Techniques of Integration

Concept

You might well wonder what "concept" is involved in a chapter on techniques of integration. After all, this is among the most mechanical of all the topics in calculus (computer algebra systems do well here). However, there are some very important critical thinking skills at work here. Students having difficulty with this material typically improve by practicing the general skills discussed below.

The first lesson is that you must pay attention to the important details. That's more complicated than it might seem, because you must learn which details are important. For example, when doing integration by substitution you learn that constants can be ignored: you would make the same substitution for an integrand of $4x \sin(x^2)$ as you would for $84x \sin(x^2)$. For substitution purposes, the important detail is that the derivative of x^2 equals a constant times x. The actual value of the constant and the role of the sine function only come into play *after* the substitution is made.

There are many new details to learn in this chapter. If the integrand is the product of two elementary functions (e.g., $x \sin x$), think about integration by parts. If the integrand is a rational function, think about partial fractions. If the integrand involves a square root of x^2 plus or minus 1, think about a trig substitution. You should have a check list of features to look for. These cues will help you decide which technique to try first. (However, realize that sometimes it may still be unclear which method will work, in which case the best advice is just to "try something.")

The second lesson is: don't try to guess the final answer before completing all the steps. If you're playing tennis, you're more likely to hit a good shot by swinging the racket with good technique than by looking at where you want the ball to go. The techniques of integration are designed to get you to the answer without having to do several steps at once. As with sports, concentrating on one step at a time is the key to success.

The final lesson also has a sports analogy. The more you practice, the more comfortable each step of the process will be. If it takes an intense effort to identify integration by parts as the method of choice on a problem, and then you have to struggle with each step of the integration by parts procedure, it will be very hard to successfully complete the problem. On the other hand, if you quickly identify the proper technique and know the technique well enough that you are essentially just filling in the blanks, then the problem will seem easy.

Read carefully through the following examples. We will show you how to get the correct integral, but we will also include a lengthy discussion of how to apply the techniques. Don't ignore the discussion. It's part of the thought process that is actually at the heart of this chapter.

Technique

Find each of the following integrals.

$$(a)\ \int \frac{x}{\sqrt{x^2-1}}\,dx \qquad (b)\ \int \frac{1}{x\sqrt{x^2-1}}\,dx \qquad (c)\ \int \frac{1}{x^2-1}\,dx$$

First, notice that all three integrands are fractions. Possible techniques for us to consider include basic formulas, substitution, partial fractions and trig substitution (but integration by parts is unlikely). We will think through each possibility.

One of the basic formulas is that if the numerator is the derivative of the denominator, then the integral is the natural log of the denominator. Check quickly to verify that none of the three given integrals is of this type. However, the derivative of the inside of the square root in part (a) almost matches the numerator. So, try a substitution.

Let $u = x^2 - 1$; then $du = 2x\,dx$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + c$$
$$= \left(x^2-1\right)^{1/2} + c$$

The same substitution does not work in part (b) because the extra x is in the denominator. Since we usually substitute for the "inside" of a composition of terms, $u = x^2-1$ is the only obvious substitution to try. However, having the square root of x^2-1 in the integrand should remind you of trig substitution. In this technique, we use the basic trig identity $\sin^2 u + \cos^2 u = 1$ to simplify the square root term. Since our square root involves x^2-1, we rearrange terms to get a square minus 1. Subtract $\cos^2 u$ and divide by $\cos^2 u$ to get

$$\sin^2 u = 1 - \cos^2 u$$
$$\tan^2 u = \sec^2 u - 1$$

(The second line is the right form for this problem since our radical involves $x^2 - 1$. Notice that the first line would be the right form for a radical involving $1 - x^2$. The other trig substitution is for radicals involving $x^2 + 1$; if you add 1 to both sides of the second line, you will get the appropriate form for this case.)

Now, if we replace x with $\sec u$, the awkward expression $x^2 - 1$ becomes $\sec^2 u - 1$ which we can simplify to $\tan^2 u$. Substitute into the integral and see what happens. We have

$$x = \sec u$$
$$dx = \sec u \tan u\,du$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec u\sqrt{\tan^2 u}}\sec u\tan u\,du$$
$$= \int \frac{\sec u\tan u}{\sec u\tan u}\,du = \int 1\,du$$
$$= u + c$$

Notice how nicely the integrand simplified. We admit that this example simplifies more than most, but realize that the goal is to get rid of the square root. Don't stop quite yet, there is one more step left. At this stage, our answer is in terms of u, so we need to convert back to x. The idea is to use the equation $x = \sec u$ to solve for whatever we need. In this case, we just need $u = \sec^{-1}x$ so the answer is

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$$

(There was an easy way to work this one, if you noticed that the given integrand is the derivative of $\sec^{-1}x$.)

Now, let's look at part (c). There are no composition terms, so substitution is unlikely. With no square roots, trig substitution is unlikely. If the denominator was x^2+1 instead of x^2-1, the integrand would be the derivative of $\tan^{-1}x$ and the problem would be easy. So, what are we left with? The integrand is a rational function, and partial fractions often works well with rational functions. Since $x^2-1=(x+1)(x-1)$, the general form is

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

(Before continuing on, note that after finding values for A and B each term on the right-hand side will be easy to integrate.) To determine the constants, multiply through by the denominator x^2-1.

$$1 = A(x-1) + B(x+1)$$

Substitute in $x = 1$, and the equation becomes $1 = 2B$, so that $B = 1/2$.
Substitute in $x = -1$, and the equation becomes $1 = -2A$, so that $A = -1/2$.
Then

$$\int \frac{1}{x^2-1}\,dx = \int \frac{-1/2}{x+1} + \frac{1/2}{x-1}\,dx$$
$$= -1/2\int \frac{1}{x+1}\,dx + 1/2\int \frac{1}{x-1}\,dx$$
$$= -1/2\ln|x+1| + 1/2\ln|x-1| + c$$