APPENDIX E

Mathematical Review

In this appendix, we review some mathematical concepts that may help you to better understand the topics covered in the book. Perhaps the most important concept in data communication is signals and their representation. We start with a brief review of trigonometric functions, as discussed in a typical precalculus book. We then briefly discuss Fourier analysis, which provides a tool for the transformation between the time and frequency domains. We finally give a brief treatment of exponential and logarithmic functions.

E.1 TRIGONOMETRIC FUNCTIONS

Let us briefly discuss some characteristics of the trigonometric functions as used in the book.

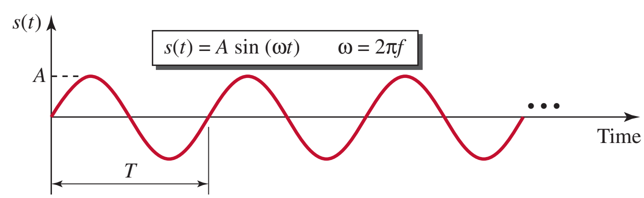
E.1.1 Sine Wave

We can mathematically describe a sine wave as

$$s(t) = A \sin(2\pi f t) = A \sin\left(\frac{2\pi}{T} t\right)$$

where s is the instantaneous amplitude, A is the peak amplitude, f is the frequency, and T is the period (phase will be discussed later). Figure E.1 shows a sine wave.

Figure E.1 *A sine wave*



Note that the value of $2\pi f$ is called the *radian frequency* and written as ω (omega), which means that a sine function can be written as $s(t) = A \sin(\omega t)$.

Example E.1

Find the peak value, frequency, and period of the following sine waves.

a. $s(t) = 5 \sin(10\pi t)$

b. $s(t) = \sin(10t)$

Solution

a. Peak amplitude: $A = 5$
Frequency: $10\pi = 2\pi f$, so $f = 5$
Period: $T = 1/f = 1/5$ s

b. Peak amplitude: $A = 1$
Frequency: $10 = 2\pi f$, so $f = 10/(2\pi) = 1.60$
Period: $T = 1/f = 1/1.60 = 0.628$ s

Example E.2

Show the mathematical representation of a sine wave with a peak amplitude of 2 and a frequency of 1000 Hz.

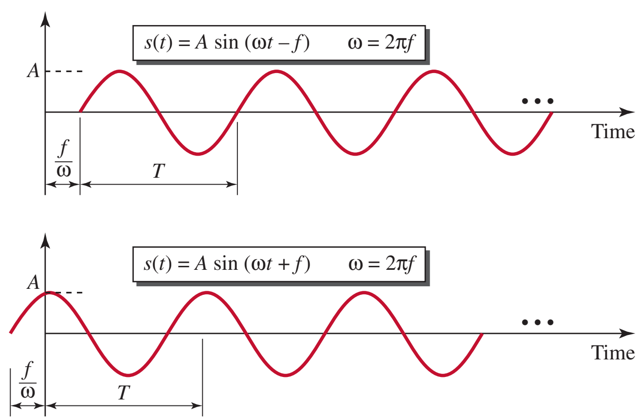
Solution

The mathematical representation is $s(t) = 2 \sin(2000\pi t)$.

Horizontal Shifting (Phase)

All the sine functions we discussed so far have an amplitude of value 0 at the origin. What if we shift the signal to the left or to the right? Figure E.2 shows two simple sine waves, one shifted to the right and one to the left.

Figure E.2 *Two horizontally shifted sine waves*



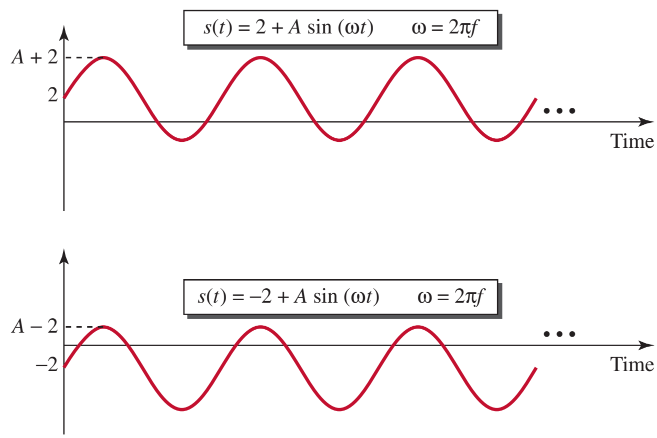
When a signal is shifted to the left or right, its first zero crossing will be at a point in time other than the origin. To show this, we need to add or subtract another constant to ωt, as shown in the figure.

Shifting a sine wave to the left or right is a positive or negative shift, respectively.

Vertical Shifting

When a sine wave is shifted vertically, a constant is added to the instantaneous amplitude of the signal. For example, if we shift a sine wave 2 units of amplitude upward, the signal becomes $s(t) = 2 + \sin(\omega t)$; if we shift it 2 units of amplitude downward, we have $s(t) = -2 + \sin(\omega t)$. Figure E.3 shows the idea.

Figure E.3 *Vertical shifting of sine waves*



E.1.2 Cosine Wave

If we shift a sine wave $T/2$ to the left, we get what is called a cosine wave (cos).

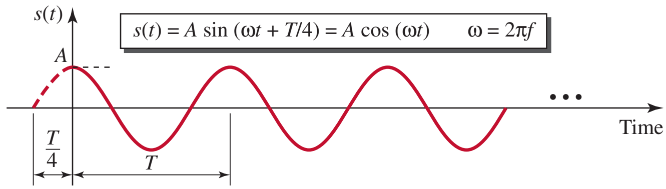
$$A \sin(\omega t + \pi/2) = A \cos(\omega t)$$

Figure E.4 shows a cosine wave.

E.1.3 Other Trigonometric Functions

There are many trigonometric functions; two of the more common are $\tan(\omega t)$ and $\cot(\omega t)$. They are defined as $\tan(\omega t) = \sin(\omega t)/\cos(\omega t)$ and $\cot(\omega t) = \cos(\omega t)/\sin(\omega t)$. Note that tan and cot are the inverse of each other.

Figure E.4 *A cosine wave*



E.1.4 Trigonometric Identities

There are several identities between trigonometric functions that we sometimes need to know. Table E.1 gives these identities for reference. Other identities can be easily derived from these.

Table E.1 *Some trigonometric identities*

Name	*Formula*
Pythagorean	$\sin^2 x + \cos^2 x = 1$
Even/odd	$\sin(-x) = -\sin(x)$ $\cos(-x) = \cos(x)$
Sum	$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
Difference	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
Product to sum	$\sin(x)\sin(y) = 1/2\,[\cos(x - y) - \cos(x + y)]$ $\cos(x)\cos(y) = 1/2\,[\cos(x - y) + \cos(x + y)]$ $\sin(x)\cos(y) = 1/2\,[\sin(x + y) + \sin(x - y)]$ $\cos(x)\sin(y) = 1/2\,[\sin(x + y) - \sin(x - y)]$

E.2 FOURIER ANALYSIS

Fourier analysis is a tool that changes a time-domain signal to a frequency-domain signal and vice versa.

E.2.1 Fourier Series

Fourier proved that a composite periodic signal with period T (frequency f) can be decomposed into a series of sine and cosine functions in which each function is an integral harmonic of the fundamental frequency f of the composite signal. The result is called the ***Fourier series.*** In other words, we can write a composite signal as shown in Figure E.5. Using the series, we can decompose any periodic signal into its harmonics. Note that A_0 is the average value of the signal over a period, A_n is the coefficient of the nth cosine component, and B_n is the coefficient of the nth sine component.

Figure E.5 *Fourier series and coefficients of terms*

Fourier series

$$s(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(2\pi nft) + \sum_{n=1}^{\infty} B_n \cos(2\pi nft)$$

Coefficients

$$A_0 = \frac{1}{T}\int_0^T s(t)\,dt \qquad A_n = \frac{2}{T}\int_0^T s(t)\cos(2\pi nft)\,dt$$

$$B_n = \frac{2}{T}\int_0^T s(t)\sin(2\pi nft)\,dt$$

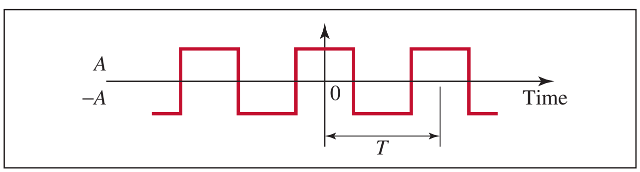
Fourier series	
Time domain: periodic	**Frequency domain: discrete**

Example E.3

Let us show the components of a square wave signal as seen in Figure E.6. The figure also shows the time domain and the frequency domain. According to the figure, such a square wave signal has only A_n coefficients. Note also that the value of $A_0 = 0$ because the average value of the signal is 0; it is oscillating above and below the time axis. The frequency domain of the signal is discrete; only odd harmonics are present and the amplitudes are alternatively positive and negative. A very important point is that the amplitude of the harmonics approaches zero as we move toward infinity. Something which is not shown in the figure is the phase. However, we know that all components are cosine waves, which means that each has a phase of 90°.

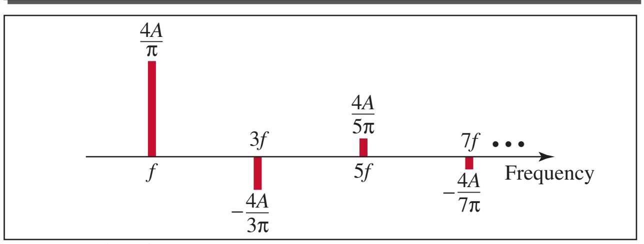
Figure E.6 *Finding the Fourier series of a periodic square function*

Time domain



$$A_0 = 0 \qquad A_n = \begin{cases} \dfrac{4A}{n\pi} & \text{for } n = 1, 5, 9, \ldots \\ \dfrac{-4A}{n\pi} & \text{for } n = 3, 7, 11, \ldots \end{cases} \qquad B_n = 0$$

$$s(t) = \frac{4A}{\pi}\cos(2\pi ft) - \frac{4A}{3\pi}\cos(2\pi 3ft) + \frac{4A}{5\pi}\cos(2\pi 5ft) - \frac{4A}{7\pi}\cos(2\pi 7ft) + \cdots$$



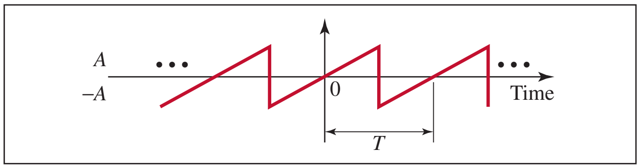
Frequency domain

Example E.4

Now let us show the components of a sawtooth signal as seen in Figure E.7. This time, we have only B_n components (sine waves). The frequency spectrum, however, is denser; we have all harmonics (f, $2f$, $3f$, …). A point which is not clear from the diagram is the phase. All components are sine waves, which means each component has a phase of 0°.

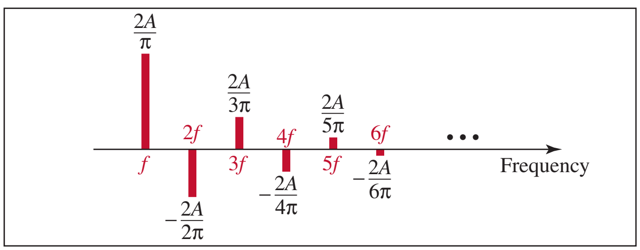
Figure E.7 *Finding the Fourier series for a sawtooth signal*

Time domain



$A_0 = 0 \qquad A_n = 0 \qquad B_n = \begin{cases} \frac{2A}{n\pi} & \text{for } n \text{ odd} \\ \frac{-2A}{n\pi} & \text{for } n \text{ even} \end{cases}$

$$s(t) = \frac{2A}{\pi}\sin(2\pi ft) - \frac{2A}{2\pi}\sin(2\pi 2ft) + \frac{2A}{3\pi}\sin(2\pi 3ft) - \frac{2A}{4\pi}\sin(2\pi 4ft) + \cdots$$



Frequency domain

E.2.2 Fourier Transform

While the Fourier series gives the discrete frequency domain of a periodic signal, the **Fourier transform** gives the continuous frequency domain of a nonperiodic signal. Figure E.8 shows how we can create a continuous frequency domain from a nonperiodic time-domain function and vice versa.

Figure E.8 *Fourier transform and inverse Fourier transform*

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt$$

Fourier transform

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}\,dt$$

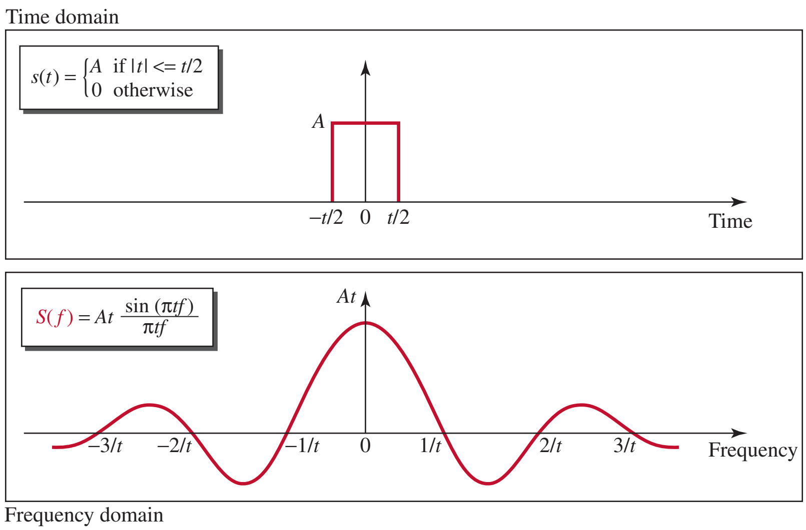
Inverse Fourier transform

Fourier transform
Time domain: nonperiodic **Frequency domain: continuous**

Example E.5

Figure E.9 shows the time and frequency domains of a single square pulse. The time domain is between –τ/2 and τ/2; the frequency domain is a continuous function that stretches from negative infinity to positive infinity. Unlike the previous examples, the frequency domain is continuous; all frequencies are there, not just the integral ones.

Figure E.9 *Finding the Fourier transform of a square pulse*



Time-limited signal: $s(t) = 0$ for $|t| \leq T$
Band-limited signal: $S(f) = 0$ for $|f| \leq B$

Time-Limited and Band-Limited Signals

Two very interesting concepts related to the Fourier transform are the time-limited and band-limited signals. A **time-limited** signal is a signal for which the amplitude of $s(t)$ is nonzero only during a period of time; the amplitude is zero everywhere else. A **band-limited** signal, on the other hand, is the signal for which the amplitude of $S(f)$ is nonzero only for a range of frequencies; the amplitude is zero everywhere else. A band-limited signal plays a very important role in the sampling theorem and Nyquist frequency because the corresponding time domain can be represented as a series of samples.

E.3 EXPONENT AND LOGARITHM

In solving networking problems, we often need to know how to handle exponential and logarithmic functions. This section briefly reviews these two concepts.

E.3.1 Exponential Function

The exponential function with **base** a is defined as $\boldsymbol{y = a^x}$. If x is an integer (integral value), we can easily calculate the value of y by multiplying the value of a by itself x times.

Example E.6

Calculate the value of the following exponential functions.

a. $y = 3^2$ **b.** $y = 5.2^6$

Solution

a. $y = 3 \times 3 = 9$ **b.** $y = 5.2 \times 5.2 \times 5.2 \times 5.2 \times 5.2 \times 5.2 = 19{,}770.609664$

If x is not an integer, we need to use a calculator.

Example E.7

Calculate the value of the following exponential functions.

a. $y = 3^{2.2}$ **b.** $y = 5.2^{6.3}$

Solution

a. $y = 11.212$ (approximately) **b.** $y = 32{,}424.60$ (approximately)

Example E.8

One very common base used in science and mathematics is the **natural base** $\boldsymbol{e}$, which has the value 2.71828183…Most calculators show this function as e^x, which can be calculated easily by entering only the value of the exponent.

Example E.9

Calculate the value of the following exponential functions.

a. $y = e^4$ **b.** $y = e^{6.3}$

Solution

a. $y = 54.56$ (approximately) **b.** $y = 544.57$ (approximately)

Properties of the Exponential Function

Exponential functions have several properties; some are useful to us in this text:

$$a^0 = 1 \qquad a^1 = a \qquad a^{-x} = 1 / a^x$$

Example E.10

The third property is useful to us because we can calculate the value of an exponential function with a negative value. We first calculate the positive value and we then invert the result.

a. $y = e^{-4}$ **b.** $y = e^{-6.3}$

Solution

Using the result of the previous example, we have

a. $y = 1/54.56 = 0.0183$ **b.** $y = 1/544.57 = 0.00183$

E.3.2 Logarithmic Function

A logarithmic function is the inverse of an exponential function, as shown below. Just as in the exponential function, *a* is called the base of the logarithmic function:

$$y = a^x \qquad x = \log_a y$$

In other words, if x is given, we can calculate y by using the exponential function; if y is given, we can calculate x by using the logarithmic function.

Exponential and logarithmic functions are the inverse of each other.

Example E.11

Calculate the value of the following logarithmic functions.

a. $x = \log_3 9$ **b.** $x = \log_2 16$

Solution

We have not yet shown how to calculate the log function in different bases, but we can solve this problem intuitively.

- **a.** Because $3^2 = 9$, we can say that $\log_3 9 = 2$, using the fact that the two functions are the inverse of each other.
- **b.** Because $2^4 = 16$, we can say that $\log_2 16 = 4$ by using the previous fact.

Two Common Bases

The two common bases for logarithmic functions, those that can be handled by a calculator, are base *e* and base 10. The logarithm in base *e* is normally shown as ln (natural logarithm); the logarithm in base 10 is normally shown as log (omitting the base).

Example E.12

Calculate the value of the following logarithmic functions.

a. $x = \log 233$ **b.** $x = \ln 45$

Solution

For these two bases we can use a calculator.

a. $x = \log 233 = 2.367$ **b.** $x = \ln 45 = 3.81$

Base Transformation

We often need to find the value of a logarithmic function in a base other than *e* or 10. If the available calculator cannot give the result in our desired base, we can use a very fundamental property of the logarithm, base transformation, as shown:

$$\log_a y = \log_b y \,/\, \log_b a$$

Note that the right-hand side is two log functions with base *b*, which is different from the base *a* at the left-hand side. This means that we can choose a base that is available in our calculator (base *b*) and find the log of a base that is not available (base *a*).

Example E.13

Calculate the value of the following logarithmic functions.

a. $x = \log_3 810$ **b.** $x = \log_5 600$

Solution

These two bases, 3 and 5, are not available on a calculator, but we can use base 10 which is available.

a. $x = \log_3 810 = \log_{10} 810 / \log_{10} 3 = 6.095$

b. $x = \log_5 600 = \log_{10} 600 / \log_{10} 5 = 3.975$

Properties of Logarithmic Functions

Like an exponential function, a logarithmic function has some properties that are useful in simplifying the calculation of a log function.

First:	$\log_a 1 = 0$
Second:	$\log_a a = 1$
Third:	$\log_a 1/x = -\log_a x$
Fourth:	$\log_a (x \times y) = \log_a x + \log_a y$
Fifth:	$\log_a x/y = \log_a x - \log_a y$
Sixth:	$\log_a x^y = y \times \log_a x$

Example E.14

Calculate the value of the following logarithmic functions.

a. $x = \log_3 1$

b. $x = \log_3 3$

c. $x = \log_{10\,(1/10)}$

d. $\log_a (x \times y)$ if we know that $\log_a x = 2$ and $\log_a y = 3$

e. $\log_2 (1024)$ without using a calculator

Solution

We use the property of log functions to solve the problems.

a. $x = \log_3 1 = 0$

b. $x = \log_3 3 = 1$

c. $x = \log_{10\,(1/10)} = \log_{10} 10^{-1} = -\log_{10} 10 = -1$

d. $\log_a (x \times y) = \log_a x + \log_a y = 2 + 3 = 5$

e. $\log_2 (1024) = \log_2 (2^{10}) = 10 \log_2 2 = 10 \times 1 = 10$

E.4 MODULAR ARITHMETIC

In integer arithmetic, if we divide a by n, we can get q and r. The relationship between these four integers can be shown as

$$a = q \times n + r$$

In this relationship, a is called the *dividend;* q, the *quotient;* n, the *divisor;* and r, the *remainder.*

Example E.15

Assume that $a = 255$ and $n = 11$. Find the value of q and r.

Solution

We can find $q = 23$ and $r = 2$ using the division algorithm we have learned in arithmetic.

E.4.1 Modulo Operator

In **modular arithmetic,** we are interested in only one of the outputs, the remainder r. We don't care about the quotient q. In other words, we want to know the value of r when we divide a by n. This implies that we can change the above relation into a binary operator with two inputs a and n and one output r. The above-mentioned binary operator is called the ***modulo operator*** and is shown as *mod*.

Example E.16

Find the result of the following operations:

a. 27 mod 5

b. 36 mod 12

c. −18 mod 14

d. −7 mod 10

Solution

We are looking for the residue r. We can divide the a by n and find q and r. We can then disregard q and keep r.

a. Dividing 27 by 5 results in $r = 2$. This means that 27 mod 5 = 2.

b. Dividing 36 by 12 results in $r = 0$. This means that 36 mod 12 = 0.

c. Dividing −18 by 14 results in $r = -4$. However, we need to add the modulus (14) to make it nonnegative. We have $r = -4 + 14 = 10$. This means that −18 mod 14 = 10.

d. Dividing −7 by 10 results in $r = -7$. After adding the modulus to −7, we have $r = 3$. This means that −7 mod 10 = 3.

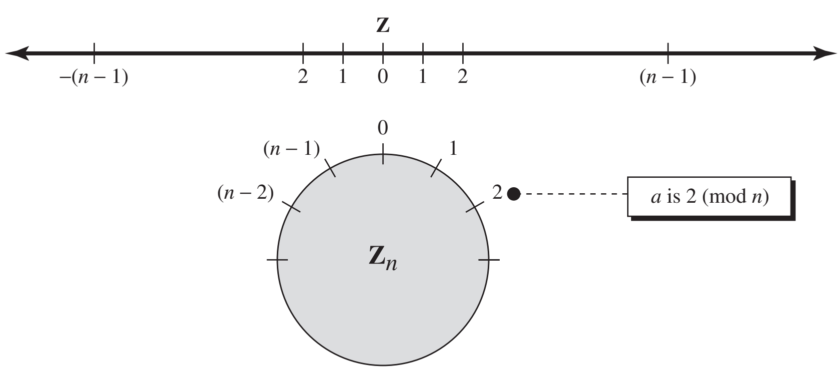
E.4.2 Set of Residues: $\mathbf{Z}_n$

The result of the modulo operation with modulus n is always an integer between 0 and $n - 1$. In other words, the result of a mod n is always a nonnegative integer less than n. We can say that the modulo operation creates a set, which in modular arithmetic is referred to as the **set of least residues modulo *n***, or $\mathbf{Z}_n$. However, we need to remember that although we have only one set of integers ($\mathbf{Z}$), we have infinite instances of the set of residues ($\mathbf{Z}_n$), one for each value of n.

E.4.3 Circular Notation

The concept of congruence can be better understood with the use of a circle. Just as we use a line to show the distribution of integers in $\mathbf{Z}$, we can use a circle to show the distribution of integers in $\mathbf{Z}_n$. Figure E.10 shows the comparison between the two. Integers 0 to $n - 1$ are spaced evenly around a circle. All integers modulo n occupy the

Figure E.10 *Comparison of $\mathbf{Z}$ and $\mathbf{Z}_n$ using graphs*



same point on the circle. Positive and negative integers from **Z** are mapped to the circle in such a way that there is a symmetry between them.

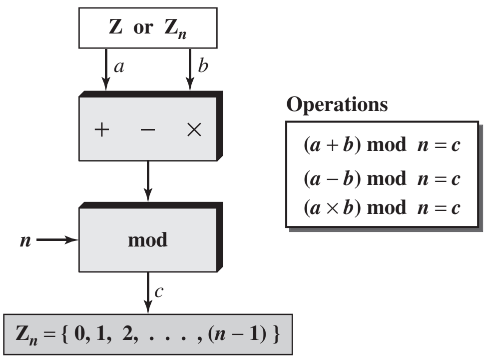
Example E.17

We use modular arithmetic in our daily life; for example, we use a clock to measure time. Our clock system uses modulo 12 arithmetic. However, instead of a 0 we use the number 12. So our clock system starts with 0 (or 12) and goes until 11. Because our days last 24 hours, we navigate around the circle two times and denote the first revolution as A.M. and the second as P.M.

E.4.4 Operations in $\mathbf{Z}_n$

The three binary operations (*addition, subtraction,* and *multiplication*) that we discussed for the set **Z** can also be defined for the set $\mathbf{Z}_n$. The result may need to be mapped to $\mathbf{Z}_n$ using the mod operator as shown in Figure E.11.

Figure E.11 *Binary operations in* $\mathbf{Z}_n$



Actually, two sets of operators are used here. The first set is one of the binary operators (+, − , ×); the second is the mod operator. We need to use parentheses to emphasize the order of operations. As Figure E.11 shows, the inputs (*a* and *b*) can be members of $\mathbf{Z}_n$ or **Z**.

Example E.18

Perform the following operations (the inputs come from $\mathbf{Z}_n$):

a. Add 7 to 14 in $\mathbf{Z}_{15}$.

b. Subtract 11 from 7 in $\mathbf{Z}_{13}$.

c. Multiply 11 by 7 in $\mathbf{Z}_{20}$.

Solution

The following shows the two steps involved in each case:

$(14 + 7) \bmod 15 \rightarrow (21) \bmod 15 = 6$

$(7 - 11) \bmod 13 \rightarrow (-4) \bmod 13 = 9$

$(7 \times 11) \bmod 20 \rightarrow (77) \bmod 20 = 17$

Example E.19

Perform the following operations (the inputs come from either $\mathbf{Z}$ or $\mathbf{Z}_n$):

a. Add 17 to 27 in $\mathbf{Z}_{14}$.

b. Subtract 43 from 12 in $\mathbf{Z}_{13}$.

c. Multiply 123 by −10 in $\mathbf{Z}_{19}$.

Solution

The following shows the two steps involved in each case:

$(17 + 27) \bmod 14$	$\rightarrow$	$(44) \bmod 14 = 2$
$(12 - 43) \bmod 13$	$\rightarrow$	$(-31) \bmod 13 = 8$
$(123 \times (-10)) \bmod 19$	$\rightarrow$	$(-1230) \bmod 19 = 5$