Visualization in Three Dimensions

Concept

The most important thing to realize about three-dimensional graphs is that they are difficult. For any difficult task, you need to break it down into small basic steps. You should work through numerous problems so that you can learn how the steps fit together. The task of representing a three-dimensional object on a two-dimensional paper or screen is challenging enough that computers and calculators still produce graphs that can be inadequate. Your ability to visualize the true graph is essential.

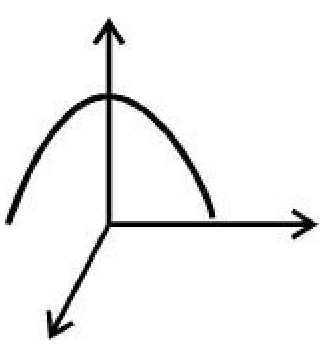
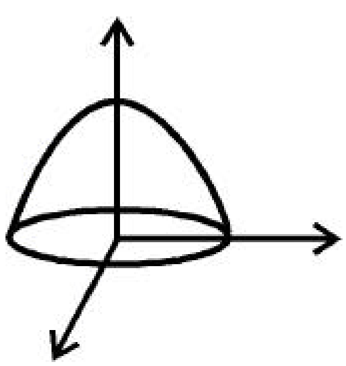
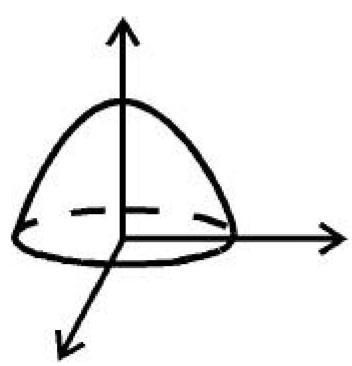
Our strategy is similar to how we found volumes of revolution in chapter 5. We think of the three-dimensional solid in terms of a number of two-dimensional cross sections, which we call **traces**. You can often get a decent idea of the graph from traces in each of the coordinate planes. For some graphs, you need to look at traces in planes parallel to the coordinate planes.

It is very important that you become familiar with all of the quadric surfaces in section 10.6. These surfaces are used throughout the rest of calculus and will give you a collection of basic shapes against which you can compare other graphs. Both of the examples below are quadric surfaces.

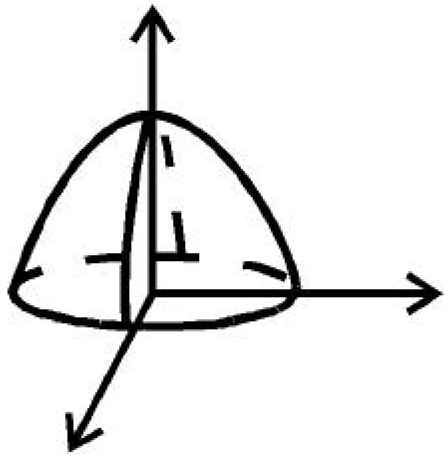
Technique

Sketch the surface $z = 4 - x^2 - y^2$.

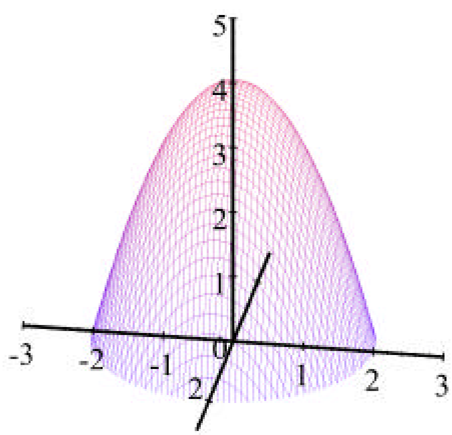
We will sketch the trace in each of the coordinate planes. Many people prefer to start with the trace in the *y*-*z* plane. Since this is the plane that you look at "head on" in your graphs, the *y*-*z* trace often shows you the outline of the figure. Setting x=0, the trace is the downward parabola $z = 4 - y^2$ and is sketched in the figure to the left. The trace in the *x*-*y* plane is often the next best trace to sketch. You may be able to visualize several parallel traces stacking up to form the three-dimensional solid. In this example, setting z=0 leaves the circle $x^2+y^2 = 4$, as shown in the figure in the middle. In a three-dimensional solid, the back half of this circle would not be visible. We use a dashed line in the figure to the right to show this.

If you use some imagination, the figure on the right above may already correctly convey the sense of a dome (or hat, or gumdrop). We can improve the three-dimensional look by adding the *x*-*z* trace. Since the *x*-axis points almost directly at you, this trace is the hardest to see and draw. Setting $y=0$, we get the downward parabola $z = 4 - x^2$. This is sketched in the figure below. To make it look more three-dimensional, we use dashed lines for the back half of the parabola which would not be visible.



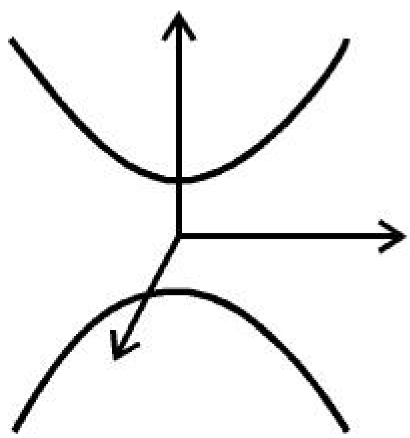
This sketch is not as detailed as the computer-generated color graph shown below, but all of the features that we will be interested in are visible. Practice drawing and reading sketches like this. It will improve your calculus skills.

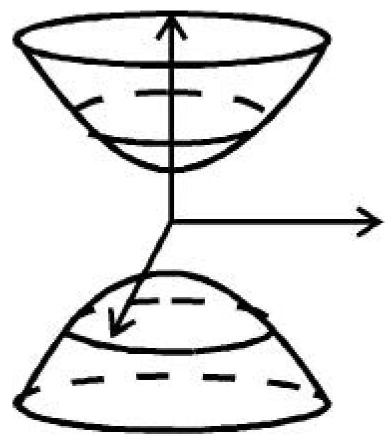


One final comment on this example. The traces we found were a parabola, another parabola and a circle. Since the parabolas are in the majority, this surface is called a paraboloid. The circular cross sections provide the adjective for the surface: this is a **circular paraboloid.**

Sketch the surface $z^2 - x^2 - y^2 = 1$.

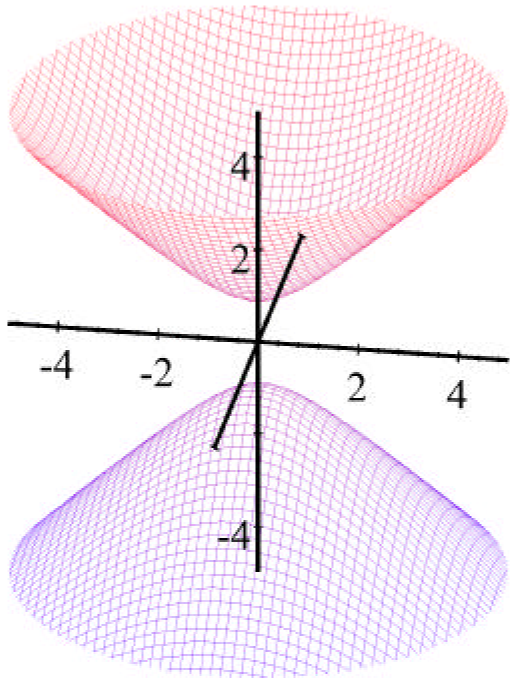
We again start with the *y*-*z* trace. Setting $x=0$, we have the hyperbola $z^2 - y^2 = 1$. This hyperbola has *z*-intercepts at $z = \pm 1$ and is sketched in the figure on the left below. We next go to the *x*-*y* trace. Setting $z=0$ gives the equation $-x^2 - y^2 = 1$ or $x^2 + y^2 = -1$. This equation has no solution, so there is no trace in the *x*-*y* plane. This may seem odd, but if you look at the hyperbolic trace we've already sketched in, you should notice that $z=0$ is in between the two halves of the hyperbola. To get a better idea of what's happening, we need to look at traces for other *z*-values. For example, setting $z=2$ gives us the equation $4 - x^2 - y^2 = 1$ or $x^2 + y^2 = 3$. Setting $z=3$ gives us the equation $9 - x^2 - y^2 = 1$ or $x^2 + y^2 = 8$. These two circles are sketched in the figure to the right. Notice that setting $z=-2$ or $z=-3$ produces the same equations, respectively. These traces are also sketched in the figure to the right.





At this point, you should have a decent mental image of the figure. It comes in two pieces, each of which somewhat resembles a circular paraboloid. We could also sketch in the *x*-*z* trace, but adding this trace (given by the hyperbola $z^2 - x^2 = 1$) may clutter up the sketch as much as it adds information.

We finish the example by naming the surface and showing a computer-generated graph. The traces are a hyperbola, circles and another hyperbola. Since hyperbolas are in the majority, the surface is a hyperboloid. The naming convention for hyperboloids is to indicate how many pieces (or sheets) the hyperboloid has. In this case, we have a **hyperboloid of two sheets**.



Extension

If you look carefully at the computer-generated picture, you should be able to identify that the overall structure of the graph is a large number of traces parallel to the *yz*- and *xz*-planes. A basic problem for software writers is how to project a given trace onto a two-dimensional screen. We chose the word “project” carefully, because the projections discussed in sections 10.3 and 10.4 are used to solve the problem. In fact, vector calculus is at the heart of many computer graphics techniques. The shading of three-dimensional figures, done in color in the above graph, is discussed in an exercise in section 12.6.