.60
Product 2      5          0            9.40 to 22.40
Product 3     32          0           12.50 to 50.00

Constraint   Slack   Shadow Price  Range of Feasibility

Labour        52          0.0      158.00 to unlimited
Material       0          2.4      170.00 to 270.91
Machine        0          0.4       50.00 to 200.00
```

a. Which decision variables are basic (i.e., in solution)?

b. By how much would the profit per unit of product 1 have to increase in order for it to have a nonzero value (i.e., for it to become a basic variable)?

c. If the profit per unit of product 2 increased by $2 to $22, would the optimal production quantities of products 2 and 3 change? Would the optimal value of the objective function change?

d. If the available amount of labour decreased by 12 hours, would that cause a change in the optimal values of the decision variables or the optimal value of the objective function? Would anything change?

e. If the available amount of material increased by 10 pounds to 210 pounds, how would that affect the optimal value of the objective function?

f. If profit per unit on product 2 increased by $1 and profit per unit on product 3 decreased by $.50, would that fall within the range of multiple changes? Would the values of the decision variables change? What would be the revised value of the objective function?

Solution

a. Products 2 and 3 are in solution (i.e., have nonzero values; the optimal value of product 2 is 5 units, and the optimal value of product 3 is 32 units).

b. The amount of increase would have to equal its *reduced cost* of $10.60.

c. No, because the change would be within its range of optimality, which has an upper limit of $22.40. The objective function value would increase by an amount equal to the quantity of product 2 and its increased unit profit. Hence, it would increase by 5($2) = $10 to $558.

d. Labour has a slack of 52 hours. Consequently, the only effect would be to decrease the slack to 40 hours.

e. The change is within the range of feasibility. The objective function value will increase by the amount of change multiplied by material's shadow price of $2.40. Hence, the objective function value would increase by 10($2.40) = $24.00. (Note: If the change had been a *decrease* of 10 pounds, which is also within the range of feasibility, the value of the objective function would have *decreased* by this amount.)

f. To determine if the changes are within the range for multiple changes, we first compute the ratio of the amount of each change to the end of the range *in the same direction.* For product 2, it is $1/$2.40 = .417; for product 3, it is − $.50/− $1.50 = .333. Next, we compute the sum of these ratios: .417 + .333 = .750. Because this does not exceed 1.00, we conclude that these changes are within the range. This means that the optimal values of the decision variables will not change. We can compute the change to the value of the objective function by multiplying each product's optimal quantity by its changed profit per unit: 5($1) + 32(− $.50) = − $11. Hence, with these changes, the value of the objective function would decrease by $11; its new value would be $548 − $11 = $537.

Discussion and Review Questions

1. For which decision environment is linear programming most suited?
2. What is meant by the term *feasible solution space*? What determines this region?
3. Explain the term *redundant constraint.*
4. What is an isocost line? An isoprofit line?
5. What does sliding an objective function line toward the origin represent? Away from the origin?
6. Briefly explain these terms:
   a. Basic variable
   b. Shadow price
   c. Range of feasibility
   d. Range of optimality
   e. Redundant constraints

Problems

1. Solve these problems using graphical linear programming and answer the questions that follow. Use simultaneous equations to determine the optimal values of the decision variables.
   a. Maximize $Z = 4x_1 + 3x_2$

Subject to

Material	$6x_1 + 4x_2 \leq 48$ kg
Labour	$4x_1 + 8x_2 \leq 80$ hr
	$x_1, x_2 \geq 0$

b. Maximize $Z = 2x_1 + 10x_2$

Subject to

R	$10x_1 + 4x_2 \geq 40$
S	$1x_1 + 6x_2 \geq 24$
T	$1x_1 + 2x_2 \leq 14$
	$x_1, x_2 \geq 0$

c. Maximize $Z = 6A + 3B$ (revenue)

Subject to

Material	$20A + 6B \leq 600$ kg
Machinery	$25A + 20B \leq 1{,}000$ hr
Labour	$20A + 30B \leq 1{,}200$ hr
	$A, B \geq 0$

(1) What are the optimal values of the decision variables and Z?
(2) Do any constraints have (nonzero) slack? If yes, which one(s) and how much slack does each have?
(3) Do any constraints have (nonzero) surplus? If yes, which one(s) and how much surplus does each have?
(4) Are any constraints redundant? If yes, which one(s)? Explain briefly.

2. Solve these problems using graphical linear programming and then answer the questions that follow. Use simultaneous equations to determine the optimal values of the decision variables.

a. Minimize $Z = 1.80S + 2.20T$

Subject to

Potassium	$5S + 8T \geq 200$ grams
Carbohydrate	$15S + 6T \geq 240$ grams
Protein	$4S + 12T \geq 180$ grams
T	$T \geq 10$ grams
	$S, T \geq 0$

b. Minimize $Z = 2x_1 + 3x_2$

Subject to

D	$4x_1 + 2x_2 \geq 20$
E	$2x_1 + 6x_2 \geq 18$
F	$1x_1 + 2x_2 \leq 12$
	$x_1, x_2 \geq 0$

(1) What are the optimal values of the decision variables and Z?
(2) Do any constraints have (nonzero) slack? If yes, which one(s) and how much slack does each have?
(3) Do any constraints have (nonzero) surplus? If yes, which one(s) and how much surplus does each have?
(4) Are any constraints redundant? If yes, which one(s)? Explain briefly.

3. An appliance manufacturer produces two models of microwave ovens: H and W. Both models require fabrication and assembly work; each H uses four hours of fabrication and two hours of assembly, and each W uses two hours of fabrication and six hours of assembly. There are 600 fabrication hours available this week and 480 hours of assembly. Each H contributes \$40 to profits, and each W contributes \$30 to profits. What quantities of H and W will maximize profits?

4. A small candy shop is preparing for the holiday season. The owner must decide how many bags of deluxe mix and how many bags of standard mix of Peanut/Raisin Delite to put up. The deluxe mix has $2/3$ kg raisins and $1/3$ kg peanuts, and the standard mix has $1/2$ kg raisins and $1/2$ kg peanuts per bag. The shop has 90 kg of raisins and 60 kg of peanuts to work with.

Peanuts cost \$.60 per kg and raisins cost \$1.50 per kg. The deluxe mix will sell for \$2.90 per kg, and the standard mix will sell for \$2.55 per kg. The owner estimates that no more than 110 bags of one type can be sold.

a. If the goal is to maximize profits, how many bags of each type should be prepared?

b. What is the expected profit?

5. A retired couple supplement their income by making fruit pies, which they sell to a local grocery store. During the month of September, they produce apple and grape pies. The apple pies are sold for \$1.50 to the grocer, and the grape pies are sold for \$1.20. The couple is able to sell all of the pies they produce owing to their high quality. They use fresh ingredients. Flour and sugar are purchased once each month. For the month of September, they have 1,200 cups of sugar and 2,100 cups of flour. Each apple pie requires 1½ cups of sugar and 3 cups of flour, and each grape pie requires 2 cups of sugar and 3 cups of flour.

   a. Determine the number of grape and the number of apple pies that will maximize revenues if the couple working together can make an apple pie in six minutes and a grape pie in three minutes. They plan to work no more than 60 hours.

   b. Determine the amounts of sugar, flour, and time that will be unused.

6. Solve each of these problems by computer and obtain the optimal values of the decision variables and the objective function.

   a. Maximize $4x_1 + 2x_2 + 5x_3$

   Subject to

   $$\begin{aligned} 1x_1 + 2x_2 + 1x_3 &\leq 25 \\ 1x_1 + 4x_2 + 2x_3 &\leq 40 \\ 3x_1 + 3x_2 + 1x_3 &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

   b. Maximize $10x_1 + 6x_2 + 3x_3$

   Subject to

   $$\begin{aligned} 1x_1 + 1x_2 + 2x_3 &\leq 25 \\ 2x_1 + 1x_2 + 4x_3 &\leq 40 \\ 1x_1 + 2x_2 + 3x_3 &\leq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

7. For Problem 6*a*, determine the following:

   a. The range of feasibility for each constraint.

   b. The range of optimality for the coefficients of the objective function.

8. For Problem 6*b*:

   a. Find the range of feasibility for each constraint, and interpret your answers.

   b. Determine the range of optimality for each coefficient of the objective function. Interpret your results.

9. A small firm makes three similar products, which all follow the same three-step process, consisting of milling, inspection, and drilling. Product A requires 12 minutes of milling, 5 minutes for inspection, and 10 minutes of drilling per unit; product B requires 10 minutes of milling, 4 minutes for inspection, and 8 minutes of drilling per unit; and product C requires 8 minutes of milling, 4 minutes for inspection, and 16 minutes of drilling. The department has 20 hours available during the next period for milling, 15 hours for inspection, and 24 hours for drilling. Product A contributes \$2.40 per unit to profit, B contributes \$2.50 per unit, and C contributes \$3.00 per unit. Determine the optimal mix of products in terms of maximizing contribution to profits for the period. Then, find the range of optimality for the profit coefficient of each variable.

10. Formulate and then solve a linear programming model of this problem, to determine how many containers of each product to produce tomorrow to maximize profits. The company makes four juice products using orange, grapefruit, and pineapple juice.

Product	Retail Price/Litre
Orange juice	\$1.00
Grapefruit juice	.90
Pineapple juice	.80
All-in-One	1.10

The All-in-One juice has equal parts of orange, grapefruit, and pineapple juice. Each product is produced in a one-litre size. On hand are 1,600 litres of orange juice, 1,200 litres of grapefruit juice, and 800 litres of pineapple juice. The cost per litre is $0.50 for orange juice, $0.40 for grapefruit juice, and $0.35 for pineapple juice.

In addition, the manager wants grapefruit juice to be used for no more than 30 percent of the number of containers produced. She wants the ratio of the number of containers of orange juice to the number of containers of pineapple juice to be at least 7 to 5.

11. A wood products firm uses leftover time at the end of each week to make goods for stock. Currently, two products on the list of items are produced for stock: a chopping board and a knife holder. Both items require three operations: cutting, gluing, and finishing. The manager of the firm has collected the following data on these products:

		TIME PER UNIT (MINUTES)		
Item	**Profit/Unit**	**Cutting**	**Gluing**	**Finishing**
Chopping board	$2	1.4	5	12
Knife holder	$6	0.8	13	3

The manager has also determined that, during each week, 56 minutes are available for cutting, 650 minutes are available for gluing, and 360 minutes are available for finishing.

a. Determine the optimal quantities of the decision variables.

b. Which resources are not completely used by your solution? How much of each resource is unused?

12. The manager of the deli section of a grocery superstore has just learned that the department has 112 kg of mayonnaise, of which 70 kg is approaching its expiration date and must be used. To use up the mayonnaise, the manager has decided to prepare two items: a ham spread and a deli spread. Each pan of the ham spread will require 1.4 kg of mayonnaise, and each pan of the deli spread will require 1.0 kg. The manager has received an order for 10 pans of ham spread and 8 pans of the deli spread. In addition, the manager has decided to have at least 10 pans of each spread available for sale. Both spreads will cost $3 per pan to make, but ham spread sells for $5 per pan and deli spread sells for $7 per pan.

a. Determine the solution that will minimize cost.

b. Determine the solution that will maximize profit.

13. A manager wants to know how many units of each product to produce on a daily basis in order to achieve the highest contribution to profit. Production requirements for the products are shown in the following table.

Product	**Material 1 (kg)**	**Material 2 (kg)**	**Labour (hours)**
A	2	3	3.2
B	1	5	1.5
C	6	—	2.0

Material 1 costs $5 per kg, material 2 costs $4 per kg, and labour costs $10 an hour. Product A sells for $80 a unit, product B sells for $90 a unit, and product C sells for $70 a unit. Available resources each day are 200 kg of material 1; 300 kg of material 2; and 150 hours of labour.

The manager must satisfy certain output requirements: The output of product A should not be more than one-third of the total number of units produced; the ratio of units of product A to units of product B should be 3 to 2; and there is a standing order for 5 units of product A each day. Formulate a linear programming model for this problem, and then solve.

14. A chocolate maker has contracted to operate a small candy counter in a fashionable store. To start with, the selection of offerings will be intentionally limited. The counter will offer a regular mix of candy made up of equal parts of cashews, raisins, caramels, and chocolates, and a deluxe mix that is one-half cashews and one-half chocolates, which will be sold in one-pound boxes. In addition, the candy counter will offer individual one-pound boxes of cashews, raisins, caramels, and chocolates.

A major attraction of the candy counter is that all candies are made fresh at the counter. However, storage space for supplies and ingredients is limited. Bins are available that can hold the amounts shown in the table:

Ingredient	Capacity (pounds per day)
Cashews	120
Raisins	200
Caramels	100
Chocolates	160

In order to present a good image and to encourage purchases, the counter will make at least 20 boxes of each type of product each day. Any leftover boxes at the end of the day will be removed and given to a nearby nursing home for goodwill.

The profit per box for the various items has been determined as follows:

Item	Profit per Box
Regular	$.80
Deluxe	.90
Cashews	.70
Raisins	.60
Caramels	.50
Chocolates	.75

a. Formulate the LP model.

b. Solve for the optimal values of the decision variables and the maximum profit.

15. Given this linear programming model, solve the model and then answer the questions that follow.

Maximize $12x_1 + 18x_2 + 15x_3$ where x_1 = the quantity of product 1 to make etc.

Subject to

Machine	$5x_1 + 4x_2 + 3x_3 \leq 160$ minutes
Labour	$4x_1 + 10x_2 + 4x_3 \leq 288$ hours
Materials	$2x_1 + 2x_2 + 4x_3 \leq 200$ pounds
Product 2	$x_2 \leq 16$ units
	$x_1, x_2, x_3 \geq 0$

a. Are any constraints binding? If so, which one(s)?

b. If the profit on product 3 were changed to $22 a unit, what would the values of the decision variables be? The objective function? Explain.

c. If the profit on product 1 were changed to $22 a unit, what would the values of the decision variables be? The objective function? Explain.

d. If 10 hours less of labour time were available, what would the values of the decision variables be? The objective function? Explain.

e. If the manager decided that as many as 20 units of product 2 could be produced (instead of 16), how much additional profit would be generated?

f. If profit per unit on each product increased by $1, would the optimal values of the decision variables change? Explain. What would the optimal value of the objective function be?

16. A garden store prepares various grades of pine bark for mulch: nuggets (x_1), mini-nuggets (x_2), and chips (x_3). The process requires pine bark, machine time, labour time, and storage space. The following model has been developed.

Maximize $9x_1 + 9x_2 + 6x_3$ (profit)

Subject to

Bark	$5x_1 + 6x_2 + 3x_3 \leq 600$ pounds
Machine	$2x_1 + 4x_2 + 5x_3 \leq 660$ minutes
Labour	$2x_1 + 4x_2 + 3x_3 \leq 480$ hours
Storage	$1x_1 + 1x_2 + 1x_3 \leq 150$ bags
	$x_1, x_2, x_3 \geq 0$

a. What is the marginal value of a pound of pine bark? Over what range is this price value appropriate?

b. What is the maximum price the store would be justified in paying for additional pine bark?

c. What is the marginal value of labour? Over what range is this value in effect?

d. The manager obtained additional machine time through better scheduling. How much additional machine time can be effectively used for this operation? Why?

e. If the manager can obtain *either* additional pine bark *or* additional storage space, which one should she choose, and how much (assuming additional quantities cost the same as usual)?

f. If a change in the chip operation increased the profit on chips from $6 per bag to $7 per bag, would the optimal quantities change? Would the value of the objective function change? If so, what would the new value(s) be?

g. If profits on chips increased to $7 per bag and profits on nuggets decreased by $.60, would the optimal quantities change? Would the value of the objective function change? If so, what would the new value(s) be?

h. If the amount of pine bark available decreased by 15 pounds, machine time decreased by 27 minutes, and storage capacity increased by five bags, would this fall in the range of feasibility for multiple changes? If so, what would the value of the objective function be?

CASE

Son, Ltd.

Son, Ltd., manufactures a variety of chemical products used by photoprocessors. Son was recently bought out by a conglomerate, and managers of the two organizations have been working together to improve the efficiency of Son's operations.

Managers have been asked to adhere to weekly operating budgets and to develop operating plans using quantitative methods whenever possible. The manager of one department has been given a weekly operating budget of $11,980 for production of three chemical products, which for convenience shall be referred to as Q, R, and W. The budget is intended to pay for direct labour and materials. Processing requirements for the three products, on a per-unit basis, are shown in the table.

The company has a contractual obligation for 85 units of product R per week.

Material A costs $4 per kg, as does material B. Labour costs $8 an hour.

Product Q sells for $122 a unit, product R sells for $115 a unit, and product W sells for $76 a unit.

The manager is considering a number of different proposals regarding the quantity of each product to produce. The manager is primarily interested in maximizing contribution. Moreover, the manager wants to know how much labour will be needed, as well as the amount of each material to purchase.

Questions

Prepare a report that addresses the following issues:

1. The optimal quantities of products and the necessary quantities of labour and materials.
2. One proposal is to make equal amounts of the products. What amount of each will maximize contribution, and what quantities of labour and materials will be needed? How much less will total contribution be if this proposal is adopted?
3. How would you formulate the constraint for material A if it were determined that there is a 5-percent waste factor for material A and equal quantities of each product are required?

Product	Labour (hours)	Material A (kg)	Material B (kg)
Q	5	2	1
R	4	2	—
W	2	1/2	2

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