49. Starting with $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$, cancel sin to get $\lim_{x\to 0} \frac{3x}{2x}$, then cancel x's to get $\frac{3}{2}$. This answer is correct. Are either of the steps used valid? Use linear approximations to argue that the first step is likely to give a correct answer.

50. Evaluate $\lim_{x\to 0} \frac{\sin nx}{\sin mx}$ for nonzero constants n and m.

51. Evaluate $\lim_{x\to 0} \frac{e^{cx} - 1}{x}$ for any constant c.

52. Evaluate $\lim_{x\to 0} \frac{\tan cx - cx}{x^3}$ for any constant c.

53. In section 1.1, we briefly discussed the position of a baseball thrown with the unusual knuckleball pitch. The left/right position (in feet) of a ball thrown with spin rate ω and a particular grip at time t seconds is $f(\omega) = (2.5/\omega)t - (2.5/4\omega^2)\sin 4\omega t$. Treating t as a constant and ω as the variable (change to x if you like), show that $\lim_{\omega\to 0} f(\omega) = 0$ for any value of t. (Hint: Find a common denominator and use L'Hôpital's Rule.) Conclude that this pitch does not move left or right at all.

54. In this exercise, we look at a knuckleball thrown with a different grip than that of exercise 53. The left or right position (in feet) of a ball thrown with spin rate ω and this new grip at time t seconds is $f(\omega) = (2.5/4\omega^2) - (2.5/4\omega^2)\sin(4\omega t + \pi/2)$. Treating t as a constant and ω as the variable (change to x if you like), find $\lim_{\omega\to 0} f(\omega)$. Your answer should depend on t. By graphing this function of t, you can see the path of the pitch (use a domain of $0 \le t \le 0.68$). Describe this pitch.

55. A water wave of length L meters in water of depth d meters has velocity v satisfying the equation

$$v^2 = \frac{4.9L}{\pi} \frac{e^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}}.$$

Treating L as a constant and thinking of v^2 as a function $f(d)$, use a linear approximation to show that $f(d) \approx 9.8d$ for small values of d. That is, for small depths the velocity of the wave is approximately $\sqrt{9.8d}$ and is independent of the wavelength L.

56. Planck's law states that the energy density of blackbody radiation of wavelength x is given by

$$f(x) = \frac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}.$$

Use the linear approximation in exercise 16 to show that $f(x) \approx 8\pi kT/x^4$, which is known as the Rayleigh-Jeans law.

57. In this exercise, we introduce **Taylor series** (explored in depth in Chapter 8). Start with the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Briefly explain why this means that for x close to 0, $\sin x \approx x$. Graph $y = \sin x$ and $y = x$ to see why this is true. If you look far enough away from $x = 0$, the graph of $y = \sin x$ eventually curves noticeably. We will find polynomials of higher order to match this curving. Show that $\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0$. This means that $\sin x - x \approx 0$ or (again) $\sin x \approx x$. Show that $\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$. This says that if x is close to 0, then $\sin x - x \approx -\frac{1}{6}x^3$ or $\sin x \approx x - \frac{1}{6}x^3$. Graph these two functions to see how well they match up. To continue, compute $\lim_{x\to 0} \frac{\sin x - (x - x^3/6)}{x^4}$ and $\lim_{x\to 0} \frac{\sin x - f(x)}{x^5}$ for the appropriate approximation $f(x)$. At this point, look at the pattern of terms you have (Hint: $6 = 3!$ and $120 = 5!$). Using this pattern, approximate $\sin x$ with an 11th-degree polynomial and graph the two functions.

3.2 NEWTON'S METHOD

We now return to the question of finding zeros of a function. In section 1.3, we introduced the method of bisections as a tedious, yet reliable, method of finding zeros of continuous functions. In this section, we explore a method which is usually much more efficient than bisections. We are again looking for values of x such that $f(x) = 0$. These values are called **roots** of the equation $f(x) = 0$ or **zeros** of the function f. If

$$f(x) = ax^2 + bx + c,$$

there is no challenge to doing this, since we have an explicit formula for the solution(s) (the quadratic formula). But, what if we want to find zeros of

$$f(x) = \tan x - x?$$

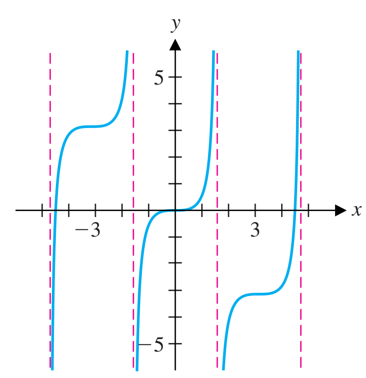


Figure 3.10
$y = \tan x - x$.

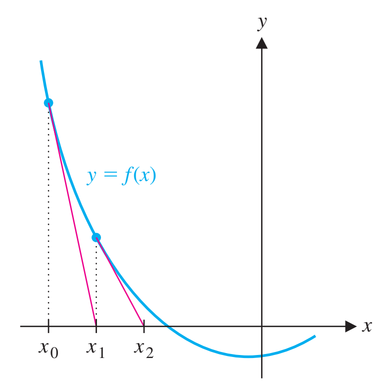


Figure 3.11
Newton's method.

This function is not algebraic and there are no formulas available for finding the zeros. Even so, we can clearly see zeros in Figure 3.10 (in fact, there are infinitely many of them). The question is, how are we to **find** them?

In general, if we wish to find approximate solutions to $f(x) = 0$, we first make a reasonable guess as to the location of a solution. We will call this an **initial guess,** denoted x_0. Once again, since the tangent line to $y = f(x)$ at $x = x_0$ tends to hug the curve, we can follow the tangent line to where it intersects the x-axis (see Figure 3.11).

Notice that this appears to provide an improved approximation to the zero. The equation of the tangent line to $y = f(x)$ at $x = x_0$ is given by the linear approximation at x_0 [see equation (1.2)],

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{2.1}$$

We denote the x-intercept of the tangent line by x_1 [found by setting $y = 0$ in (2.1)]. We then have

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

and, solving this for x_1, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If we repeat this process, using x_1 as our new guess, we should produce an improved approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

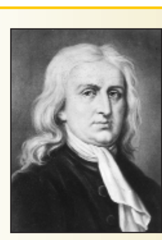
and so on (see Figure 3.11). In this way, we generate a sequence of **successive approximations** determined by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0, 1, 2, 3, \ldots.} \tag{2.2}$$

This procedure is called the **Newton-Raphson method,** or simply **Newton's method.** If Figure 3.11 is any indication, x_n should get closer and closer to a zero as n increases.

Newton's method is generally a very fast, accurate method for approximating the zeros of a function, as we illustrate with the following example.

Historical Notes



Sir Isaac Newton (1642–1727) An English mathematician and scientist known as the co-inventor of calculus. In a 2-year period from 1665 to 1667, Newton made major discoveries in several areas of calculus, as well as optics and the law of gravitation. Newton's mathematical results were not published in a timely fashion. Instead, techniques such as Newton's method were quietly introduced as useful tools in his scientific papers. Newton's *Mathematical Principles of Natural Philosophy* is widely regarded as one of the greatest achievements of the human mind.

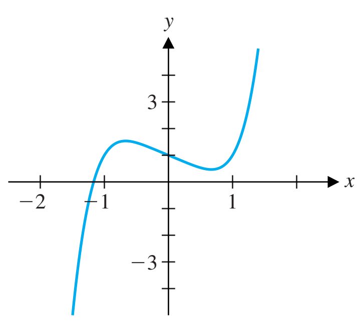


Figure 3.12
$y = x^5 - x + 1.$

Example 2.1 **Using Newton's Method to Approximate a Zero**

Find a zero of $f(x) = x^5 - x + 1$.

Solution From Figure 3.12, it appears as if the only zero of f is located between $x = -2$ and $x = -1$. We further observe that $f(-1) = 1 > 0$ and $f(-2) = -29 < 0$. Since f is continuous (all polynomials are continuous!), the Intermediate Value Theorem (Theorem 3.4 in section 1.3) says that f must have a zero on the interval $(-2, -1)$. Further, because the zero appears to be closer to $x = -1$, we make the initial guess $x_0 = -1$. Finally, $f'(x) = 5x^4 - 1$ and so, Newton's method gives us

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^5 - x_n + 1}{5x_n^4 - 1}, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Using the initial guess $x_0 = -1$, we get

$$\begin{aligned} x_1 &= -1 - \frac{(-1)^5 - (-1) + 1}{5(-1)^4 - 1} \\ &= -1 - \frac{1}{4} = -\frac{5}{4}. \end{aligned}$$

Likewise, from $x_1 = -\frac{5}{4}$, we get the improved approximation

$$\begin{aligned} x_2 &= -\frac{5}{4} - \frac{\left(-\frac{5}{4}\right)^5 - \left(-\frac{5}{4}\right) + 1}{5\left(-\frac{5}{4}\right)^4 - 1} \\ &= -1.178459394 \end{aligned}$$

and so on, we find that

$$\begin{aligned} x_3 &= -1.167537384, \\ x_4 &= -1.167304083 \end{aligned}$$

and

$$x_5 = -1.167303978 = x_6.$$

Since $x_5 = x_6$, we will make no further progress by calculating additional steps. As a final check on the accuracy of our approximation, we compute

$$f(x_6) \approx 3 \times 10^{-12}.$$

Since this is very close to zero, we say that $x_6 = -1.167303978$ is an **approximate zero** of f.

You can bring Newton's method to bear on a variety of approximation problems. As we illustrate in the following example, you may first need to rephrase the problem as a rootfinding problem.

Example 2.2 Using Newton's Method to Approximate a Cube Root

Use Newton's method to approximate $\sqrt[3]{7}$.

Solution Recall that we can use a linear approximation for a problem like this. However, Newton's method will quickly provide us with an accurate approximation. Recall that Newton's method is used to solve equations of the form $f(x) = 0$. We can rewrite the current problem in this form, as follows. Suppose $x = \sqrt[3]{7}$. Then, $x^3 = 7$, which can be rewritten as

$$f(x) = x^3 - 7 = 0.$$

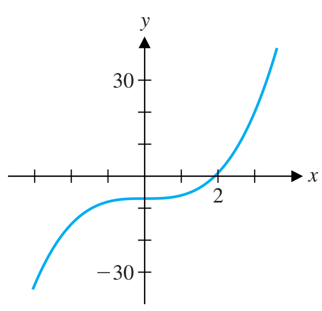


Figure 3.13
$y = x^3 - 7$.

Here, $f'(x) = 3x^2$ and we obtain an initial guess from a graph of $y = f(x)$ (see Figure 3.13). Notice that there is a zero near $x = 2$ and so we take $x_0 = 2$. Newton's method then yields

$$x_1 = 2 - \frac{2^3 - 7}{3(2^2)} = \frac{23}{12} \approx 1.916666667.$$

Continuing this process, we have

$$x_2 \approx 1.912938458$$

and

$$x_3 \approx 1.912931183 \approx x_4.$$

Further,

$$f(x_4) \approx -5 \times 10^{-12}$$

and so, x_4 is an approximate zero of f. This also says that

$$\sqrt[3]{7} \approx 1.912931183.$$

Compare this with the value of $\sqrt[3]{7}$ produced by your calculator, to see how very accurate Newton's method was here.

Remark 2.1

Although it seemed to be very efficient in the last two examples, Newton's method does not always work. We urge you to make sure that the values coming from the method are getting progressively closer and closer together (zeroing-in, we hope, on the desired solution). Don't stop until you've reached the limits of accuracy of your computing device. Also, be sure to compute the value of the function at the suspected approximate zero. If the function value is not close to zero, do not accept the value as an approximate zero.

As we illustrate in the following example, Newton's method requires a good initial guess in order to find an accurate approximation.

Example 2.3 The Effect of a Bad Guess on Newton's Method

Use Newton's method to find an approximate zero of $f(x) = x^3 - 3x^2 + x - 1$.

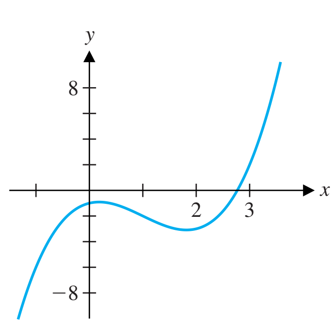


Figure 3.14
$y = x^3 - 3x^2 + x - 1$.

Solution From the graph in Figure 3.14, there appears to be a zero on the interval (2, 3). If you were to use the (not particularly good) initial guess $x_0 = 1$, you would get $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and so on. Try this for yourself. Newton's method is sensitive to the initial guess and you just made a bad initial guess. If you had instead started with the improved initial guess $x_0 = 2$, Newton's method would have quickly converged to the approximate zero 2.769292354. (Again, try this for yourself.)

n	x_n
1	−9.5
2	−65.9
3	−2302
4	−2654301
5	-3.5×10^{12}
6	-3.1×10^{24}

Newton's method iterations for $x_0 = -2$.

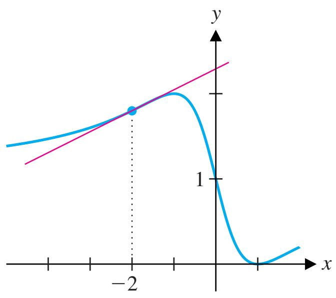


Figure 3.15
$y = \frac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -2$.

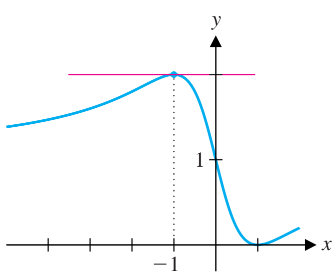


Figure 3.16
$y = \frac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -1$.

As we saw in example 2.3, a poor initial guess may have disastrous consequences. However, simply picking a good initial guess will not guarantee the rapid convergence of Newton's method. For some functions, the convergence will be slow no matter how good your initial guess is. By slow convergence, we mean that it takes many iterations to see significant improvement in the approximation.

Example 2.4 Unusually Slow Convergence for Newton's Method

Use Newton's method with (a) $x_0 = -2$, (b) $x_0 = -1$ and (c) $x_0 = 0$ to try to locate the zero of $f(x) = \frac{(x-1)^2}{x^2+1}$.

Solution Of course, there's no mystery here: f has only one zero, located at $x = 1$. But, watch what happens if we try to use Newton's method with the specified guesses.

(a) If we take $x_0 = -2$ and apply Newton's method, we calculate the values in the accompanying table.

Obviously, the Newton's method iterations are blowing up for the given initial guess. To see why, look at Figure 3.15, which shows the graphs of both $y = f(x)$ and the tangent line at $x = -2$. Notice that if you follow the tangent line to where it intersects the x-axis, you will be going *away* from the zero (far away). Since all of the tangent lines for $x \le -2$ have positive slope [compute $f'(x)$ to see why this is true], each subsequent step takes you farther from the zero. (Draw your own graph showing several tangent lines to see why this is true.)

(b) If we use the improved initial guess $x_0 = -1$, note that we cannot even compute x_1. In this case, $f'(x_0) = 0$ and so, Newton's method fails. Notice that graphically, this means that the tangent line to $y = f(x)$ at $x = -1$ is horizontal (see Figure 3.16), so that the tangent line never intersects the x-axis.

(c) With the even better initial guess $x_0 = 0$, we obtain the successive approximations in the following table.

n	x_n	n	x_n
1	0.5	7	0.9881719
2	0.70833	8	0.9940512
3	0.85653	9	0.9970168
4	0.912179	10	0.9985062
5	0.95425	11	0.9992525
6	0.976614	12	0.9996261

Newton's method iterations for $x_0 = 0$.

Finally, we happened upon an initial guess for which Newton's method converged to the root $x = 1$. What is unusual here is that the successive approximations shown in the table are converging to 1 much more slowly than in previous examples. By comparison, note that in example 2.1, the iterations stop changing at x_5. Here, x_5 is not particularly close to the desired zero of $f(x)$. In fact, in this example, x_{12} is not as close to the zero as x_5 is in example 2.1. We look further into this type of behavior in the exercises.

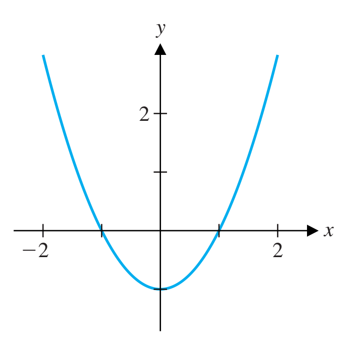
Despite the small problems experienced in examples 2.3 and 2.4, you should view Newton's method as a generally reliable and efficient method of locating zeros approximately. Just use a bit of caution and common sense. If the successive approximations are converging to some value that does not appear consistent with the graph, then you need to scrutinize your results more carefully and perhaps try some other initial guesses.

EXERCISES 3.2

1. In example 2.2, we mentioned that you might think of using a linear approximation instead of Newton's method. Explain the relationship between a linear approximation to $\sqrt[3]{7}$ and a Newton's method approximation to $\sqrt[3]{7}$. (Hint: Compare the first step of Newton's method to a linear approximation.)

2. Explain why Newton's method fails computationally if $f'(x_0) = 0$. In terms of tangent lines intersecting the x-axis, explain why having $f'(x_0) = 0$ is a problem.

3. Given the graph of $y = f(x)$, draw in the tangent lines used in Newton's method to determine x_1 and x_2 after starting at $x_0 = 2$. Which of the zeros will Newton's method converge to?



4. Repeat exercise 3 with $x_0 = -2$ and $x_0 = 0.4$.

5. What would happen to Newton's method in exercise 3 if you had a starting value of $x_0 = 0$?

6. Consider the use of Newton's method in exercise 3 with $x_0 = 0.2$ and $x_0 = 10$. Obviously, $x_0 = 0.2$ is much closer to a zero of the function, but which initial guess would work better in Newton's method? Explain.

In exercises 7–10, use Newton's method with the given x_0 to (a) compute x_1 and x_2 by hand and (b) use a computer or calculator to find the root to at least five-digit accuracy.

7. $x^3 + 3x^2 - 1 = 0,\ x_0 = 1$

8. $x^3 + 4x^2 - x - 1 = 0,\ x_0 = -1$

9. $x^4 - 3x^2 + 1 = 0,\ x_0 = 1$

10. $x^4 - 3x^2 + 1 = 0,\ x_0 = -1$

In exercises 11–20, use Newton's method to find an approximate root (at least six-digit accuracy). Sketch the graph and explain how you determined your initial guess.

11. $x^3 + 4x^2 - 3x + 1 = 0$
12. $x^4 - 4x^3 + x^2 - 1 = 0$
13. $x^5 + 3x^3 + x - 1 = 0$
14. $x^3 + 2x + 1 = 0$
15. $\cos x - x = 0$
16. $x - e^{-x} = 0$
17. $\sin x = x^2 - 1$
18. $\cos x^2 = x$
19. $e^x = -x$
20. $e^{-x} = \sqrt{x}$

In exercises 21–26, Newton's method fails. Explain why the method fails and, if possible, find a root by correcting the problem.

21. $4x^3 - 7x^2 + 1 = 0,\ x_0 = 0$

22. $4x^3 - 7x^2 + 1 = 0,\ x_0 = 1$

23. $x^2 + 1 = 0,\ x_0 = 0$

24. $x^2 + 1 = 0,\ x_0 = 1$

25. $\dfrac{4x^2 - 8x + 1}{4x^2 - 3x - 7} = 0,\ x_0 = -1$

26. $\left(\dfrac{x+1}{x-2}\right)^{1/3} = 0,\ x_0 = 0.5$

27. Show that Newton's method applied to $x^2 - c = 0$ (where $c > 0$ is some constant) produces the iterative scheme $x_{n+1} = \frac{1}{2}(x_n + c/x_n)$ for approximating $\sqrt{c}$. This scheme has been known for over 2000 years. To understand why it works, suppose that your initial guess (x_0) at $\sqrt{c}$ is a little too small. How would c/x_0 compare to $\sqrt{c}$? Explain why the average of x_0 and c/x_0 would give a better approximation to $\sqrt{c}$.

28. Show that Newton's method applied to $x^n - c = 0$ (where n and c are positive constants) produces the iterative scheme $x_{n+1} = \frac{1}{n}[(n-1)x_n + cx_n^{1-n}]$ for approximating $\sqrt[n]{c}$.

In exercises 29–36, use Newton's method [state the function $f(x)$ you use] to estimate the given number. (Hint: See exercises 27–28.)

29. $\sqrt{11}$ 30. $\sqrt{23}$ 31. $\sqrt[3]{11}$ 32. $\sqrt[3]{23}$

33. $\sqrt[4]{24}$ 34. $\sqrt[5]{33}$ 35. $\sqrt[4.4]{24}$ 36. $\sqrt[4.6]{24}$

37. Suppose that a species reproduces as follows: with probability p_0, an organism has no offspring; with probability p_1, an organism has one offspring, with probability p_2, an organism has two offspring, and so on. The probability that the species goes extinct is given by the smallest nonnegative solution of the equation $p_0 + p_1 x + p_2 x^2 + \cdots = x$ (see Sigmund's *Games of Life*). Find the positive solutions of the equations $0.1 + 0.2x + 0.3x^2 + 0.4x^3 = x$ and $0.4 + 0.3x + 0.2x^2 + 0.1x^3 = x$. Explain in terms of species going extinct why the first equation has a smaller solution than the second.

38. For the extinction problem in exercise 37, show algebraically that if $p_0 = 0$, the probability of extinction is 0. Explain this result in terms of species reproduction. Show that a species with $p_0 = 0.35$, $p_1 = 0.4$ and $p_2 = 0.25$ (all other p_n's are 0) goes extinct with certainty (probability 1). This will be explored more in the exercises for section 3.5.

39. The spruce budworm is an enemy of the balsam fir tree. In one model of the interaction between these organisms, possible long-term populations of the budworm are solutions of the equation $r(1 - x/k) = x/(1 + x^2)$ for positive constants r and k (see Murray's *Mathematical Biology*). Find all positive solutions of the equation with $r = 0.5$ and $k = 7$.

40. Repeat exercise 39 with $r = 0.5$ and $k = 7.5$. For a small change in the environmental constant k (from 7 to 7.5), how did the solution change from exercise 39 to exercise 40? The largest solution corresponds to an "infestation" of the spruce budworm.

41. Newton's theory of gravitation states that the weight of a person at elevation x feet above sea level is $W(x) = PR^2/(R + x)^2$, where P is the person's weight at sea level and R is the radius of the earth (approximately 20,900,000 feet). Find the linear approximation of $W(x)$ at $x = 0$. Use the linear approximation to estimate the elevation required to reduce the weight of a 120-pound person by 1%.

42. One important aspect of Einstein's theory of relativity is that mass is not constant. For a person with mass m_0 at rest, the mass will equal $m = m_0/\sqrt{1 - v^2/c^2}$ at velocity v (here, c is the speed of light). Thinking of m as a function of v, find the linear approximation of $m(v)$ at $v = 0$. Use the linear approximation to show that mass is essentially constant for small velocities.

43. In the figure, a beam is clamped at its left end and simply supported at its right end. A force P (called an **axial load**) is applied at the right end.

If enough force is applied, the beam will buckle. Obviously, it is important for engineers to be able to compute this **buckling load.** For certain types of beams, the buckling load is the smallest positive solution of the equation $\tan\sqrt{x} = \sqrt{x}$. The shape of the buckled beam is given by $y = \sqrt{L} - \sqrt{L}x - \sqrt{L}\cos\sqrt{L}x + \sin\sqrt{L}x$, where L is the buckling load. Find L and the shape of the buckled beam.

44. The spectral radiancy S of an ideal radiator at constant temperature can be thought of as function $S(f)$ of the radiant frequency f. The function $S(f)$ attains its maximum when $3e^{-cf} + cf - 3 = 0$ for the constant $c = 10^{-13}$. Use Newton's method to approximate the solution.

In exercises 45–49, we explore the convergence of Newton's method for $f(x) = x^3 - 3x^2 + 2x$.

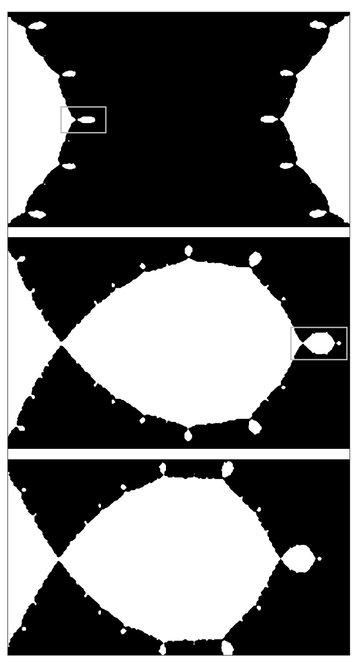
45. Show that the zeros of f are $x = 0$, $x = 1$ and $x = 2$.

46. Determine which of the three zeros Newton's method iterates converge to for (a) $x_0 = 0.1$, (b) $x_0 = 1.1$ and (c) $x_0 = 2.1$.

47. Determine which of the three zeros Newton's method iterates converge to for (a) $x_0 = 0.54$, (b) $x_0 = 0.55$ and (c) $x_0 = 0.56$.

48. The results of exercise 46 should make sense, but exercise 47 is probably surprising. Compute slopes of f at each of the starting points in exercises 46 and 47 and try to explain graphically why the results in exercise 47 are confusing.

49. In this exercise, you will extend the work of exercises 45–48. First, a definition: the **basin of attraction** of a zero is the set of starting values x_0 for which Newton's method iterates converge to the zero. As exercises 45–48 indicate, the basin boundaries are more complicated than you might expect. For example, you have seen that the interval [0.54, 0.56] contains points in all three basins of attraction. Show that the same is true of the interval [0.552, 0.553]. The picture gets even more interesting when you use complex numbers. These are numbers of the form $a + bi$ where $i = \sqrt{-1}$. The remainder of the exercise requires a CAS or calculator that is programmable and performs calculations with complex numbers. First, try Newton's method with starting point $x_0 = 1 + i$. The formula is exactly the same! Use your computer to show that $x_1 = x_0 - \dfrac{x_0^3 - 3x_0^2 + 2x_0}{3x_0^2 - 6x_0 + 2} = 1 + \frac{1}{2}i$. Then verify



that $x_2 = 1 + \frac{1}{7}i$ and $x_3 = 1 + \frac{1}{182}i$. It certainly appears that the iterates are converging to the zero $x = 1$. Now, for some programming: set up a double loop with the parameter a running from 0 to 2 in steps of 0.02 and b running from -1 to 1 in steps of 0.02. Within the double loop, set $x_0 = a + bi$ and compute 10 Newton's method iterates. If x_{10} is close to 0, say $|x_{10} - 0| < 0.1$, then we can conjecture that the iterates converge to 0. (Note: For complex numbers, $|a + bi| = \sqrt{a^2 + b^2}$.) Color the pixel at the point (a, b) black if the iterates converge to 0 and white if not. You can change the ranges of a and b and the step size to "zoom in" on interesting regions. The pictures to the left show the basin of attraction (in black) for $x = 1$. In the first figure, we display the region with $-1.5 \le x \le 3.5$. In the second figure, we have zoomed in to the portion for $0.26 \le x \le 0.56$. The third shows an even tighter zoom: $0.5 \le x \le 0.555$.

50. Another important question involving Newton's method is how fast it converges to a given zero. Intuitively, we can distinguish between the rate of convergence for $f(x) = x^2 - 1$ (with $x_0 = 1.1$) and that for $g(x) = x^2 - 2x + 1$ (with $x_0 = 1.1$). But how can we measure this? One method is to take successive approximations x_{n-1} and x_n and compute the difference $\Delta_n = x_n - x_{n-1}$. To discover the importance of this quantity, run Newton's method with $x_0 = 1.5$ and then compute the ratios Δ_3/Δ_2, Δ_4/Δ_3, Δ_5/Δ_4 and so on, for each of the following functions

$$F_1(x) = (x-1)(x+2)^3 = x^4 + 5x^3 + 6x^2 - 4x - 8,$$
$$F_2(x) = (x-1)^2(x+2)^2 = x^4 + 2x^3 - 3x^2 - 4x + 4,$$
$$F_3(x) = (x-1)^3(x+2) = x^4 - x^3 - 3x^2 + 5x - 2,$$
$$F_4(x) = (x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1.$$

In each case, conjecture a value for the limit $r = \lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n}$. If the limit exists and is nonzero, we say that Newton's method **converges linearly.** How does r relate to your intuitive sense of how fast the method converges? For $f(x) = (x-1)^4$, we say that the zero $x = 1$ has **multiplicity** 4. For $f(x) = (x-1)^3(x+2)$, $x = 1$ has multiplicity 3, and so on. How does r relate to the multiplicity of the zero? Based on this analysis, why did Newton's method converge faster for $f(x) = x^2 - 1$ than for $g(x) = x^2 - 2x + 1$? Finally, use Newton's method to compute the rate r and hypothesize the multiplicity of the zero $x = 0$ for $f(x) = x \sin x$ and $g(x) = x \sin x^2$.

3.3 MAXIMUM AND MINIMUM VALUES

It seems that no matter where we turn today, we hear about the need to maximize this or minimize that. In order to remain competitive in a global economy, businesses need to minimize waste and maximize the return on their investment. Managers for massively complex projects like the International Space Station must constantly readjust their programs to squeeze the most out of dwindling resources. In the extremely competitive personal computer industry, companies must continually evaluate how low they can afford to set their prices and still earn a profit adequate to survive. With this backdrop, it should be apparent that one of the main thrusts of our increasingly mathematical society is to use mathematical methods to maximize and minimize various quantities of interest. In this section, we investigate the notion of maximum and minimum from a purely mathematical standpoint. In section 3.7, we examine how to apply these notions to problems of an applied nature.

First, we give careful mathematical definitions of some familiar terms.