Optimization Problems

Concept

To some extent, section 3.6 duplicates section 3.2, where you learned that an absolute maximum or minimum of a function always occurs at a critical number or at an endpoint of an interval. One difference is that most of the problems in section 3.6 are stated in a real world context. There is one other very important difference in the problems in these two sections. Most of the problems in section 3.6 actually involve **two** variables. To solve such a problem, you must somehow reduce your problem to a function of **one** variable. This can be done by identifying a relationship between the variables, and using the relationship to substitute for one of the variables. The extra step of substituting for one of two variables is the main characteristic that makes the problems in section 3.6 different.

There are a couple of ways to prevent this extra step from confusing you. One is to draw and label a picture, since most of the problems have a geometric component. Another way is to work enough problems that the overall structure of the problems becomes familiar. You will usually (1) draw a picture, (2) identify a function of two variables to optimize, (3) identify an equation defining the relationship between the variables, (4) solve the equation for one of the variables and substitute into the function to be optimized, (5) find critical point(s) of this function of one variable and (6) verify that you have found the desired extremum. As with any multi-step process, don't get in too much of a hurry.

Technique

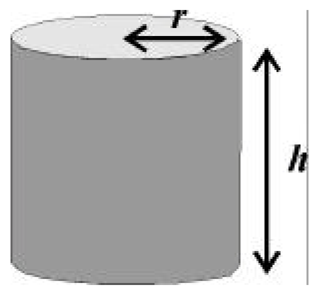
A can is to have a total surface area of 20 in^3. Find the dimensions that will maximize the enclosed volume.

At first reading, it may be unclear what the variables are supposed to be or what kind of a function you're supposed to work with. This is where a picture with labels helps. We sketch a cylindrical can with radius *r* and height *h* (these are the variables). The problem statement refers to both surface area and volume, so let's compute these for a general can. The can actually consists of three surfaces (top, bottom, sides) so we must add up the areas of each piece. The top and bottom are circles of radius *r* (area πr^2) and the side could be unrolled into a rectangle of dimensions $2\pi r$ by *h* (area $2\pi rh$). Thus,

$$\text{Surface area} = \pi r^2 + \pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi rh$$

The volume of a cylinder equals the area of the circular cross-section times the height:

$$\text{Volume} = \pi r^2 h$$



Now, read the problem again. We have been told that the surface area is supposed to equal 20. This gives us the equation

$$2\pi r^2 + 2\pi rh = 20$$

We are not given a value for the volume; we are supposed to maximize it. So the volume $\pi r^2 h$ is our function of two variables. We use the surface area equation to replace one variable. You are allowed to solve for either variable, so choose the one that looks easier. In our equation, *h* only appears once, so we solve for *h*:

$$2\pi rh = 20 - 2\pi r^2$$
$$h = (20 - 2\pi r^2)/\ 2\pi r = 10/\pi r - r$$

Substituting into the formula for volume, we have the function

$$V = \pi r^2 h = \pi r^2(10/\pi r - r) = 10r - \pi r^3$$

To find critical points, compute the derivative of $V(r)$ and set it equal to 0.

$$V\,'(r) = 10 - 3\pi r^2 = 0$$
$$r = (10/3\pi)^{1/2} = 1.03$$

We ignore the negative square root since *r* represents a radius. A radius of about 1.03 is critical, but does it give a maximum volume, minimum volume or neither? There are two methods for verifying that we have found the maximum point. First, note that for $0 < r < 1.03$, $V\,'(r) > 0$; also, if $r > 1.03$, $V\,'(r) < 0$. By the First Derivative Test, this says that there is a (relative) maximum at r=1.03. Since there is only one critical point in the domain $r > 0$, the relative maximum must also be the absolute maximum. A different (but less reliable) method is to graph $V(r)$ and note that the maximum occurs at r=1.03.

We're not quite finished! Read the problem one last time to see if the question has been answered. Since the problem asks for the "dimensions" of the can of maximum volume, that's what we need for an answer. Our value of $r = 1.03$ is one dimension, but we also need the height of the can. From $h = 10/\pi r - r$, we can compute $h = 2.06$. A radius of 1.03" and a height of 2.06" gives the maximum volume.

Extension

Each of the examples in section 3.6 covers a different situation with its own unique mathematical complications. For example, the highway problem of example 6.6 only has one variable, so you do not have the extra substitution step. However, a natural question arises in this problem that is not present in other problems. Specifically, we compare the broken-line path of minimum cost to a straight-line path. Most of the broken-line problems (oil pipeline, water line, endurance contest, Washington crossing the Delaware, etc.) have similar follow-up questions.

You should complete examples of each type of optimization problem that you are going to be responsible for in your class. Pay special attention to the sequence of steps done to solve each problem so that you will know how to get started on any problems you end up working for a grade. As we noted in the example above, you should read the problem more than once so that you remain clear on what you are supposed to do.