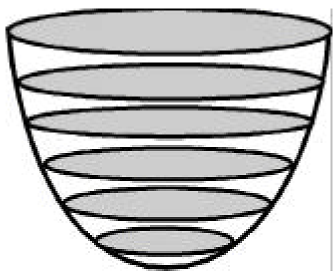
Volumes of Revolution

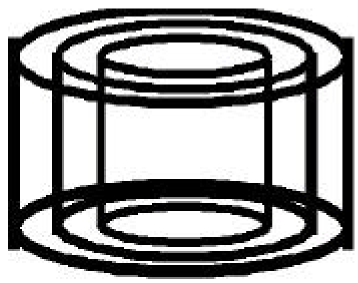
Concept

The secret to making this topic easy is to visualize the disks or shells stacking up to make the three-dimensional solid. If you master this, you only need to remember the formulas for area of a circle and surface area of a cylinder. Without a good sense of the geometry, this topic can be a confusing set of formulas.

In the disk method, you should visualize many filled-in circles of differing radii being stacked up (bottom to top, or left to right) to form a solid. If you can identify the radius of each circle in terms of the given function(s), you simply integrate πr^2 (the area of a circle) to find the volume. The washer method is almost identical, but is used for figures with cavities (holes). Here, you use the disk method to find the volume of the filled-in solid and subtract the volume of the hole.



Disk method



Shell method

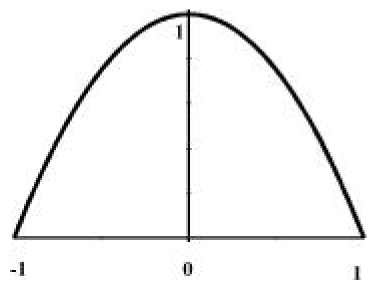
In the shell method, you should visualize many cylinders being nested (inside out) to form the solid. If you can identify the radius and height of each cylinder in terms of the given function(s), you integrate $2\pi rh$ (the surface area of a cylinder) to find the volume.

Along with learning each method, you need to be able to decide which method to use on a given problem. In many cases, both methods work equally well. In some cases, one method can be highly preferable to the other. Usually, the geometry of the region being revolved determines which method to use. If the region is well defined by two functions of x (one for the top, one for the bottom), you want to integrate with respect to x. Here's where the visualization is needed: if the solid is formed by disks stacking up horizontally (along the x-axis), use the disk/washer method. If the solid is formed by cylinders nesting inside each other left-to-right (along the x-axis), use the shell method. Conversely, if the region being revolved is well defined by two functions of y (one for the left, one for the right), you want to integrate with respect to y. If the solid is formed by disks stacking up vertically (along the y-axis), use the disk/washer method. If the solid is formed by cylinders nesting inside each other vertically (along the y-axis), use the shell method.

Technique

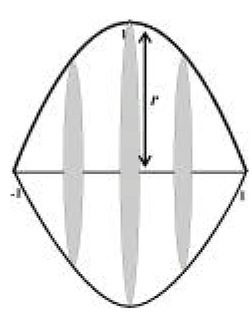
Find the volume of the solid formed by revolving the region *R* about the given axis.
***R* is the region bounded by $y=1-x^2$ and the *x*-axis.**
Revolve about (a) the *x*-axis, (b) $y=2$ and (c) $x=2$.

First, sketch a picture so that you can see the region.



Notice here that the top of the figure is defined by the curve $y = 1-x^2$ and the bottom of the figure is defined by $y = 0$ (the *x*-axis). Thus, integration with respect to *x* will be easy to set up. On the other hand, the left and the right boundaries of the region are both defined by the parabola. To use a *y*-integration, we would need to find separate equations for the left and right halves of the parabola. While this is not impossible to do [you should get $x = (1-y)^{1/2}$ and $x = -(1-y)^{1/2}$], it's a complication that we can avoid by integrating with respect to *x*.

For part (a), imagine revolving this region about the *x*-axis. You should visualize a solid somewhat like a bean or a football.



Notice that the circular cross sections (the disks) align themselves left to right along the *x*-axis. Thus, the disk/washer method would produce the *x*-integration we want. Finally, notice that the solid does not have a cavity or hole. This indicates that we should use the disk method. For the disk method, we integrate πr^2, where *r* is the radius shown above. Since the radius extends from $y = 0$ up to $y = 1-x^2$, we get $r = 1-x^2$. The region extends from $x = -1$ to $x = 1$; these are the limits of integration. The volume is given by

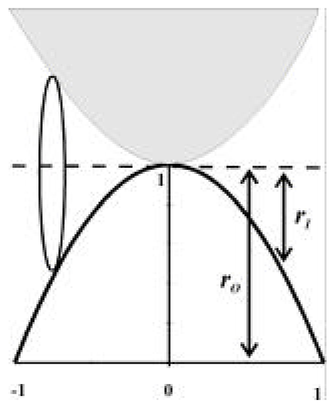
$$\int_{-1}^{1} \pi r^2\, dx = \int_{-1}^{1} \pi \left(1 - x^2\right)^2 dx = \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx$$

$$= \pi \left(x - 2x^3/3 + x^5/5\right)\Big|_{x=-1}^{x=1}$$

$$= \pi(1 - 2/3 + 1/5) - \pi(-1 + 2/3 - 1/5)$$

$$= 16\pi/15$$

For part (b), imagine revolving the region about the line $y = 2$. The circular cross sections again line up left to right (along the x-axis), so we again want to use the disk/washer method to compute the volume. This time, notice that the solid has a large cavity, seen in the figure as a gap between the original region and its shaded reflection.



The cavity means that we will use the method of washers, and need to identify an inner radius and outer radius. As seen in the figure, the larger outer radius extends from the axis of rotation $y = 2$ down to the bottom of the figure given by $y = 0$. Thus, $r_O = 2 - 0 = 2$. The smaller inner radius extends from the axis of rotation $y = 2$ down to the upper boundary of the region given by $y = 1-x^2$. Thus, $r_I = 2 - (1-x^2) = 1 + x^2$. The volume is given by

$$\int_{-1}^{1} \pi \left(r_O^2 - r_I^2\right) dx = \int_{-1}^{1} \pi \left[2^2 - \left(1 + x^2\right)^2\right] dx$$

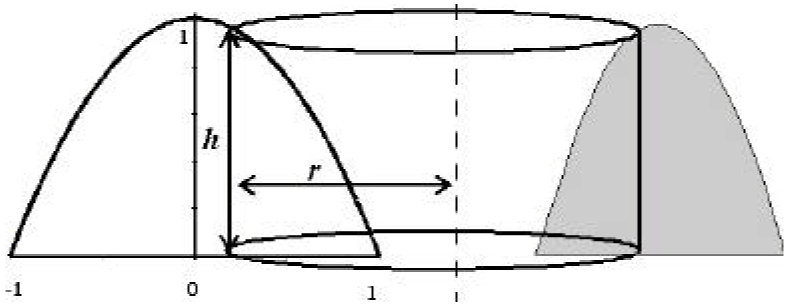
$$= \pi \int_{-1}^{1} \left(4 - 1 - 2x^2 - x^4\right) dx$$

$$= \pi \left(3x - 2x^3/3 - x^5/5\right)\Big|_{x=-1}^{x=1}$$

$$= \pi(3 - 2/3 - 1/5) - \pi(-3 + 2/3 + 1/5)$$

$$= 64\pi/15$$

For part (c), imagine revolving the region about the line $x = 2$. The circular cross sections line up vertically along the *y*-axis, so the disk/washer method requires a *y*-integration. Since we prefer an *x*-integration in this problem, we use the shell method.



For the shell method, we visualize one of the cylindrical shells and identify its radius and height. As shown above, the height extends from the top of the region $y = 1 - x^2$ to the bottom of the region $y = 0$. Thus, $h = 1 - x^2$. The radius extends from the axis of rotation at $x = 2$ back to a location at a general *x*. Thus, $r = 2 - x$. Putting this together, the volume is given by

$$\int_{-1}^{1} 2\pi r h \, dx = \int_{-1}^{1} 2\pi (2 - x)(1 - x^2)\, dx$$

$$= 2\pi \int_{-1}^{1} (2 - 2x^2 - x + x^3)\, dx$$

$$= 2\pi \left(2x - 2x^3/3 - x^2/2 + x^4/4\right)\Big|_{x=-1}^{x=1}$$

$$= 2\pi(2 - 2/3 - 1/2 + 1/4) - 2\pi(-2 + 2/3 - 1/2 + 1/4)$$

$$= 16\pi/3$$

Extension

Be sure to realize that the above example does not cover all cases. For some regions, a *y*-integration will be easier. If we had chosen to integrate with respect to *y* in the above example, we would have used the shell method for parts (a) and (b) and the method of washers for part (c). We strongly urge you to resist the temptation to try to memorize rules for which method to use. Understand the thought process! This will serve you best as you progress on to more complicated three-dimensional figures later in calculus.

There are aspects of these calculations that you can expect to see in each problem. Using the method of washers, both the inside radius and outside radius will equal the difference between the axis of rotation and one boundary of the figure. Using the shell method, the height *h* is determined by the boundaries of the region and the radius *r* is given by the difference between the axis of rotation and the variable of integration. You can use this as a template for each specific problem you work.