Chapter 8

A **sequence** is an infinite collection af real numbers, written in a specific order.

The sequence $\{a_n\}$ **converges** to L if an only if given any number $\varepsilon > 0$, there is a number $N > n_o$ for which $|a_n - L| < \varepsilon$, for every $n > N$. If there is no such number L, we say the the sequence diverges.

Supppose that $\{a_n\}$ and $\{b_n\}$ both converge to L_1 and L_2 respnectively. Then
(i) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L_1 + L_2$
(ii) $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = L_1 - L_2$
(iii) $\lim\limits_{n\to\infty} (a_n\, b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n) = L_1 L_2$
(iv) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} = \frac{L_1}{L_2}$ if $L_2 \neq 0$

Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then, $\lim\limits_{n\to\infty} f(n) = L$, also.

Squeeze Theorem: Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences, both converging to the limit, L. If there is an integer $n_1 \geq n_o$ such that for all $n \geq n_1$, $a_n \leq c_n \leq b_n$, then $\{c_n\}$ converges to L also.

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$, also.

The sequence $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1}$ for all n.
The sequence $\{a_n\}$ is **decreasing** if $a_n \geq a_{n-1}$ for all n.
A sequence that is increasing or decreasing is **monotonic**.

A sequence $\{a_n\}$ is **bounded** if there is a number $M > 0$ for which $|a_n| \leq M$, for all n.

Every bounded, monotonic sequence converges.

The nth partial sum, $S_n = a_1 + a_2 + \ldots + a_n$

$\sum\limits_{k=1}^{\infty} a_k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} = \lim\limits_{n\to\infty} S_n$

For $a \neq 0$, the **geometric series**, $\sum\limits_{k=0}^{\infty} a\, r_k$ converges to $\frac{a}{1-r}$, if $|r| < 1$ and diverges if $|r| \geq 1$.

$\sum\limits_{k=1}^{\infty} a_k$ converges, then $\lim\limits_{k\to\infty} a_k = 0$.

The nth Term Test for Divergence: If $\lim\limits_{l\to\infty} a_k \neq 0$, then the series $\sum\limits_{k=1}^{\infty} a_k$ diverges.

(i) If $\sum\limits_{k=1}^{\infty} a_k$ converges to A and $\sum\limits_{k=1}^{\infty} b_k$ converges to B, then $\sum\limits_{k=1}^{\infty} (a_k \pm b_k)$ converges to $A \pm B$ and $\sum\limits_{k=1}^{\infty} c\, a_k$ converges to cA , for any constant, c.
(ii) If $\sum\limits_{k=1}^{\infty} a_k$ converges and $\sum\limits_{k=1}^{\infty} b_k$ diverges, then $\sum\limits_{k=1}^{\infty} (a_k \pm b_k)$ diverges.

The Integral Test: If $f(k) = a_k$ for all k = 1,2,... And f is continuous, decreasing, and positive for $X \geq 1$, then $\int_1^{\infty} f(x)\,dx$ and $\sum\limits_{k=1}^{\infty} a_k$ either both converge or both diverge.

Comparison Test : Suppose that $0 \leq a_k \leq b_k$ for all k. (i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$.converges. (ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, too.

Limit Comparison Test: Suppose that $a_k, b_k > 0$ and that for some finite value, L, $\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0$. Then, either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

An **alternating serties** is any series of the form $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \ldots$

The Alternating Series Test: Suppose that $\lim_{k \to \infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$ for all $k \geq 1$. Then the The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Suppose that an alternating series converges to some number S and that the error in approximating S by the nth partial sum S_n satisifies $|S - S_n| \leq a_n$.

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

The Ratio Test: Given $\sum_{k=1}^{\infty} a_k$, suppose that $\lim_{k \to \infty} |\frac{a_{k-1}}{a_k}| = L$. Then (i) if $L < 1$, the series converges absolutely, (ii) if $L > 1$, the series diverges and (iii) if $L = 1$, there is no conclusion.

A **power series** is any series of the form $\sum_{k=0}^{\infty} b_k(x - c)^k = b_0 + b_1(x - c) + b_2(x - c)^2 + b_3(x - 3)^3 + \ldots$

Given any power series, $\sum_{k=0}^{\infty} b_k(x - c)^k$, there are exactly three possiblilities: (i) The series converges for all x in $(-\infty, \infty)$ and the radius of convergence is $r = \infty$; (ii) The sereies converges only for $x = c$ and the radius of convergence is $r = 0$ or (iii) the series converges for x in $(c - r, c + r)$.

$\sum_{k=1}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$ Is called the **Taylor series** expansion for f. If c = o it is called a **Maclaurin series**.

$P_n(x) = \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$ The error in using $P_n(x)$ to approximate $f(x)$ is $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}$ For some numbers between x and c.

$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

A **Fourier series** is a series of the form $\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$

Suppose that f is periodic of period $2l$ and that f and f′ are continuous on the interval $\{-l, l\}$, except for at most a finite number of jump discontinuities. Then, f has a convergent Fourier series expansion. Further the series converges to f(x), when f is continuous at x and to $1/2[\lim_{t \to x^+} f(t) + \lim_{t \to x^-} f(t)]$ at any points x where f is discontinuous.