Approximations

Concept

Calculations of derivatives and integrals consume a large percentage of your time in calculus. When you're busy memorizing the derivative and integral formulas that make precise calculations (relatively) simple, it can be easy to lose sight of the fact that we can't solve **all** problems precisely. In practice, many quantities **must** be approximated, sometimes because the functions involved are too complicated and sometimes because you don't have a function to which you can apply your rules (you may only have data representing a small number of function values). To be an effective user of calculus, you need to be able to recognize when an approximation is the best you can do and, of course, you should also have some knowledge of how to obtain a **good** approximation.

Several different types of approximations are discussed in calculus. One type is the decimal approximation of irrational numbers like π, and most exponential and logarithmic values. Another type of approximation is the numerical approximation of limits, including the limits that define derivatives and integrals. There are numerical methods like Newton's method that approximate solutions by solving simpler but closely related problems. Finally, there are sophisticated numerical methods like Simpson's rule for approximating integrals. This type of method is derived from simpler methods through a careful analysis of the strengths of each method.

A general issue that causes some confusion is how accurate an approximation needs to be. Unfortunately, there is no single rule to go by. How close an approximation has to be to the exact answer to be judged "good" depends on the purpose of the calculation. If you're building a fence in the back yard, a measurement to within one inch may be satisfactory, whereas an engineer burning information onto a microchip could not tolerate an error of one micron. We often give you guidelines on how far to take a calculation (e.g., "three digits of accuracy") and your professor may do the same. Otherwise, you should try to go as far as common sense and your calculator allow.

Technique

Here, we will look at four approximations used in calculus.

Use numerical evidence to conjecture a value of $\lim_{x \to 0} \sin 2x / x$.

Since the limit is taken as *x* approaches 0, we want to compute values of the function sin2*x* / *x* for *x*-values getting closer and closer to 0. We want to emphasize two points. First, be sure to compute a *sequence* of function values so that you can observe patterns. Often, the pattern in the function values for a sequence like *x* = 0.1, *x* = 0.01 and *x* = 0.001 provides better information than computing the function value at a very small value like *x* = 0.00000001. Second, you should use *x*-values on both sides of the limiting value (in this example, use both positive and negative values). Since a limit does not exist if the half-limits disagree, you need to check both sides to gather evidence that the limit exists.

Based on the table of values shown, it is reasonable to conjecture that the limit equals 2.

x	Sin2*x* / *x*		*x*	Sin2*x* / *x*
0.1	1.98		–0.1	1.98
0.01	1.9998		–0.01	1.9998
0.001	1.999998		–0.001	1.999998

Notice that we showed different numbers of digits for different calculations. We decided to do this **after** completing the calculations. The pattern we observed was a 1 followed by a number of nines and then an 8. The number of nines increases by two in each new calculation, providing strong evidence that the limit is 1.9999999---, which equals 2.

Estimate the derivative *f* ’(2) from the given table of function values.

x	*f*(*x*)		*x*	*f*(*x*)
1	2.2		2.5	3.6
1.5	2.8		3.0	4.2
2	3.2		3.5	5.0

Here, we have no formula for the function, so the best we can do is use the small amount of data given. We know that the derivative at *x*=2 equals the slope of the tangent line at *x*=2. In turn, the slope of the tangent line can be approximated by slopes of secant lines through the point at *x*=2. That is,

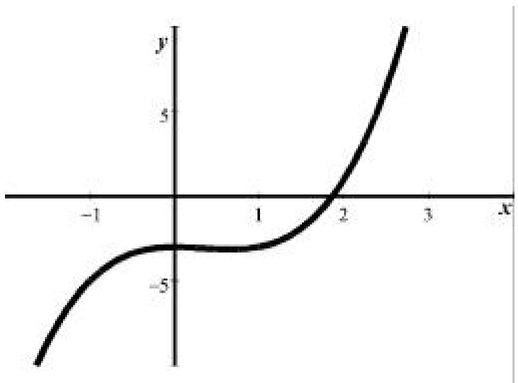
$$f'(2) \approx \frac{f(x)-f(2)}{x-2}$$

for values of *x* close to 2. The closest values we have in the table are *x*=1.5 and *x*=2.5. Using *x*=1.5, we estimate the slope to be (3.2–2.8)/(2–1.5)=0.8. Using *x*=2.5, we estimate the slope to be (3.6–3.2)/(2.5–2)=0.8. Since these estimates are equal, a reasonable estimate of the derivative is 0.8.

However, there are other reasonable answers. Notice that if we use the point at x=1, we estimate the slope to be (3.2–2.2)/(2–1)=1.0. This estimate is probably worse than our previous estimate, since x=1 is farther from x=2 than is x=1.5. However, the derivative is defined as a limit of slopes of secant lines, two of which are 1.0 and 0.8. If the pattern continues, we would expect the derivative to be less than 0.8, say 0.7 or 0.6. An estimate of this type should be presented almost as a short essay: your best estimate along with the evidence that indicates this value.

Use Newton's method to approximate the zero of $x^3 – x^2 – 3$.

A graph indicates that there is a single zero near x=2.



We can use Newton's method with an initial guess of $x_0 = 2$. Recall that the formula for Newton's method is

$$x_{n+1} = x_n – f(x_n) / f'(x_n)$$

Starting with $x_0 = 2$, this looks like

$$x_1 = 2 – f(2) / f'(2)$$

From $f(x) = x^3 – x^2 – 3$, we get $f'(x) = 3x^2 – 2x$, so $f(2) = 8–4–3 = 1$ and $f'(2) = 12–4 = 8$. This gives us

$$x_1 = 2 – 1 / 8 = 15 / 8$$

We now repeat the above steps with the new approximation 15/8. As we start to get fractions like 15/8, it becomes harder to complete the steps without a calculator. (Most graphing calculators allow you to do these repetitive calculations easily. For example, TI calculators have Ans and Enter features that you may want to investigate.)

$$\begin{aligned} x_2 &= 15/8 – f(15/8) / f'(15/8) \\ &= 15/8 – (39/512) / (435/64) \\ &= 1081 / 580 \end{aligned}$$

The next step would be to plug 1081/580 back into the Newton's method formula. Unless you have a CAS to help you with this (you should get x_3 = 1216999751 / 652999670), you should approximate 1081/580 with a decimal like 1.86379 (or 1.86). Be aware that rounding may (slightly) affect the value of the next calculation or two, but will not typically change the value of the zero you find. Using x_2 = 1.86379, we get

$$x_3 = 1.86379 - f(1.86379) / f\,'(1.863792)$$
$$= 1.86371$$

You should verify that x_4 = 1.86371 also. When you get the same number back, it is time to stop (or find a machine with more digits displayed). From working several examples, you should have confidence that if Newton's method converges to a number, that number is likely to be a solution. As a check, you can look back at the graph to see if the answer makes sense (it does) and compute the value of the function at the current *x*-value to see if it's close to zero: here, $f(1.86371) \approx 0.000023$.

Use Simpson's Rule with *n*=8 to approximate

$$\int_1^3 \sin\left(x^3 + 1\right) dx$$

The general form of Simpson's rule is

$$[\, f(x_0) + 4\, f(x_1) + 2\, f(x_2) + 4\, f(x_3) + \ldots + 2\, f(x_{n-2}) + 4\, f(x_{n-1}) + f(x_n)\,]\ \Delta x / 3$$

We need to compute Δx, determine the *x*'s at which to evaluate *f* and then substitute in. As with Riemann sums, Δx equals the length of each subinterval. In this case, the integral is from *x*=1 to *x*=3 and we are dividing the interval into *n*=8 pieces. Each piece has width (3–1)/8 = 0.25. The *x*'s are the endpoints of the subintervals, starting at 1 and ending at 3. With subintervals every 0.25 unit, the endpoints are 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3. So Simpson's rule is

$$[f(1) + 4\, f(1.25) + 2f(1.5) + 4\, f(1.75) + 2f(2) + 4\, f(2.25) + 2f(2.5) + 4\, f(2.75) + f(3)]\ 0.25/3$$

Now evaluate the function. Since $f(x)=\sin(x^3+1)$, we have $f(1) = \sin(1+1)=.909$, then $f(1.25) = \sin(1.25^3+1)=.187$, and so on. It looks like

$$[.909+ 4(.187)+ 2(-.943)+ 4(.076)+ 2(.412)+ 4(-.175)+ 2(-.794)+ 4(.193)+ .271]\ 0.25/3$$

This computes out to approximately –0.1746. Since the first four digits of the exact integral are +0.0169, this is not an especially accurate approximation. (The sine function oscillates rapidly. Can you explain why the approximation for a small value of *n* is not great?) A better approximation can be obtained with a larger value of *n*.