Parametric and Polar Graphs

Concept

Compared to the function graphs that the first eight chapters of the book focus on, the loops and patterns of parametric and polar graphs are very different and complex. The surprise is that many of these intricate curves are the graphs of simple-looking equations. The key to understanding how these curves are produced is to start from scratch. These are different types of graphs and you must go back to first principles to understand them.

Try to remember how you first sketched graphs in algebra. You made a chart with several x-values, computed the corresponding y-values and then plotted the points. Over time, you learned to recognize that the graph of $y = mx+b$ is a line, while the graph of $y = ax^2+bx+c$ is a parabola. The starting point for parametric and polar graphs is making charts and plotting points. As you work through several examples, you can start to recognize which equations correspond to which graphs.

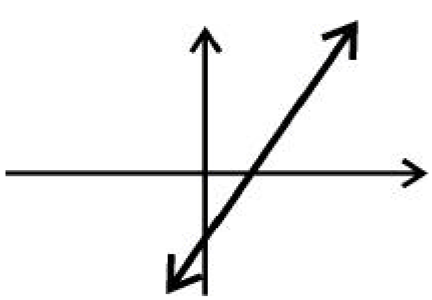
The good news is that your knowledge of graphs of functions $y = f(x)$ will allow you to quickly recognize the graphs of some equations, so the process of going from charts of x- and y-values to graphs can be fast. This is where some confusion starts. On a given problem, we won't immediately know what the graph will look like, but we can use the properties of the given functions to quickly build up an image of the graph. Since different properties are useful in different problems, you must understand the process instead of trying to memorize examples.

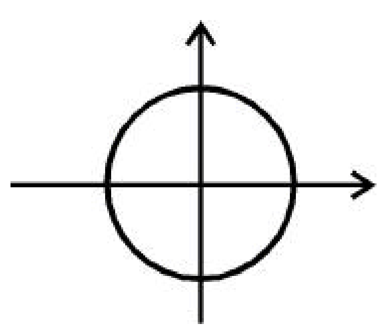
Technique

There are two different types of parametric graphs where you should learn to immediately sketch a graph. Some equations can be memorized; these include lines (where both x and y have the form $mx+b$) and circles (where x and y have the form $r\cos(at)$ and $r\sin(at)$ for constants r and a).

Identify the graphs of (a) $x(t) = 2t+1, y(t) = 3t$ and (b) $x(t) = 2\cos t, y(t) = 2\sin t$

Part (a) gives an equation of the line through the points (1,0) and (3,3). (These points occur with t=0 and t=1, respectively). Part (b) gives an equation of a circle with radius 2 and center at the origin.

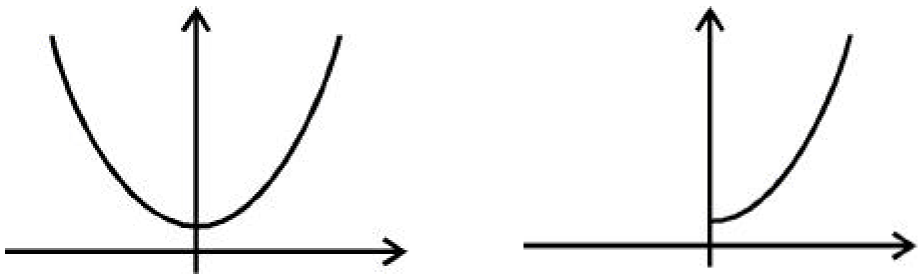




For some parametric equations, you can eliminate t to get y as a familiar function of x. Then you can sketch the portion of this familiar graph indicated by the domain.

Identify the graphs of (a) $x(t) = t, y(t) = t^2+1$ and (b) $x(t) = t^2, y(t) = t^4+1$

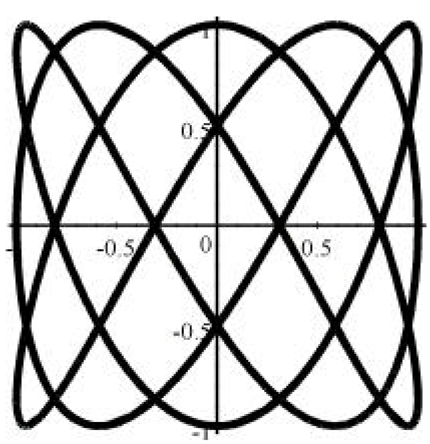
Notice that in both cases, we can write $y = x^2+1$. [In part (b), $x^2=(t^2)^2=t^4$.] So the graph will be some portion of a parabola with vertex (0,1). In part (a), $x = t$ so if t can be any real number, then x can be any real number. Thus, in part (a) we get the entire parabola. However, in part (b) we have $x = t^2 \geq 0$. Any point with $x > 0$ is graphed on the right side of the y-axis, so in part (b) we only get the right half of the parabola.



For most parametric equations, you will not be able to solve for y as a function of x. Often, the best you can do is get a general idea of some properties of the graph. For a full graph, you would need to construct a large chart and plot lots of points. Fortunately, this is one task that graphing calculators are quite good at.

Describe the graph of $x(t) = \cos 3t, y(t) = \sin 5t$.

Even if you try several trig identities, there is no simple way to write $\sin 5t$ in terms of $\cos 3t$. So, we won't find an x-y equation for this graph. We can determine some properties of the graph, however. First, since sine and cosine are both bounded functions, the graph will lie entirely in the square with x and y between -1 and 1. Second, both sine and cosine are periodic. In this case, both $\sin 5t$ and $\cos 3t$ repeat every 2π units, so the graph will be completed with $0 \leq t \leq 2\pi$. You should expect the graph to form some type of closed loop. These properties are present in the computer-generated graph shown.



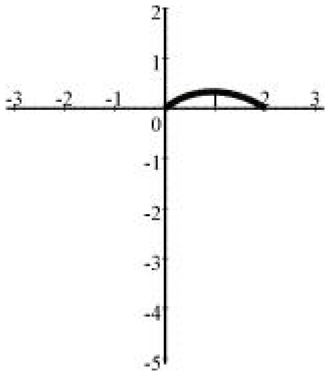
Graphs in polar coordinates are typically constructed piece-by-piece. While graphing calculators will provide fine graphs of most polar curves, you do need to understand how to do these by hand. This understanding will help you recognize whether or not your calculator has drawn the entire graph. More importantly, you will need this understanding to compute areas and other integrals in polar coordinates. In this case, it is essential that you be able to identify which pieces of a curve correspond to which values of θ.

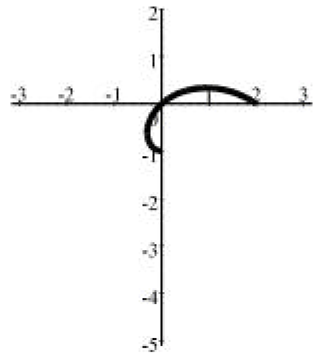
Sketch the graph of the polar curve $r = 2 - 3\sin\theta$.

We look for points at which r is a maximum, minimum or 0. We know that the sine function reaches a maximum at $\theta=\pi/2$ and drops to a minimum at $\theta=3\pi/2$. Since the sine term is subtracted, this says that r reaches a minimum of $2-3 = -1$ at $\theta = \pi/2$ and has a maximum of $2+3=5$ at $\theta=3\pi/2$. We have $r=0$ if $3\sin\theta = 2$ or $\sin\theta = 2/3$. This happens when $\theta=\sin^{-1}(2/3)\approx 0.73$ (about 42°) and $\theta=\pi - \sin^{-1}(2/3)\approx 2.41$ (about 138°). Summarizing the important information, we have

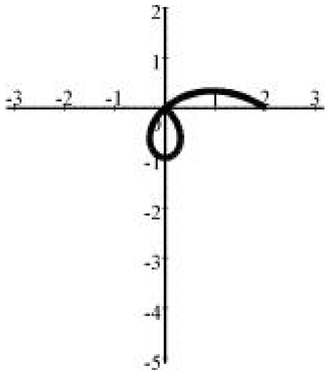
$$\text{At } \theta = 0,\ r = 2$$
$$\text{At } \theta = \sin^{-1}(2/3),\ r = 0$$
$$\text{At } \theta = \pi/2,\ r = -1$$
$$\text{At } \theta = \pi - \sin^{-1}(2/3),\ r = 0$$
$$\text{At } \theta = \pi,\ r = 2$$
$$\text{At } \theta = 3\pi/2,\ r = 5$$
$$\text{At } \theta = 2\pi,\ r = 2$$

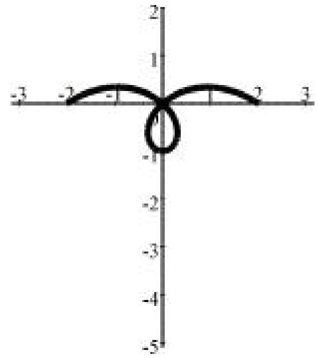
From here, we essentially plot each point and connect the dots. Start at $\theta = 0$, $r = 2$; this is the point (2,0). Staying in the first quadrant, swing over to the origin at $\theta = \sin^{-1}(2/3)$, just short of 45° (see the figure to the left). At this point, r becomes negative so the graph moves into the third quadrant. The sketch to the right follows the plot to where r reaches its most negative value at $\theta = \pi/2$.



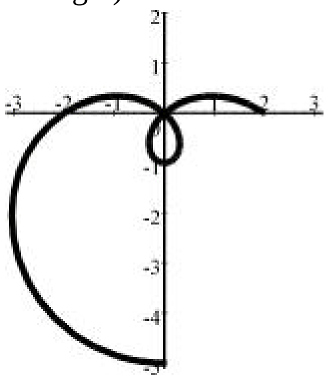


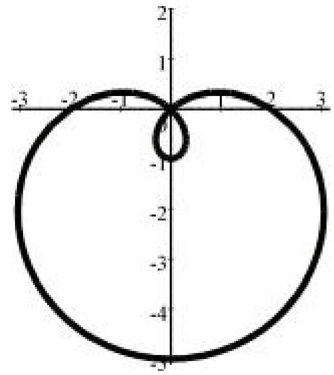
The angles from $\theta = \pi/2$ to $\theta = \pi - \sin^{-1}(2/3)$ are second quadrant angles. Since r remains negative in this interval, the curve is plotted in the fourth quadrant (see the figure to the left). From $\theta = \pi - \sin^{-1}(2/3)$ to $\theta = \pi$, r is positive and the plot is in the second quadrant (see the figure to the right).





Finally, we follow the curve in the third quadrant to its maximum *r*-value at θ = 3π/2 (see the figure to the left) and in the fourth quadrant back to the beginning point (see the figure on the right).





This process may seem a little slow, but we urge you to follow along with each step. If you do, your understanding of polar coordinates and your ability to successfully use polar coordinates in later chapters will improve dramatically.

Extension

Parametric equations and polar coordinates each provide alternatives to rectangular coordinates. If you are modeling a physical process, you now have options other than finding one variable *y* as a function of another variable *x*. If you can write each spatial coordinate (*x*, *y* and possibly *z*) as a function of time, you should consider using parametric equations. This is especially true with projectile motion, as seen in section 5.5, and examples like the Scrambler. When your physical process has a component of circular motion or symmetry, you should consider using polar coordinates. This idea is used extensively in the multiple integrals discussed in chapters 13 and 14.