10

Block Designs and Latin Squares

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Prerequisites: The prerequisites for this chapter are properties of integers and basic counting techniques. See Sections 3.4 and 5.1 of *Discrete Mathematics and Its Applications*.

Introduction

Suppose that you were given \$25,000 to conduct experiments in the following situation. Five different chemicals can be added during a manufacturing process. This process can take place at five different temperatures. Finally, the process can be performed on any of five different machines. Also, performing the process once costs \$1000.

To obtain the most information possible, you would want to perform the process with each different chemical as an additive at each different temperature on each different machine. By the product rule of counting, this would require 125 experiments, and you would need five times as much money as you have been allotted.

Fortunately, statistical analysts have come up with a tool called *analysis of variance* which can draw conclusions from less information. If each chemical can be used at each different temperature and on each different machine, much information can be obtained. Consider the matrix

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{ccccc} T_1 & T_2 & T_3 & T_4 & T_5 \end{array} \\ \begin{pmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 1 & 2 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \\ 5 & 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} \end{array}$$

Let the rows represent the five chemical additives C_1, C_2, C_3, C_4, C_5; the columns represent the temperatures T_1, T_2, T_3, T_4, T_5; and let the (i, j)th entry of the matrix be the machine number on which the process is performed. Since each of the numbers 1, 2, 3, 4, 5 appears exactly once in each row and exactly once in each column, every chemical appears with every machine, every temperature appears with every machine, and every (*chemical,temperature*) pair appears exactly once. So the experiments above can be performed for the allotted $25,000, and statistics can be used to gain meaningful information about the relations among additive, temperature, and machine. A matrix of the type above is known as a *Latin square*.

Latin Squares

To generalize this idea to an arbitrary number n, we need an arrangement of rows and columns in which each of the numbers $1, 2, \ldots, n$ appears exactly once in each row and column, permitting n tests to create a mix of the n states represented by the rows and the n states represented by the columns. Hence we have the following definition.

Definition 1 A *Latin square of order* n is an $n \times n$ matrix whose entries are the integers $1, 2, \ldots, n$, arranged so that each integer appears exactly once in each row and exactly once in each column. □

Example 1 One Latin square of order 5 is the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

It is easy to generalize this example to any n by making the first row $1, 2, \ldots, n$; the second row $2, 3, \ldots, n, 1$; and so on. □

In more complicated experiments, it may be necessary to have more than one Latin square of a given order. But we need some concept of what "different" means. Certainly

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

should not be different. To help clarify the situation, we introduce the idea of a *reduced* Latin square.

Definition 2 A Latin square is *reduced* if in the first row and in the first column the numbers $1, 2, \ldots, n$ appear in natural order. □

Example 2 The Latin square

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

can be transformed into reduced form by reversing the first two rows, obtaining

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$ □

Since interchanges of rows and columns only amount to changing the names of those rows and columns, it is always permissible to do this. Thus we can always transform a Latin square to reduced form. Of course, interchanging numbers inside the matrix only amounts to a renumbering of the "machines", which still is not essentially different.

Definition 3 Two Latin squares are *equivalent* if one can be transformed into the other by rearranging rows, rearranging columns, and renaming elements. □

Even reduced Latin squares can be equivalent, as illustrated in the following example.

Example 3 Show that the Latin squares

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

are equivalent.

Solution: We first interchange rows 3 and 4 of A to obtain

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

We then interchange columns 3 and 4, obtaining

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Finally, by interchanging the "names" 3 and 4, we obtain B. □

Evidently, it is not obvious whether or not two Latin squares are equivalent. But, since equivalence is an equivalence relation on Latin squares, there is some finite number of distinct classes of non-equivalent Latin squares of order n.

Orthogonal Latin Squares

Now reconsider the problem of the Introduction. Suppose in addition to chemical, temperature, and machine, the day of the week on which the process is performed is also significant. Still each chemical and temperature must go together — now not only on each machine exactly once, but also on each day of the week exactly once.

If we use two Latin squares, one for the machine number and the other for the day of the week, the machines and days of the week will still be properly mixed if the two Latin squares are related properly, referred to as *orthogonal*.

Definition 4 Two $n \times n$ Latin squares $L_1 = (a_{ij})$ and $L_2 = (b_{ij})$ are called *orthogonal* if every ordered pair of symbols (k_1, k_2), $1 \le k_1 \le n$, $1 \le k_2 \le n$, occurs among the n^2 ordered pairs (a_{ij}, b_{ij}). □

Example 4 Show that the two Latin squares

$$L_1 = \begin{pmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 1 & 2 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \\ 5 & 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 1 & 2 & 4 & 3 & 5 \\ 3 & 4 & 1 & 5 & 2 \\ 2 & 3 & 5 & 4 & 1 \\ 5 & 1 & 3 & 2 & 4 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

are orthogonal.

Solution: Simply looking at the 25 pairs (a_{ij}, b_{ij}), where a_{ij} is the entry in row i and column j of L_1 and b_{ij} is the entry in row i and column j of L_2 establishes the fact. Note, for instance, that the pair $(1, 1)$ comes from row i, column 1; $(1, 2)$ from row 4, column 4; $(1, 3)$ from row 5, column 5; $(1, 4)$ from row 2, column 2; and $(1, 5)$ from row 3, column 3. □

Suppose we use the two Latin squares in Example 4 to schedule our 25 experiments. Looking at row i, column j tells us when to use chemical C_i and temperature T_j. The entry from L_1 tells the machine, and the entry from L_2 tells the day of the week. Hence, since in row 2 and column 4, L_1 has entry 4 and L_2 has entry 5, chemical C_2 would be used with temperature T_4 on machine 4 on day 5 (Friday). Since L_1 and L_2 are orthogonal, each (*machine,day*) pair occurs exactly once, as of course does each (*chemical,temperature*) pair.

Of course, this is just an illustration when $n = 5$. But the same principles clearly apply for every n. Thus, we obtain the following theorem.

Theorem 1 If there are two orthogonal Latin squares of order n, it is possible to schedule a series of n^2 experiments with four different variable elements, each of which has n possible states, such that any ordered pair of two states appears exactly once with each ordered pair of the other two states. ■

The problem of determining for which n there are orthogonal Latin squares of order n has a long history. There are, in fact, no two orthogonal Latin squares of orders 2 or 6. For a long time it was conjectured that there were no orthogonal Latin squares of order $2n$ if n is odd. But in fact such squares have been found for $2n \ge 10$, where n is odd.

Finite Projective Planes

In fact, it is sometimes possible to find $n-1$ mutually orthogonal Latin squares of order n. The simplest such example consists of the orthogonal Latin squares

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

of order 3. With each set of $n-1$ mutually orthogonal Latin squares of order n, it is possible to associate a geometric object called a *finite projective plane*.

Definition 5 A *projective plane* consists of two sets of elements called points and lines (each line is a subset of points) such that each two points belong to exactly one line and each two lines intersect in exactly one point. □

The $n-1$ mutually orthogonal Latin squares of order n, when they exist, can be used to generate a projective plane of order n. We state the following theorem without proof and then illustrate it with an example.

Theorem 2 A finite projective plane necessarily has the same number of points, $n+1$, on every line. (Such a plane is said to be of *order* n.) It also has $n+1$ lines through every point, and a total of n^2+n+1 points and n^2+n+1 lines. ■

Example 5 Find the projective plane associated with $n-1$ mutually orthogonal Latin squares $L_1, \ldots, L_{n-1}$ of order n, and illustrate it when $n = 3$.

Solution: Let the points be

$$\{a_{ij} \,|\, 1 \le i,\, j \le n\} \cup R \cup C \cup \{P_i \,|\, 1 \le i \le n-1\}.$$

The a_{ij} correspond to the entries in the matrix, R is for row, C is for column, and the P_i correspond to the $n-1$ Latin squares.

The lines are

1) $a_{11}, a_{12}, \ldots, a_{1n}, R$

$\vdots$

n) $a_{n1}, a_{n2}, \ldots, a_{nn}, R$

n+1) $a_{11}, a_{21}, \ldots, a_{n1}, C$

$\vdots$

2n) $a_{1n}, a_{2n}, \ldots, a_{nn}, C$

These correspond to the rows and columns of the matrices.

There are also lines joining the points whose entries in L_1 are 1 and P_1, whose entries in L_1 are 2 and P_1, ..., whose entries in L_1 are n and P_1. There is a similar set of n lines for each Latin square (lines found using the kth Latin square include the point P_k). Finally, there is the line containing R, C, P_1, ..., P_{n-1}.

In particular, when $n = 3$ there are 13 points:

$$a_{11},\ a_{12},\ a_{13},\ a_{21},\ a_{22},\ a_{23},\ a_{31},\ a_{32},\ a_{33},\ R,\ C,\ P_1, \text{ and } P_2.$$

The thirteen lines are

1) a_{11}, a_{12}, a_{13}, R
2) a_{21}, a_{22}, a_{23}, R
3) a_{31}, a_{32}, a_{33}, R
4) a_{11}, a_{21}, a_{31}, C
5) a_{12}, a_{22}, a_{32}, C
6) a_{13}, a_{23}, a_{33}, C

from L_1:

7) a_{11}, a_{23}, a_{32}, P_1
8) a_{12}, a_{21}, a_{33}, P_1
9) a_{13}, a_{22}, a_{31}, P_1

from L_2:

10) a_{11}, a_{22}, a_{33}, P_2
11) a_{12}, a_{23}, a_{31}, P_2
12) a_{13}, a_{21}, a_{32}, P_2

from the extra points:

13) R, C, P_1, P_2

Examination shows that every line contains four points, every point is on four lines, every two lines intersect in a point, every two points determine a line. □

The theory of finite projective planes is extremely rich. Both Ryser [5] and Hall [3] have entire chapters on the subject. Ryser shows that there is always a projective plane of order p^k is p is a prime and k is a positive integer. He also shows that if $n \equiv 1$ or 2 (mod 4) and some prime factor of n which is congruent to 3 (mod 4) occurs to an odd power in the prime factorization of n, then there is no projective plane of order n.

Block Designs

We now return to the situation of the Introduction, but we suppose that there are only four temperatures that are possible, instead of five. Still, deleting the last column of the Latin square gives a reasonable experimental design.

Example 6 The matrix

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 3 & 1 & 2 & 4 \\ 2 & 5 & 1 & 3 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{pmatrix}$$

is obtained from the Latin square of the Introduction by deleting the last column. This effectively removes one temperature. Still, each machine number occurs four times, once in each column, and it appears in exactly four of the five rows. So, while it is not complete, it is still balanced. □

For the general situation, the machine numbers are called **varieties**, the rows **blocks**, and the number of columns k.

Definition 6 A *balanced incomplete block design* (BIBD), also called a (b, v, r, k, λ)*-design*, comprises a set of v varieties arranged in b blocks in such a way that

i) each block has the same number $k < v$ of varieties, with no variety occurring twice in the same block;
ii) each variety occurs in exactly r blocks;
iii) each pair of varieties occurs together in exactly λ blocks. □

For instance, the 5×4 matrix of Example 6 is a $(5, 5, 4, 4, 3)$-design.

Since each pair of varieties occurs together in the same number of blocks, the design is called *pairwise balanced*. In general, if each set of t varieties occurs together in the same number of blocks, the design is called a t**-design**. The design in Example 6 is both a 4-design (every ordered 4-tuple appears once) and a 3-design (every ordered triple appears twice). Note that the only thing that prevents a Latin square from being a BIBD is its completeness ($k = v$). Since the symmetry occurring in a Latin square is rarely present in actual experiments, the construction of BIBDs is an important ability in designing experiments.

Usually, BIBDs are specified by listing their blocks. For instance, the $(5, 5, 4, 4, 3)$–design in Example 6 is

$$\{\{1, 2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}.$$

There are several relationships among the parameters b, v, r, k, and λ, as stated in the following theorem.

Theorem 3 In a (b, v, r, k, λ)-design,

i) $bk = vr$

ii) $\lambda(v-1) = r(k-1)$

iii) $r > \lambda$

iv) $b \geq v$.

Proof: Only the first three parts will be proved; the fourth is beyond the scope of the book; see [5] for a proof.

i) Since each of the b blocks has k varieties, bk counts the total number of varieties; but since each variety is in r blocks, vr also counts this. Hence $bk = vr$.

ii) Since each pair of varieties occurs in exactly λ blocks, the total number of varieties (other than v_1) occurring in the blocks containing v_1 is $\lambda(v-1)$. But since v_1 occurs in exactly r blocks, that number is also $r(k-1)$.

iii) Since $k < v$, we have $k - 1 < v - 1$. But, $r(k-1) = \lambda(v-1)$, so we must have $r > \lambda$. ■

As noted in the discussion preceding Theorem 3, if every set of t varieties occurs in the same number of blocks, λ_t, the design is called a t-design. So, of course, every BIBD is a 1-design (with $\lambda_1 = r$) and a 2-design (with $\lambda_2 = \lambda$), from Definition 6. Usually, a design is called a t-design for the largest t possible for that design. In fact, a t-design is often referred to as a t-(b, v, k, λ_t) design, and the r and k of Definition 6 are not given. The following theorem shows that they can be determined from t, b, v, k, and λ_t.

Theorem 4 If $0 < s < t$, and D is a t–(b, v, k, λ_t) design, then D is also an s–(b, v, k, λ_s) design, where λ_s can be determined from λ_t.

Proof: We need only show D is a $(t-1)$-design. To do this, let S be a fixed set of $t-1$ varieties, and suppose S is a subset of λ_{t-1} blocks of D. Each block containing S also contains $k - (t-1)$ other varieties. Any one of these together with S forms a t-set containing S. Since S is in λ_{t-1} blocks, there are $\lambda_{t-1}(k-t+1)$ such t-sets altogether.

But we can count this number of t-sets in another way. The design D contains $v - (t-1)$ varieties other than the ones in S. Each one of these together with S forms a t-set. By hypothesis, each such t-set is in λ_t blocks. Thus, the number of t-sets is also $\lambda_t(v-t+1)$.

Hence $\lambda_{t-1}(k-t+1) = \lambda_t(v-t+1)$, and so λ_{t-1} is independent of the choice of S. ■

Corollary 1 Every t-design is a BIBD with $r = \lambda_1$, $\lambda = \lambda_2$. ■

For instance, the BIBD obtained at the beginning of this section is a 4-design with $b = 5$, $v = 5$, $k = 4$, $\lambda_4 = 1$. From the formula of the proof,

$$\lambda_3(4 - 4 + 1) = \lambda_4(5 - 4 + 1), \text{ so } \lambda_3 = 2$$

$$\lambda_2(4 - 3 + 1) = \lambda_3(5 - 3 + 1), \text{ so } \lambda_2 = 3$$

$$\lambda_1(4 - 2 + 1) = \lambda_3(5 - 2 + 1), \text{ so } \lambda_1 = 4$$

Thus, this 4-design is indeed a $(5, 5, 4, 4, 3)$-design.

Of course, any Latin square of order n can be used to obtain an $(n - 1)$-design just by deleting one column.

Some other interesting designs that occur are the $(n^2 + n + 1,\ n^2 + n + 1,\ n + 1,\ n + 1,\ 1)$ designs obtained from the projective plane of order n by making the varieties the points and making the blocks the lines. Since every line contains $q+1$ points, every point is on $q+1$ lines, and two points determine a unique line, this is indeed a BIBD.

Steiner Triple Systems

Another often-studied class of BIBDs is the class with $k = 3$ and $\lambda = 1$.

Definition 7 A $(b, v, r, 3, 1)$-design is called a *Steiner triple system*. □

While these are named for Jacob Steiner, they first arose in a problem called *Kirkman's Schoolgirl Problem* (1847).

Example 7 Suppose that 15 girls go for a walk in groups of three. They do this each of the seven days of the week. Can they choose their walking partners so that each girl walks with each other girl exactly once in a week?

Solution: This amounts to finding a Steiner triple system with $v = 15$ (girls), $b = 35$ (there are five groups of girls each day for seven days), and $r = 7$ (each girl walks on seven days). Further, the 35 blocks must be divided into seven groups of five so that each girl appears exactly once in each group.

Number the girls $0, 1, 2, \ldots, 14$. Let the first group be

$$\{14, 1, 2\}, \{3, 5, 9\}, \{11, 4, 0\}, \{7, 6, 12\}, \{13, 8, 10\}.$$

Then let group i $(1 \le i \le 6)$ be obtained as follows:

a) 14 remains in the same place,
b) if $0 \le k \le 13$, k is replaced by $(k + 2i)$ **mod** 14.

It is left as an exercise to show that the seven groups obtained actually give 35 different blocks of the type desired. □

Steiner conjectured in 1853 that Steiner triple systems exist exactly when $v \geq 3$ and $v \equiv 1$ or $3 \pmod 6$. This has proven to be the case. In the case where $v = 6n + 3$, we have $b = (2n + 1)(3n + 1)$. (See Exercise 9.)

If the b triples can be partitioned into $3n+1$ components with each variety appearing exactly once in each component, the system is called a *Kirkman triple system*. Notice that the solution to the Kirkman schoolgirl problem has $n = 2$. For more results about Steiner triple systems, see [3] or [5].

Suggested Readings

1. P. Dembowski, *Finite Geometries*, Springer, New York, 1997.
2. J. Dénes and A. D. Keedwell, *Latin Squares and their Applications*, Academic Press, Burlington, MA, 1991.
3. M. Hall, *Combinatorial Theory*, Second Edition, John Wiley & Sons, Hoboken, N.J., 1998.
4. J. Riordan, *An Introduction to Combinatorial Analysis*, Dover Publications, Mineola, N.Y., 2002.
5. H. Ryser, *Combinatorial Mathematics*, Carus Monograph #14, Mathematical Association of America, 1963.

Exercises

1. Show that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

are equivalent.

2. Find a reduced Latin square equivalent to

$$\begin{pmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 1 & 2 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \\ 5 & 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}$$

⋆**3.** Show that "equivalent" is an equivalence relation on the set of $n \times n$ Latin squares.

4. Find a Latin square orthogonal to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

⋆**5.** a) Find a set of three mutually orthogonal Latin squares of order 4.
b) How many points are there in a finite projective plane of order 4?
c) How many points are on a typical line?
d) List all the points and lines.

6. Suppose a BIBD has 69 blocks, 24 varieties, and 8 varieties in a block. What are all the parameters?

7. Delete a column from the Latin square of Exercise 4 to obtain a 3-design. What are its parameters?

8. Construct a BIBD from the finite projective plane in the text. What are its parameters?

9. Suppose a Steiner triple system has $6n+3$ varieties. Show that $r = 3n+1$ and $b = (2n+1)(3n+1)$.

10. a) Find the other six groups of the solution to the Kirkman schoolgirl problem.
b) With whom does schoolgirl 5 walk on Thursday (assume Sunday is day 0).
c) When does schoolgirl 14 walk with schoolgirl 0? Who is their companion?

⋆**11.** Alice, Betty, Carol, Donna, Evelyn, Fran, Georgia, Henrietta, and Isabel form a swimming group. They plan to swim once a week, with one group of three swimming each of Monday, Wednesday, and Friday. Over the course of four weeks, each woman wants to swim with each other woman. Construct a schedule for them.

⋆**12.** There is a unique Steiner triple system with $v = 7$.
a) What are its parameters as a BIBD?
b) Construct it.
c) Use it to construct a finite projective plane. What order is it?

Computer Projects

1. Write a computer program to determine whether or not two Latin squares are orthogonal.

2. Write a computer program to determine r and λ in a BIBD if b, v, and k are given. The output should list all of b, v, k, r, λ and should say "this is an impossible configuration" if r or λ turns out not to be an integer.