Infinite Series

Concept

Although infinite series have numerous applications in all areas of science and engineering, your introduction to series in calculus has a large theoretical component. Some students are uncomfortable at first with problems that do not ask for a numerical answer. The final answer to many series problems is either “converges” or “diverges” but you should expect to do the same amount of computation and analysis that you do to evaluate an integral. The main difference is that the series computations are used as evidence for making an either/or decision.

With an infinite series, you are taking a sequence of numbers and adding them one at a time, keeping track of the running total (or **partial sum**). You need to decide whether your running total will, in the long run, converge to some specific number or not. Your decision is rarely made by actually computing sums (if this seems odd, recall that even though an integral is a limit of Riemann sums, you rarely compute a Riemann sum to evaluate an integral). Instead, we have a number of tests (much like we have several techniques of integration) which may be used to determine the answer.

So, how can you decide if the partial sums will converge or not? One case is obvious: if you continue to add large numbers to the running total, the total will not converge. This idea is called the kth Term Test for Divergence, which states that if the sequence does not converge to 0, then the series diverges. But, what happens if you add in smaller and smaller numbers? The result is analogous to adding water to a glass. Whether the water overflows the glass or not depends on *how fast* you turn the water off. Similarly, whether a series converges or not depends on how fast the sequence goes to 0. The trick is to learn how fast is fast enough. You do this through experience. Filling a glass with water is again analogous. You first try a couple of times and note whether the glass overflows or not. Then you compare. If you turn the water off more slowly than you did one time when the water overflowed, you can be sure that it will overflow again. With series, it is very important that you learn (memorize) several specific series which converge and several which diverge. Then learn the tests for series, all of which are based on a comparison of new series to old series.

By the end of the chapter, you should approach the convergence of a series very much like you would an integral. There are properties of the sequence that will suggest which of several tests to apply to the series. Then, you need to correctly apply the test. As you study series, be sure to make special note of the examples and types of sequences that are characteristic of each test so that you can quickly identify the right one to use.

Technique

Determine whether each series converges or diverges.

$$(a)\ \sum_{n=1}^{\infty} \frac{n}{n^2+1} \qquad (b)\ \sum_{n=1}^{\infty} \frac{4}{n^2+1} \qquad (c)\ \sum_{n=1}^{\infty} \frac{4+n}{n!} \qquad (d)\ \sum_{n=1}^{\infty} e^{-n}$$

You must identify a good test to use, and then use it. To emphasize how important the identification step is, we will start by completing that step for each of the four series. Notice that the sequences in parts (a) and (b) are given by rational functions. The most common test to use on a rational function is the Limit Comparison Test, comparing the series to a *p*-series. Part (c) is different due to the factorial in the denominator. Usually a factorial indicates the Ratio Test. The sequence in part (d) is given by an exponential function. The Ratio Test usually works well with exponential functions (it would work well here), but since the exponential function is one that is easy to integrate, we will use the Integral Test. Be sure you understand the choices we made. Without a good test to use, you won't be able to solve the problem correctly.

Now, for the details. We chose the Limit Comparison Test for part (a). We first identify the power function that approximates the rational function for large n. In this case, if n is large, the fraction $n / (n^2+1)$ will be very close to n / n^2, which simplifies to $1 / n$. The limit below verifies that we have "simplified" correctly.

$$\lim_{n\to\infty} \frac{n \cdot (n^2+1)}{1 \cdot n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Since the limit exists and is nonzero, the Limit Comparison Test guarantees that the two series behave the same: they either both converge or both diverge.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

We know that the harmonic series on the right diverges, so we conclude that part (a) also diverges.

Part (b) is similar. The rational function $4 / (n^2+1)$ is approximately equal to $4 / n^2$ for large n. If you like, you can further simplify the comparison by replacing $4/n^2$ with $1/n^2$. The limit verifies that we have a good comparison.

$$\lim_{n\to\infty} \frac{4 \cdot (n^2+1)}{1 \cdot n^2} = \lim_{n\to\infty} \frac{4n^2}{n^2+1} = 4$$

Since the limit exists and is nonzero, the Limit Comparison Test guarantees that the two series behave the same: they either both converge or both diverge.

$$\sum_{n=1}^{\infty} \frac{4}{n^2+1} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series on the right is a *p*-series with $p = 2 > 1$, so it converges. By the Limit Comparison Test, the series in part (b) also converges.

We chose the Ratio Test to use in part (c). This test is commonly used when the sequence involves factorials. To use the test effectively here, you need to understand the common simplification $n! / (n+1)! = 1 / (n+1)$. Then the following steps should be clear. You start by replacing each n in the sequence $(4+n) / n!$ with $n+1$. You should get $(4+n+1) / (n+1)!$, or $(5+n) / (n+1)!$. Divide this expression by the original term $(4+n) / n!$, simplify and find the limit as n approaches ∞.

$$\lim_{n\to\infty} \frac{(5+n)\cdot(n+1)!}{(4+n)\cdot n!} = \lim_{n\to\infty} \frac{5+n}{4+n}\,\frac{n!}{(n+1)!}$$
$$= \lim_{n\to\infty} \frac{5+n}{4+n}\,\frac{1}{n+1}$$
$$= 1\,(0) = 0$$

Since the limit is less than 1, the series converges.

As mentioned above, there are two tests that work well for part (d). We illustrate the Integral Test here. The idea is to match the sequence e^{-n} with the corresponding function e^{-x} and integrate the function from 1 to ∞. Notice that the resulting integral is an improper integral as covered in section 7.7. We have

$$\int_1^{\infty} e^{-x}\,dx = \lim_{R\to\infty} \int_1^{R} e^{-x}\,dx = \lim_{R\to\infty} \left[-e^{-x}\right]_{x=1}^{x=R}$$
$$= \lim_{R\to\infty} \left(-e^{-R} + e^{-1}\right) = 0 + e^{-1} = e^{-1}$$

The value of the integral doesn't matter at all, only whether it exists or not. Since the integral exists, the corresponding series converges.

Pay careful attention to how each test works. In parts (a) and (b), you do a direct comparison and part of your job is to determine a good comparison series. In part (c), you must learn how to set up the limit and then compare your limit to 1. In part (d), you compute an improper integral and see if the result is finite or not.

Extension

At the end of chapter 8, you get a brief look at why series are so important. Taylor series and Fourier series are used throughout science and engineering. A series of functions is more complicated than a series of numbers, so a good understanding of the principles of convergence is essential.