Chapter 7

Integration by parts: $\int u\, dv = uv - \int v\, dv$

$$\int_{x=a}^{x=b} u\, dv = uv \Big|_{x=b}^{x=a} - \int_{x=a}^{x=b} v\, du$$

$\int \sin^m x \cos^n x\, dx$ Case 1: m or n is odd. If m is odd isolate one power of sin x to use for du. Replace the factors of $\sin^2 x$ with $1 - \cos^2 x$. Let u = cos x. If n is odd isolate one power of cos x to use for du. Replace the factors of $\cos^2 x$ with $1 - \sin^2 x$. Let u = sin x. Case 2: m and n are both even integers. Use the double angle formulas, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to reduce the powers in the integrand.

$\int \tan^m x \sec^n x\, dx$ Case 1: m is odd. Isolate one factor of sec x tanx for du. Replace any factors of $\tan^2 x$ with $\sec^2 x$ and let u = sec x. Case 2: n is even. Isolate one factor of $\sec^2 x$ for du. Then, replace any factors of $\sec^2 x$ with $1 + \tan^2 x$. Let u = tan x.

Trigonometric substitutions:

If the integral contains: $\sqrt{a^2 - x^2}$ Substitute: $x = a \sin\theta\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sqrt{a^2 + x^2}$ Substitute: $x = a \tan\theta\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sqrt{x^2 - a^2}$ Substitute: $x = a \sec\theta\ 0 \le \theta < \frac{\pi}{2}$

Partial fractions decomposition:

$$\frac{a_1x+b_1}{(a_2x+b_2)(a_3x+b_3)} = \frac{A}{a_2x+b_2} + \frac{B}{a_3x+3}$$

$$\frac{P(x)}{(ax+b)^n} = \frac{c_1}{(ax+b)} + \frac{c_2}{(ax+b)^2} + \dots + \frac{c_n}{(ax+b)^n}$$

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\dots(a_nx^2+b_nx+c_n)} = \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x^2+B_2}{(a_2x^2+b_2x+c_2)} + \dots + \frac{A_nx^2+B_n}{(a_nx^2+b_nx+c_n)}$$

l'Hopital's Rule: Suppose that f and g are differentiable on the interval (a,b), except possibly at some fixed point c in (a,b) and that $g'(x) \neq 0$, on (a,b), except possibly at c. Suppose further that $\lim_{x\to c} \frac{f(x)}{g(x)}$ has the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ (or $\pm\infty$). Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

If f is continuous on the interval [a,b) and $|f(x)| \to \infty$, as $x \to b^-$, we define $\int_a^b f(x)\, dx = \lim_{R\to b^-} \int_a^R f(x)\, dx$.
If f is continuous on the interval (a,b] and $|f(x)| \to \infty$ as $x \to a^+$, we define $\int_a^b f(x)\, dx = \lim_{R\to a^+} \int_a^R f(x)\, dx$.
If f is continuous on the interval [a,b] except at some c in (a,b) and $|f(x)| \to \infty$, as $x \to c$, then we write $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_a^b f(x)\, dx$. If both integrals converge to L_1 and L_2 respectively the we say the improper integral $\int_a^b f(x)\, dx$ converges to $L_1 + L_2$. Otherwise the improper initegral diverges.

We define $\int_a^\infty f(x)\, dx = \lim_{R\to\infty} \int_a^R f(x)\, dx$. If the limit exists we say that the improper integral converges. Otherwise it diverges. Similarly, we define $\int_{-\infty}^b f(x)dx = \lim_{R\to-\infty} \int_R^b f(x)dx$.
We define $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx$. If both integrals converge then $\int_{-\infty}^\infty f(x)\, dx$ converges. Otherwise it diverges.

The Comparison Test: Suppose that f and g are continuous on $[a,\infty)$ and $0 \le f(x) \le g(x)$ for all x in $[a,\infty)$. (i) If $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ converges also. (ii) If $\int_a^\infty f(x)\, dx$ diverges, then $\int_a^\infty g(x)\, dx$ diverges also.