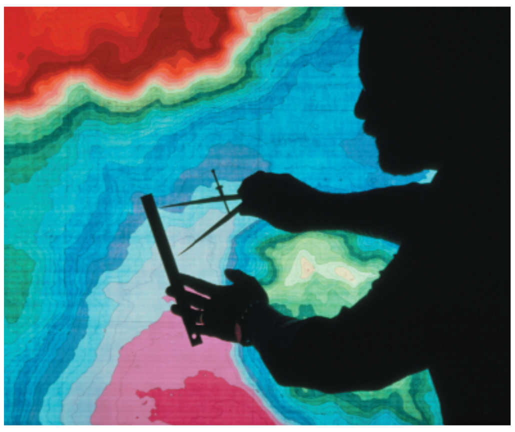


CHAPTER 3

APPLICATIONS OF DIFFERENTIATION

One relatively new test available to physicians for diagnosing injuries and disease is the MRI. Magnetic resonance imaging (MRI) is used to visualize internal structures, such as torn cartilage in a knee. The ability to *see* the physical status of a knee or an internal organ without surgery is an invaluable aid to physicians and their patients. However, it still takes an experienced physician to distinguish the important features of an MRI from insignificant ones. If you have ever looked at an MRI or even a conventional x-ray, you have probably been amazed at the details that your physician could quickly identify. In the MRI below, can you identify any damage to the knee? Of course, it always helps to know what you are looking for.

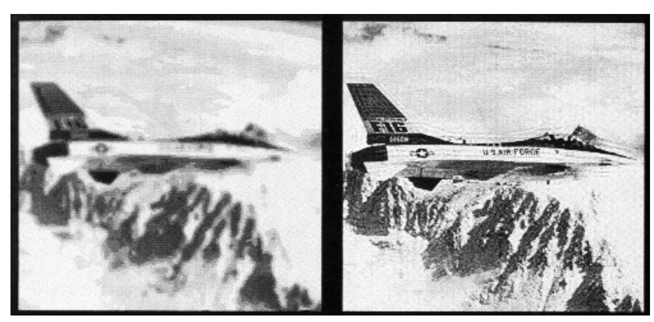
The ability to accurately read graphs is one of the primary goals of this chapter. By the end of section 3.6, you should have a good idea of what the significant features of a graph are. Although we will be looking only at two-dimensional graphs of functions, the language and skills that you acquire here will transfer to plots of seismic readings, sonar mappings of the ocean floor and other graphical displays of information that you may encounter.

Most people do not recognize the vast amount of mathematical computation required to produce a viewable image from an MRI. In an MRI, magnetic fields and pulses of radio waves are used to determine the distribution of hydrogen atoms in the body (see *Visualization* by R. Friedhoff and W. Benzon for more details). The presence of hydrogen atoms, in turn, is deduced from the release of energy during the magnetization process. (This is a long way from a standard x-ray image!) By solving countless equations and performing lengthy calculations, a computer transforms the energy data into an accurate image of the interior of a human body.

MRI of a knee.

Likewise, it may surprise you how many calculations we must perform to draw an accurate graph of a function. At each stage of the graphing process, we must solve equations to identify significant features of the graph. Because of the central role that equation solving plays in this chapter, we devote the second section to a discussion of a powerful method that you can use to approximate solutions of difficult equations.

But, you may ask, if a computer or calculator can do all of the calculations, why do you need to know what it's doing? One answer is that most computer algorithms are imperfect approximations that can occasionally result in significant errors on certain types of problems. By understanding how such algorithms work, you can anticipate and identify when a computer is in error. For example, the picture on the right below is a computer enhancement of the out-of-focus picture on the left.



Notice that the picture on the right shows a faint halo around the airplane. This is not a real ghost or aura or even sound waves, but instead a by-product of the computer algorithm used to sharpen the picture. If you understand the algorithm, you will not misinterpret the ghost. A ghost on an airplane is not serious, but the mathematics used to sharpen the picture is also used to produce MRIs, where a misinterpreted ghost could have serious consequences. (How would you feel if an MRI appeared to show a tumor that was not actually there?)

3.1 LINEAR APPROXIMATIONS AND L'HÔPITAL'S RULE

For what purpose do you use a scientific calculator? If you think about it, you'll discover that there are two distinctly different jobs that calculators do for you. First, they perform arithmetic operations (addition, subtraction, multiplication and division) much faster than any of us could hope to do them. It's not that you don't know how to multiply 1024 by 1673, but rather that it is time-consuming to carry out this (albeit well-understood) calculation with pencil and paper. For such problems, calculators are a tremendous convenience, which none of us would like to live without. Perhaps more significantly, we also use our calculators to compute values of transcendental functions such as sine, cosine, tangent, exponentials and logarithms. In the case of these function evaluations, the calculator is much more than a mere convenience.

If asked to calculate sin(1.2345678) without a calculator, you would probably draw a blank. Don't worry, there's nothing wrong with your background. (Also, don't worry that anyone will ever ask you to do this without a calculator.) The problem is that the sine function is not *algebraic*. That is, there is no formula for $\sin x$ involving only the arithmetic operations. So, how does your calculator "know" that $\sin(1.2345678) \approx 0.9440056953$? In short, it doesn't *know* this at all. Rather, the calculator has a built-in program that generates **approximate** values of the sine and other transcendental functions.

In this section, we take a small step into the (very large) world of approximation by developing a simple approximation method. Although somewhat crude, it points the way

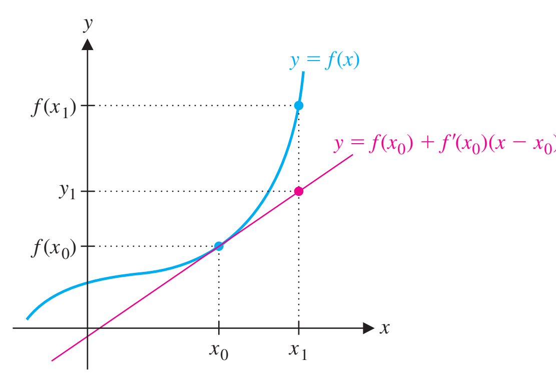


Figure 3.1
Linear approximation of $f(x_1)$.

toward more sophisticated approximation techniques to follow later in the text. Our primary intent here is to give you a taste of how you might approach the problem of approximation.

Linear Approximations

Suppose we wanted to find an approximation for $f(x_1)$, where $f(x_1)$ is unknown, but where $f(x_0)$ is known for some x_0 "close" to x_1. For instance, the value of $\cos(1)$ is unknown, but we do know that $\cos(\pi/3) = \frac{1}{2}$ exactly and $\pi/3 \approx 1.047$ is "close" to 1. We could always use $\frac{1}{2}$ as an approximation to $\cos(1)$, but we can do better.

Recall that the tangent line to the curve $y = f(x)$ at $x = x_0$ stays close to the curve near the point of tangency. Referring to Figure 3.1, notice that if x_1 is "close" to x_0 and we follow the tangent line at $x = x_0$ to the point corresponding to $x = x_1$, then the y-coordinate of that point (y_1) should be "close" to the y-coordinate of the point on the curve $y = f(x)$ [i.e., $f(x_1)$].

Since the slope of the tangent line to $y = f(x)$ at $x = x_0$ is $f'(x_0)$, the equation of the tangent line to $y = f(x)$ at $x = x_0$ is found from

$$m_{\tan} = f'(x_0) = \frac{y - f(x_0)}{x - x_0}. \tag{1.1}$$

Solving equation (1.1) for y gives us

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{1.2}$$

Notice that (1.2) is the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$. We give the linear function defined by this equation a name, as follows.

Definition 1.1
The **linear** (or **tangent line**) **approximation** of $f(x)$ at $x = x_0$ is the function $L(x) = f(x_0) + f'(x_0)(x - x_0)$.

Notice that the y-coordinate y_1 of the point on the tangent line corresponding to $x = x_1$ is simply found by substituting $x = x_1$ in equation (1.2). This gives us

$$y_1 = f(x_0) + f'(x_0)(x_1 - x_0). \tag{1.3}$$

We define the **increments** Δx and Δy by

$$\Delta x = x_1 - x_0$$

and

$$\Delta y = f(x_1) - f(x_0).$$

Using this notation, equation (1.3) gives us the approximation

$$\boxed{f(x_1) \approx y_1 = f(x_0) + f'(x_0)\,\Delta x.} \tag{1.4}$$

We illustrate this in Figure 3.2. We sometimes rewrite (1.4) by subtracting $f(x_0)$ from both sides, to yield

$$\Delta y = f(x_1) - f(x_0) \approx f'(x_0)\,\Delta x = dy, \tag{1.5}$$

where $dy = f'(x_0)\,\Delta x$ is called the **differential** of y. When using this notation, we also define dx, the differential of x, by $dx = \Delta x$, so that by (1.5),

$$dy = f'(x_0)\,dx.$$

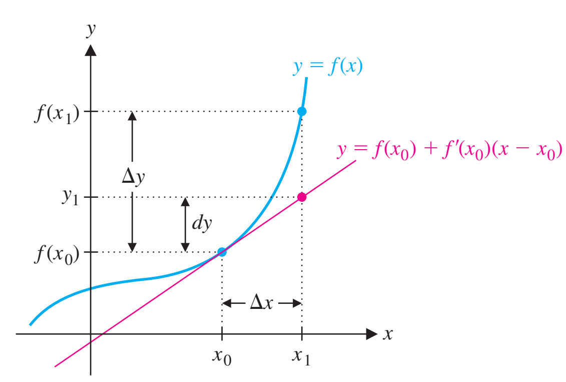


Figure 3.2
Increments and differentials.

We can use linear approximations to produce approximate values of transcendental functions, as in the following example.

Example 1.1 Finding a Linear Approximation

Find the linear approximation to $f(x) = \cos x$ at $x_0 = \pi/3$ and use it to approximate $\cos(1)$.

Solution From Definition 1.1, the linear approximation is defined as $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Here, $x_0 = \pi/3$, $f(x) = \cos x$ and $f'(x) = -\sin x$. So, we have

$$L(x) = \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

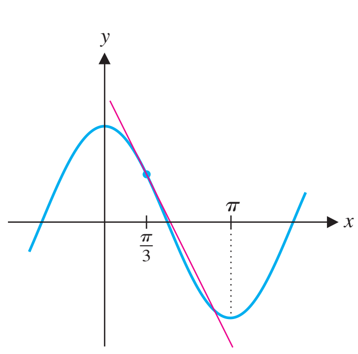


Figure 3.3
$y = \cos x$ and the linear approximation at $x_0 = \pi/3$.

In Figure 3.3, we show a graph of $y = \cos x$ and the linear approximation to $\cos x$ for $x_0 = \pi/3$. Notice that the linear approximation (i.e., the tangent line at $x_0 = \pi/3$) stays close to the graph of $y = \cos x$ only for x close to $\pi/3$. In fact, for $x < 0$ or $x > \pi$, the linear approximation is obviously quite poor. This is typical of linear approximations (tangent lines). They generally stay close to the curve only nearby the point of tangency.

Although we gave you the value of x_0 here, observe that if you had been given the choice, one reason you might choose $x_0 = \frac{\pi}{3}$ is that $\frac{\pi}{3}$ is the value closest to 1 at which we know the value of the cosine exactly. Finally, an estimate of cos(1) is

$$L(1) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(1 - \frac{\pi}{3}\right) \approx 0.5409.$$

Notice that on your calculator you would find $\cos(1) \approx 0.5403$, so that we have found a fairly good approximation to the desired value.

In the following example, we derive a useful approximation to $\sin x$, valid for x close to 0. This approximation is significant in part because it is often used in applications in physics and engineering to simplify equations involving $\sin x$.

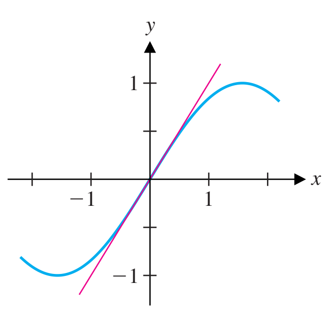


Figure 3.4
$y = \sin x$ and $y = x$.

Example 1.2 **Linear Approximation of sin x**

Find the linear approximation of $f(x) = \sin x$, for x close to 0.

Solution Here, $f'(x) = \cos x$, so that from Definition 1.1, we have

$$\sin x \approx L(x) = f(0) + f'(0)(x - 0) = \sin 0 + \cos 0(x) = x.$$

This says that for x close to 0, $\sin x \approx x$. We illustrate this in Figure 3.4, where we show graphs of both $y = \sin x$ and $y = x$.

Pay close attention to Figure 3.4; notice that the graph of $y = x$ stays close to the graph of $y = \sin x$ only in the vicinity of $x = 0$. This indicates that the approximation $\sin x \approx x$ is valid only for x close to 0. Also note that the farther x gets from 0, the worse the approximation becomes. Take another look at Figure 3.1, to convince yourself that this is generally true. This becomes even more apparent in the following example, where we also illustrate the use of the increments Δx and Δy.

Example 1.3 **Linear Approximation to Some Cube Roots**

Use a linear approximation to approximate $\sqrt[3]{8.02}$, $\sqrt[3]{8.07}$, $\sqrt[3]{8.15}$ and $\sqrt[3]{25.2}$.

Solution Here we are approximating values of the function $f(x) = \sqrt[3]{x} = x^{1/3}$. So, $f'(x) = \frac{1}{3}x^{-2/3}$. Of course, the closest number to any of 8.02, 8.07 or 8.15 whose cube root we know exactly is 8. Now,

$$\begin{aligned} f(8.02) &= f(8) + [f(8.02) - f(8)] \quad \text{Add and subtract } f(8). \\ &= f(8) + \Delta y. \end{aligned} \tag{1.6}$$

From (1.5), we have

$$\begin{aligned} \Delta y \approx dy &= f'(8)\,\Delta x \\ &= \left(\frac{1}{3}\right) 8^{-2/3}(8.02 - 8) = \frac{1}{600}. \quad \text{Since } \Delta x = 8.02 - 8. \end{aligned} \tag{1.7}$$

Using (1.6) and (1.7), we get

$$f(8.02) \approx f(8) + dy = 2 + \frac{1}{600} \approx 2.0016667,$$

while your calculator accurately returns $\sqrt[3]{8.02} \approx 2.0016653$. Similarly, we get

$$f(8.07) \approx f(8) + \frac{1}{3}8^{-2/3}(8.07 - 8) \approx 2.0058333$$

and

$$f(8.15) \approx f(8) + \frac{1}{3}8^{-2/3}(8.15 - 8) \approx 2.0125,$$

while your calculator returns $\sqrt[3]{8.07} \approx 2.005816$ and $\sqrt[3]{8.15} \approx 2.012423$, respectively.

Finally, notice that to approximate $\sqrt[3]{25.2}$, 8 is not the closest number to 25.2 whose cube root we know exactly. Since 25.2 is much closer to 27 than to 8, we write

$$f(25.2) = f(27) + \Delta y \approx f(27) + dy = 3 + dy.$$

In this case,

$$dy = f'(27)\Delta x = \frac{1}{3}27^{-2/3}(25.2 - 27) = \frac{1}{3}\left(\frac{1}{9}\right)(-1.8) = -\frac{1}{15}$$

and we have

$$f(25.2) \approx 3 + dy = 3 - \frac{1}{15} = 2.9333333,$$

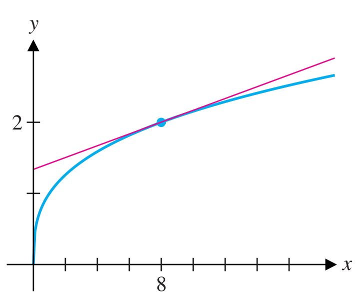


Figure 3.5
$y = \sqrt[3]{x}$ and the linear approximation at $x_0 = 8$.

compared to the value of 2.931794, produced by your calculator. It is important to recognize here that the farther the value of x gets from the point of tangency, the worse the approximation tends to be. You can see this clearly in Figure 3.5, where the linear approximation gets farther away from $\sqrt[3]{x}$, as x gets farther from 8.

Our first three examples were intended to familiarize you with the technique and to give you a feel for how good (or bad) linear approximations tend to be. In the following example, there is no exact answer to compare with the approximation. Our use of the linear approximation here is referred to as **linear interpolation.**

Example 1.4 Using a Linear Approximation to Perform Linear Interpolation

x	6	10	14
$f(x)$	84	60	32

The price of an item affects consumer demand for that item. Suppose that based on market research, a company estimates that $f(x)$ thousand small cameras can be sold at the price of $\$x$, as given in the accompanying table. Estimate the number of cameras that can be sold at $7.

Solution The closest x-value to $x = 7$ in the table is $x = 6$. (In other words, this is the closest value of x at which we know the value of $f(x)$.) The linear approximation of $f(x)$ at $x = 6$ would look like

$$L(x) = f(6) + f'(6)(x - 6).$$

From the table, we know that $f(6) = 84$, but we do not know $f'(6)$. Further, we can't compute $f'(x)$, since we don't have a formula for $f(x)$. The best we can do with the given data is approximate the derivative by

$$f'(6) \approx \frac{f(10) - f(6)}{10 - 6} = \frac{60 - 84}{4} = -6.$$

The linear approximation is then

$$L(x) = 84 - 6(x - 6).$$

Using this, we estimate that the number of cameras sold at $x = 7$ would be $L(7) = 84 - 6 = 78$ thousand. That is, we would expect to sell approximately 78 thousand cameras at a price of \$7. We show a graphical interpretation of this in Figure 3.6, where the straight line is the linear approximation (in this case, the secant line joining the first two data points).

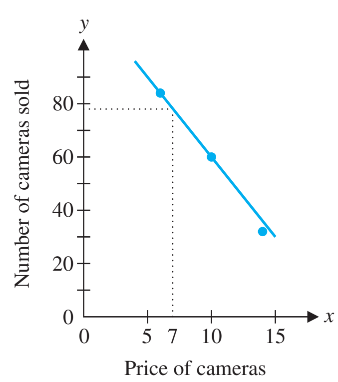


Figure 3.6
Linear interpolation.

L'Hôpital's Rule

We close this section by using linear approximations to suggest a simple method for computing some challenging limits. We develop a special case of an important result known as L'Hôpital's Rule, which we more thoroughly develop in section 7.6.

Look back at section 2.5, where we struggled with the limit $\lim_{x\to 0} \frac{\sin x}{x}$, ultimately resolving it only with an intricate geometric argument. This limit has the indeterminate form $\frac{0}{0}$ (i.e., the limits of the numerator and the denominator are both 0), but there is no way to simplify the numerator or denominator to simplify the expression. More generally, we'd like to evaluate $\lim_{x\to c} \frac{f(x)}{g(x)}$, where $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$. We can use linear approximations to suggest a solution, as follows.

If both functions are differentiable at $x = c$, then they are also continuous at $x = c$, so that $f(c) = \lim_{x\to c} f(x) = 0$ and $g(c) = \lim_{x\to c} g(x) = 0$. We now have the linear approximations

$$f(x) \approx f(c) + f'(c)(x - c) = f'(c)(x - c)$$

and

$$g(x) \approx g(c) + g'(c)(x - c) = g'(c)(x - c),$$

where we have used the fact that $f(c) = 0$ and $g(c) = 0$. As we have seen, the approximation should improve as x approaches c, so we would expect that if the limits exist,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(c)(x - c)}{g'(c)(x - c)} = \lim_{x\to c} \frac{f'(c)}{g'(c)} = \frac{f'(c)}{g'(c)},$$

assuming that $g'(c) \neq 0$. Note that if $f'(x)$ and $g'(x)$ are continuous at $x = c$ and $g'(c) \neq 0$, then $\frac{f'(c)}{g'(c)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$. We summarize this in the following result.

Theorem 1.1 (L'Hôpital's Rule)

Suppose that f and g are differentiable on the interval (a, b) except possibly at some fixed point c in (a, b) and that $g'(x) \neq 0$ on (a, b) except possibly at $x = c$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Here, we prove only the special case where f, f', g and g' are all continuous on all of (a, b) and $g'(c) \neq 0$, while leaving the more intricate general case for Appendix A. First, recall the alternative form of the definition of derivative (found in section 2.2):

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

Historical Notes

Guillaume de l'Hôpital (1661–1704) A French mathematician who first published the result now known as L'Hôpital's Rule. Born into nobility, l'Hôpital was taught calculus by the brilliant mathematician Johann Bernoulli. A competent mathematician, l'Hôpital is best known as the author of the first calculus textbook. L'Hôpital was a friend and patron of many of the top mathematicians of the seventeenth century.

Working backward, we have by continuity that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=\frac{f'(c)}{g'(c)}=\frac{\displaystyle\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}{\displaystyle\lim_{x\to c}\frac{g(x)-g(c)}{x-c}}=\lim_{x\to c}\frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}}=\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}.$$

Further, since f and g are continuous at $x=c$, we have that

$$f(c)=\lim_{x\to c}f(x)=0 \quad\text{and}\quad g(c)=\lim_{x\to c}g(x)=0.$$

It now follows that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}=\lim_{x\to c}\frac{f(x)}{g(x)},$$

which is what we wanted.

With this result, certain limits become quite easy to evaluate.

Example 1.5 Revisiting an Old Limit

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$.

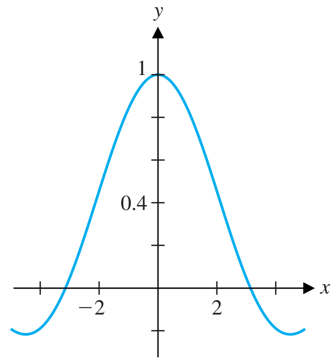


Figure 3.7
$y=\dfrac{\sin x}{x}$.

Solution Again, this limit has the indeterminate form $\frac{0}{0}$ and $f(x)=\sin x$ and $g(x)=x$ are both continuous and differentiable everywhere. Finally, $g'(x)=\dfrac{d}{dx}(x)=1\neq 0$, so that all of the hypotheses of L'Hôpital's Rule are satisfied. From the graph in Figure 3.7, it appears that the limit is approximately 1. We can confirm this suspicion with L'Hôpital's Rule. We have

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\dfrac{d}{dx}(\sin x)}{\dfrac{d}{dx}(x)}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1,$$

as we proved using a complicated geometric argument in section 2.5.

For some limits, you must apply L'Hôpital's Rule more than once.

Example 1.6 A Limit Requiring Two Applications of L'Hôpital's Rule

Evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

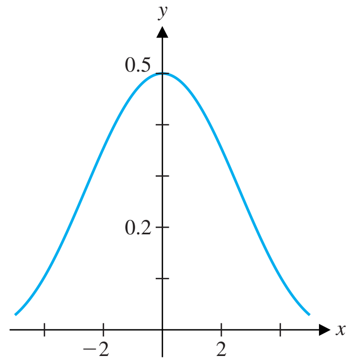


Figure 3.8
$y=\dfrac{1-\cos x}{x^2}$.

Solution Again, this has the indeterminate form $\frac{0}{0}$ and it is a simple matter to verify that the hypotheses of L'Hôpital's Rule are satisfied. In this case, the graph in Figure 3.8 indicates the limit to be approximately 0.5. From L'Hôpital's Rule, we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x},$$

which again has the indeterminate form $\frac{0}{0}$. In this case, we can verify that the hypotheses of L'Hôpital's Rule are satisfied for this new limit problem. Applying this again, it then follows that

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}.$$

Remark 1.1

A very common error is to apply L'Hôpital's Rule indiscriminately, without first checking that the limit has the indeterminate form $\frac{0}{0}$. Be very careful here.

Example 1.7 **An Erroneous Use of L'Hôpital's Rule**

Find the mistake in the string of equalities

$$\lim_{x\to 0} \frac{x^2}{e^x - 1} = \lim_{x\to 0} \frac{2x}{e^x} = \lim_{x\to 0} \frac{2}{e^x} = \frac{2}{1} = 2.$$

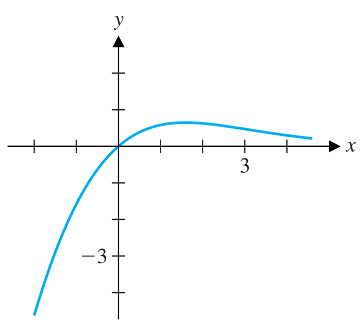


Figure 3.9

$y = \dfrac{x^2}{e^x - 1}$.

Solution From the graph in Figure 3.9, we can see that the limit is approximately 0, so 2 appears to be incorrect. The first limit $\lim_{x\to 0} \frac{x^2}{e^x - 1}$ has the form $\frac{0}{0}$ and the functions $f(x) = x^2$ and $g(x) = e^x - 1$ satisfy the hypotheses of L'Hôpital's Rule. Therefore, the first equality: $\lim_{x\to 0} \frac{x^2}{e^x - 1} = \lim_{x\to 0} \frac{2x}{e^x}$ holds. However, notice that $\lim_{x\to 0} \frac{2x}{e^x} = \frac{0}{1} = 0$ and L'Hôpital's Rule does **not** apply here. The correct evaluation is then

$$\lim_{x\to 0} \frac{x^2}{e^x - 1} = \lim_{x\to 0} \frac{2x}{e^x} = \frac{0}{1} = 0.$$

EXERCISES 3.1

1. We constructed a variety of linear approximations in this section. Approximations can be "good" approximations or "bad" approximations. Explain why it can be said that $y = x$ is a good approximation to $y = \sin x$ near $x = 0$ but $y = 1$ is not a good approximation to $y = \cos x$ near $x = 0$. (Hint: Look at the graphs of $y = \sin x$ and $y = x$ on the same axes, then do the same with $y = \cos x$ and $y = 1$.)

2. Briefly explain in terms of tangent lines why the approximation in example 1.2 gets worse as x gets farther from 8.

3. A friend is struggling with L'Hôpital's Rule. When asked to work a problem, your friend says, "First, I plug in for x and get 0 over 0. Then I use the quotient rule to take the derivative. Then I plug x back in." Explain to your friend what the mistake is and how to correct it.

4. Suppose that two runners begin a race from the starting line, with one runner initially going twice as fast as the other. If $f(t)$ and $g(t)$ represent the positions of the runners at time $t \ge 0$, explain why we can assume that $f(0) = g(0) = 0$ and $\lim_{t\to 0+} \frac{f'(t)}{g'(t)} = 2$. Explain in terms of the runners' positions why L'Hôpital's Rule holds: that is, $\lim_{t\to 0} \frac{f(t)}{g(t)} = 2$.

In exercises 5–12, find the linear approximation to $f(x)$ at $x = x_0$. Graph the function and its linear approximation.

5. $f(x) = \sqrt{x}$, $x_0 = 1$
6. $f(x) = (x+1)^{1/3}$, $x_0 = 0$
7. $f(x) = \sqrt{2x+9}$, $x_0 = 0$
8. $f(x) = 2/x$, $x_0 = 1$
9. $f(x) = \sin 3x$, $x_0 = 0$
10. $f(x) = \sin x$, $x_0 = \pi$
11. $f(x) = e^{2x}$, $x_0 = 0$
12. $f(x) = e^{x^2}$, $x_0 = 0$

In exercises 13–16, find the linear approximation at $x = 0$ to show that the following commonly used approximations are valid for "small" x. Compare the approximate and exact values for $x = 0.01$, $x = 0.1$ and $x = 1$.

13. $\tan x \approx x$
14. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$
15. $\sqrt{4+x} \approx 2 + \frac{1}{4}x$
16. $e^x \approx 1 + x$

In exercises 17–22, use linear approximations to estimate the quantity.

17. $\sin 1$
18. $\sin \frac{9}{4}$
19. $\sqrt[4]{16.04}$
20. $\sqrt[4]{16.08}$
21. $\sqrt[4]{16.16}$
22. $\ln 2.8$ (Hint: $\ln e = 1$.)

23. For exercises 19–21, compute the error (the absolute value of the difference between the exact value and the linear approximation).

24. Thinking of exercises 19–21 as numbers of the form $\sqrt[4]{16+\Delta x}$, denote the errors as $e(\Delta x)$ (where $\Delta x = 0.04$, $\Delta x = 0.08$ and $\Delta x = 0.16$). Based on these three computations, determine a constant c such that $e(\Delta x) \approx c(\Delta x)^2$.

25. Use a computer algebra system (CAS) to determine the range of x's in exercise 13 for which the approximation is accurate to within 0.01. That is, find x such that $|\tan x - x| < 0.01$.

26. Use a CAS to determine the range of x's in exercise 16 for which the approximation is accurate to within 0.01. That is, find x such that $|e^x - (1+x)| < 0.01$.

In exercises 27–30, use linear interpolation to estimate the desired quantity.

27. A company estimates that $f(x)$ thousand software games can be sold at the price of $\$x$ as given in the table.

x	20	30	40
$f(x)$	18	14	12

Estimate the number of games that can be sold at (a) \$24 and (b) \$36.

28. A vending company estimates that $f(x)$ cans of soft drink can be sold in a day if the temperature is x°F as given in the table.

x	60	80	100
$f(x)$	84	120	168

Estimate the number of cans that can be sold at (a) 72° and (b) 94°.

29. An animation director enters the position $f(t)$ of a character's head after t frames of the movie as given in the table.

t	200	220	240
$f(t)$	128	142	136

If the computer software uses interpolation to determine the intermediate positions, determine the position of the head at frame number (a) 208 and (b) 232.

30. A sensor measures the position $f(t)$ of a particle t microseconds after a collision as given in the table.

t	5	10	15
$f(t)$	8	14	18

Estimate the position of the particle at time (a) $t = 8$ and (b) $t = 12$.

In exercises 31–42, use L'Hôpital's Rule to evaluate the limit.

31. $\lim_{x\to 2} \frac{x-2}{x^2-4}$

32. $\lim_{x\to -1} \frac{x+1}{x^2+4x+3}$

33. $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

34. $\lim_{x\to 0} \frac{x^3}{x - \tan x}$

35. $\lim_{x\to 1} \frac{x-1}{\ln x}$

36. $\lim_{x\to 1} \frac{\ln x}{x^2}$

37. $\lim_{x\to 0} \frac{e^x - 1}{\cos x - 1}$

38. $\lim_{x\to 0+} \frac{\sqrt{x}}{\ln(x+1)}$

39. $\lim_{x\to 0} \frac{x^2}{\cos x - x}$

40. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

41. $\lim_{x\to 1} \frac{\ln(\ln x)}{\ln x}$

42. $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$

43. Compute $\lim_{x\to 0} \frac{\sin x^2}{x^2}$ and compare your result to that of example 1.5.

44. Compute $\lim_{x\to 0} \frac{1-\cos x^2}{x^4}$ and compare your result to that of example 1.6.

45. Use your results from exercises 43 and 44 to evaluate $\lim_{x\to 0} \frac{\sin x^3}{x^3}$ and $\lim_{x\to 0} \frac{1-\cos x^3}{x^6}$ without doing any calculations.

46. If $\lim_{x\to 0} \frac{f(x)}{g(x)} = L$, what can be said about $\lim_{x\to 0} \frac{f(x^2)}{g(x^2)}$? Explain why knowing that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ for $a \neq 0$ does not tell you anything about $\lim_{x\to a} \frac{f(x^2)}{g(x^2)}$.

47. Find all errors in the string

$$\lim_{x\to 0} \frac{\cos x}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}.$$

Then determine the correct value of the limit.

48. Find all errors in the string

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Then determine the correct value of the limit.

49. Starting with $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x}$, cancel sin to get $\lim\limits_{x\to 0} \dfrac{3x}{2x}$, then cancel x's to get $\frac{3}{2}$. This answer is correct. Are either of the steps used valid? Use linear approximations to argue that the first step is likely to give a correct answer.

50. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin nx}{\sin mx}$ for nonzero constants n and m.

51. Evaluate $\lim\limits_{x\to 0} \dfrac{e^{cx}-1}{x}$ for any constant c.

52. Evaluate $\lim\limits_{x\to 0} \dfrac{\tan cx - cx}{x^3}$ for any constant c.

53. In section 1.1, we briefly discussed the position of a baseball thrown with the unusual knuckleball pitch. The left/right position (in feet) of a ball thrown with spin rate ω and a particular grip at time t seconds is $f(\omega) = (2.5/\omega)t - (2.5/4\omega^2)\sin 4\omega t$. Treating t as a constant and ω as the variable (change to x if you like), show that $\lim\limits_{\omega\to 0} f(\omega) = 0$ for any value of t. (Hint: Find a common denominator and use L'Hôpital's Rule.) Conclude that this pitch does not move left or right at all.

54. In this exercise, we look at a knuckleball thrown with a different grip than that of exercise 53. The left or right position (in feet) of a ball thrown with spin rate ω and this new grip at time t seconds is $f(\omega) = (2.5/4\omega^2) - (2.5/4\omega^2)\sin(4\omega t + \pi/2)$. Treating t as a constant and ω as the variable (change to x if you like), find $\lim\limits_{\omega\to 0} f(\omega)$. Your answer should depend on t. By graphing this function of t, you can see the path of the pitch (use a domain of $0 \le t \le 0.68$). Describe this pitch.

55. A water wave of length L meters in water of depth d meters has velocity v satisfying the equation

$$v^2 = \frac{4.9L}{\pi}\,\frac{e^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}}.$$

Treating L as a constant and thinking of v^2 as a function $f(d)$, use a linear approximation to show that $f(d) \approx 9.8d$ for small values of d. That is, for small depths the velocity of the wave is approximately $\sqrt{9.8d}$ and is independent of the wavelength L.

56. Planck's law states that the energy density of blackbody radiation of wavelength x is given by

$$f(x) = \frac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}.$$

Use the linear approximation in exercise 16 to show that $f(x) \approx 8\pi kT/x^4$, which is known as the Rayleigh-Jeans law.

57. In this exercise, we introduce **Taylor series** (explored in depth in Chapter 8). Start with the limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. Briefly explain why this means that for x close to 0, $\sin x \approx x$. Graph $y = \sin x$ and $y = x$ to see why this is true. If you look far enough away from $x = 0$, the graph of $y = \sin x$ eventually curves noticeably. We will find polynomials of higher order to match this curving. Show that $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^2} = 0$. This means that $\sin x - x \approx 0$ or (again) $\sin x \approx x$. Show that $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3} = -\frac{1}{6}$. This says that if x is close to 0, then $\sin x - x \approx -\frac{1}{6}x^3$ or $\sin x \approx x - \frac{1}{6}x^3$. Graph these two functions to see how well they match up. To continue, compute $\lim\limits_{x\to 0} \dfrac{\sin x - (x - x^3/6)}{x^4}$ and $\lim\limits_{x\to 0} \dfrac{\sin x - f(x)}{x^5}$ for the appropriate approximation $f(x)$. At this point, look at the pattern of terms you have (Hint: $6 = 3!$ and $120 = 5!$). Using this pattern, approximate $\sin x$ with an 11th-degree polynomial and graph the two functions.

3.2 NEWTON'S METHOD

We now return to the question of finding zeros of a function. In section 1.3, we introduced the method of bisections as a tedious, yet reliable, method of finding zeros of continuous functions. In this section, we explore a method which is usually much more efficient than bisections. We are again looking for values of x such that $f(x) = 0$. These values are called **roots** of the equation $f(x) = 0$ or **zeros** of the function f. If

$$f(x) = ax^2 + bx + c,$$

there is no challenge to doing this, since we have an explicit formula for the solution(s) (the quadratic formula). But, what if we want to find zeros of

$$f(x) = \tan x - x?$$