49. Starting with $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$, cancel sin to get $\lim_{x\to 0} \frac{3x}{2x}$, then cancel x's to get $\frac{3}{2}$. This answer is correct. Are either of the steps used valid? Use linear approximations to argue that the first step is likely to give a correct answer.

50. Evaluate $\lim_{x\to 0} \frac{\sin nx}{\sin mx}$ for nonzero constants n and m.

51. Evaluate $\lim_{x\to 0} \frac{e^{cx} - 1}{x}$ for any constant c.

52. Evaluate $\lim_{x\to 0} \frac{\tan cx - cx}{x^3}$ for any constant c.

53. In section 1.1, we briefly discussed the position of a baseball thrown with the unusual knuckleball pitch. The left/right position (in feet) of a ball thrown with spin rate ω and a particular grip at time t seconds is $f(\omega) = (2.5/\omega)t - (2.5/4\omega^2) \sin 4\omega t$. Treating t as a constant and ω as the variable (change to x if you like), show that $\lim_{\omega\to 0} f(\omega) = 0$ for any value of t. (Hint: Find a common denominator and use L'Hôpital's Rule.) Conclude that this pitch does not move left or right at all.

54. In this exercise, we look at a knuckleball thrown with a different grip than that of exercise 53. The left or right position (in feet) of a ball thrown with spin rate ω and this new grip at time t seconds is $f(\omega) = (2.5/4\omega^2) - (2.5/4\omega^2) \sin(4\omega t + \pi/2)$. Treating t as a constant and ω as the variable (change to x if you like), find $\lim_{\omega\to 0} f(\omega)$. Your answer should depend on t. By graphing this function of t, you can see the path of the pitch (use a domain of $0 \le t \le 0.68$). Describe this pitch.

55. A water wave of length L meters in water of depth d meters has velocity v satisfying the equation

$$v^2 = \frac{4.9L}{\pi} \frac{e^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}}.$$

Treating L as a constant and thinking of v^2 as a function $f(d)$, use a linear approximation to show that $f(d) \approx 9.8d$ for small values of d. That is, for small depths the velocity of the wave is approximately $\sqrt{9.8d}$ and is independent of the wavelength L.

56. Planck's law states that the energy density of blackbody radiation of wavelength x is given by

$$f(x) = \frac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}.$$

Use the linear approximation in exercise 16 to show that $f(x) \approx 8\pi kT/x^4$, which is known as the Rayleigh-Jeans law.

57. In this exercise, we introduce **Taylor series** (explored in depth in Chapter 8). Start with the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Briefly explain why this means that for x close to 0, $\sin x \approx x$. Graph $y = \sin x$ and $y = x$ to see why this is true. If you look far enough away from $x = 0$, the graph of $y = \sin x$ eventually curves noticeably. We will find polynomials of higher order to match this curving. Show that $\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0$. This means that $\sin x - x \approx 0$ or (again) $\sin x \approx x$. Show that $\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$. This says that if x is close to 0, then $\sin x - x \approx -\frac{1}{6}x^3$ or $\sin x \approx x - \frac{1}{6}x^3$. Graph these two functions to see how well they match up. To continue, compute $\lim_{x\to 0} \frac{\sin x - (x - x^3/6)}{x^4}$ and $\lim_{x\to 0} \frac{\sin x - f(x)}{x^5}$ for the appropriate approximation $f(x)$. At this point, look at the pattern of terms you have (Hint: $6 = 3!$ and $120 = 5!$). Using this pattern, approximate $\sin x$ with an 11th-degree polynomial and graph the two functions.

3.2 NEWTON'S METHOD

We now return to the question of finding zeros of a function. In section 1.3, we introduced the method of bisections as a tedious, yet reliable, method of finding zeros of continuous functions. In this section, we explore a method which is usually much more efficient than bisections. We are again looking for values of x such that $f(x) = 0$. These values are called **roots** of the equation $f(x) = 0$ or **zeros** of the function f. If

$$f(x) = ax^2 + bx + c,$$

there is no challenge to doing this, since we have an explicit formula for the solution(s) (the quadratic formula). But, what if we want to find zeros of

$$f(x) = \tan x - x?$$

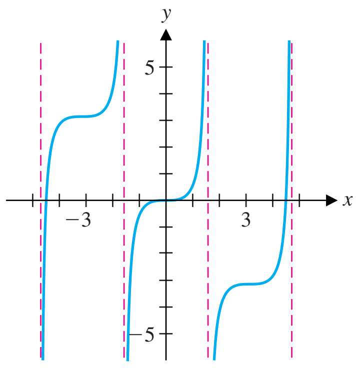


Figure 3.10

$y = \tan x - x$.

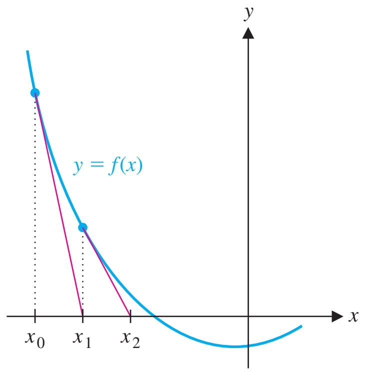


Figure 3.11

Newton's method.

This function is not algebraic and there are no formulas available for finding the zeros. Even so, we can clearly see zeros in Figure 3.10 (in fact, there are infinitely many of them). The question is, how are we to **find** them?

In general, if we wish to find approximate solutions to $f(x) = 0$, we first make a reasonable guess as to the location of a solution. We will call this an **initial guess,** denoted x_0. Once again, since the tangent line to $y = f(x)$ at $x = x_0$ tends to hug the curve, we can follow the tangent line to where it intersects the x-axis (see Figure 3.11).

Notice that this appears to provide an improved approximation to the zero. The equation of the tangent line to $y = f(x)$ at $x = x_0$ is given by the linear approximation at x_0 [see equation (1.2)],

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{2.1}$$

We denote the x-intercept of the tangent line by x_1 [found by setting $y = 0$ in (2.1)]. We then have

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

and, solving this for x_1, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If we repeat this process, using x_1 as our new guess, we should produce an improved approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

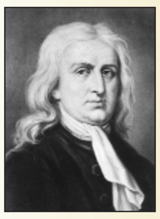
and so on (see Figure 3.11). In this way, we generate a sequence of **successive approximations** determined by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0, 1, 2, 3, \ldots.} \tag{2.2}$$

This procedure is called the **Newton-Raphson method,** or simply **Newton's method.** If Figure 3.11 is any indication, x_n should get closer and closer to a zero as n increases.

Newton's method is generally a very fast, accurate method for approximating the zeros of a function, as we illustrate with the following example.

HISTORICAL NOTES



Sir Isaac Newton (1642–1727) An English mathematician and scientist known as the co-inventor of calculus. In a 2-year period from 1665 to 1667, Newton made major discoveries in several areas of calculus, as well as optics and the law of gravitation. Newton's mathematical results were not published in a timely fashion. Instead, techniques such as Newton's method were quietly introduced as useful tools in his scientific papers. Newton's *Mathematical Principles of Natural Philosophy* is widely regarded as one of the greatest achievements of the human mind.

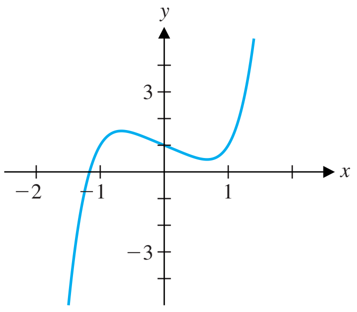


Figure 3.12
$y = x^5 - x + 1$.

Example 2.1 Using Newton's Method to Approximate a Zero

Find a zero of $f(x) = x^5 - x + 1$.

Solution From Figure 3.12, it appears as if the only zero of f is located between $x = -2$ and $x = -1$. We further observe that $f(-1) = 1 > 0$ and $f(-2) = -29 < 0$. Since f is continuous (all polynomials are continuous!), the Intermediate Value Theorem (Theorem 3.4 in section 1.3) says that f must have a zero on the interval $(-2, -1)$. Further, because the zero appears to be closer to $x = -1$, we make the initial guess $x_0 = -1$. Finally, $f'(x) = 5x^4 - 1$ and so, Newton's method gives us

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^5 - x_n + 1}{5x_n^4 - 1}, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Using the initial guess $x_0 = -1$, we get

$$\begin{aligned} x_1 &= -1 - \frac{(-1)^5 - (-1) + 1}{5(-1)^4 - 1} \\ &= -1 - \frac{1}{4} = -\frac{5}{4}. \end{aligned}$$

Likewise, from $x_1 = -\frac{5}{4}$, we get the improved approximation

$$\begin{aligned} x_2 &= -\frac{5}{4} - \frac{\left(-\frac{5}{4}\right)^5 - \left(-\frac{5}{4}\right) + 1}{5\left(-\frac{5}{4}\right)^4 - 1} \\ &= -1.178459394 \end{aligned}$$

and so on, we find that

$$x_3 = -1.167537384,$$

$$x_4 = -1.167304083$$

and

$$x_5 = -1.167303978 = x_6.$$

Since $x_5 = x_6$, we will make no further progress by calculating additional steps. As a final check on the accuracy of our approximation, we compute

$$f(x_6) \approx 3 \times 10^{-12}.$$

Since this is very close to zero, we say that $x_6 = -1.167303978$ is an **approximate zero** of f.

■

You can bring Newton's method to bear on a variety of approximation problems. As we illustrate in the following example, you may first need to rephrase the problem as a rootfinding problem.

Example 2.2 Using Newton's Method to Approximate a Cube Root

Use Newton's method to approximate $\sqrt[3]{7}$.

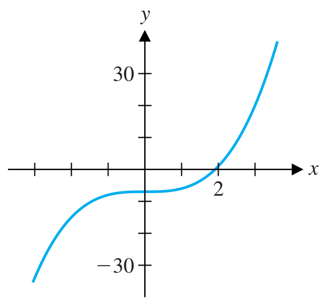


Figure 3.13

$y = x^3 - 7$.

Solution Recall that we can use a linear approximation for a problem like this. However, Newton's method will quickly provide us with an accurate approximation. Recall that Newton's method is used to solve equations of the form $f(x) = 0$. We can rewrite the current problem in this form, as follows. Suppose $x = \sqrt[3]{7}$. Then, $x^3 = 7$, which can be rewritten as

$$f(x) = x^3 - 7 = 0.$$

Here, $f'(x) = 3x^2$ and we obtain an initial guess from a graph of $y = f(x)$ (see Figure 3.13). Notice that there is a zero near $x = 2$ and so we take $x_0 = 2$. Newton's method then yields

$$x_1 = 2 - \frac{2^3 - 7}{3(2^2)} = \frac{23}{12} \approx 1.916666667.$$

Continuing this process, we have

$$x_2 \approx 1.912938458$$

and

$$x_3 \approx 1.912931183 \approx x_4.$$

Further,

$$f(x_4) \approx -5 \times 10^{-12}$$

and so, x_4 is an approximate zero of f. This also says that

$$\sqrt[3]{7} \approx 1.912931183.$$

Compare this with the value of $\sqrt[3]{7}$ produced by your calculator, to see how very accurate Newton's method was here.

Remark 2.1

Although it seemed to be very efficient in the last two examples, Newton's method does not always work. We urge you to make sure that the values coming from the method are getting progressively closer and closer together (zeroing-in, we hope, on the desired solution). Don't stop until you've reached the limits of accuracy of your computing device. Also, be sure to compute the value of the function at the suspected approximate zero. If the function value is not close to zero, do not accept the value as an approximate zero.

As we illustrate in the following example, Newton's method requires a good initial guess in order to find an accurate approximation.

Example 2.3 The Effect of a Bad Guess on Newton's Method

Use Newton's method to find an approximate zero of $f(x) = x^3 - 3x^2 + x - 1$.

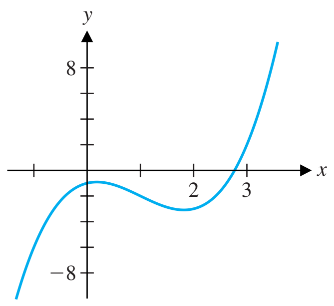


Figure 3.14

$y = x^3 - 3x^2 + x - 1$.

Solution From the graph in Figure 3.14, there appears to be a zero on the interval (2, 3). If you were to use the (not particularly good) initial guess $x_0 = 1$, you would get $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and so on. Try this for yourself. Newton's method is sensitive to the initial guess and you just made a bad initial guess. If you had instead started with the improved initial guess $x_0 = 2$, Newton's method would have quickly converged to the approximate zero 2.769292354. (Again, try this for yourself.)

As we saw in example 2.3, a poor initial guess may have disastrous consequences. However, simply picking a good initial guess will not guarantee the rapid convergence of Newton's method. For some functions, the convergence will be slow no matter how good your initial guess is. By slow convergence, we mean that it takes many iterations to see significant improvement in the approximation.

Example 2.4 Unusually Slow Convergence for Newton's Method

Use Newton's method with (a) $x_0 = -2$, (b) $x_0 = -1$ and (c) $x_0 = 0$ to try to locate the zero of $f(x) = \dfrac{(x-1)^2}{x^2+1}$.

Solution Of course, there's no mystery here: f has only one zero, located at $x = 1$. But, watch what happens if we try to use Newton's method with the specified guesses.

(a) If we take $x_0 = -2$ and apply Newton's method, we calculate the values in the accompanying table.

n	x_n
1	-9.5
2	-65.9
3	-2302
4	-2654301
5	-3.5×10^{12}
6	-3.1×10^{24}

Newton's method iterations for $x_0 = -2$.

Obviously, the Newton's method iterations are blowing up for the given initial guess. To see why, look at Figure 3.15, which shows the graphs of both $y = f(x)$ and the tangent line at $x = -2$. Notice that if you follow the tangent line to where it intersects the x-axis, you will be going *away* from the zero (far away). Since all of the tangent lines for $x \leq -2$ have positive slope [compute $f'(x)$ to see why this is true], each subsequent step takes you farther from the zero. (Draw your own graph showing several tangent lines to see why this is true.)

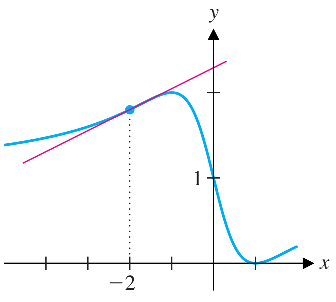


Figure 3.15

$y = \dfrac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -2$.

(b) If we use the improved initial guess $x_0 = -1$, note that we cannot even compute x_1. In this case, $f'(x_0) = 0$ and so, Newton's method fails. Notice that graphically, this means that the tangent line to $y = f(x)$ at $x = -1$ is horizontal (see Figure 3.16), so that the tangent line never intersects the x-axis.

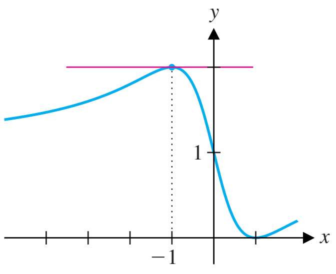


Figure 3.16

$y = \dfrac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -1$.

(c) With the even better initial guess $x_0 = 0$, we obtain the successive approximations in the following table.

n	x_n	n	x_n
1	0.5	7	0.9881719
2	0.70833	8	0.9940512
3	0.85653	9	0.9970168
4	0.912179	10	0.9985062
5	0.95425	11	0.9992525
6	0.976614	12	0.9996261

Newton's method iterations for $x_0 = 0$.

Finally, we happened upon an initial guess for which Newton's method converged to the root $x = 1$. What is unusual here is that the successive approximations shown in the table are converging to 1 much more slowly than in previous examples. By comparison, note that in example 2.1, the iterations stop changing at x_5. Here, x_5 is not particularly close to the desired zero of $f(x)$. In fact, in this example, x_{12} is not as close to the zero as x_5 is in example 2.1. We look further into this type of behavior in the exercises.