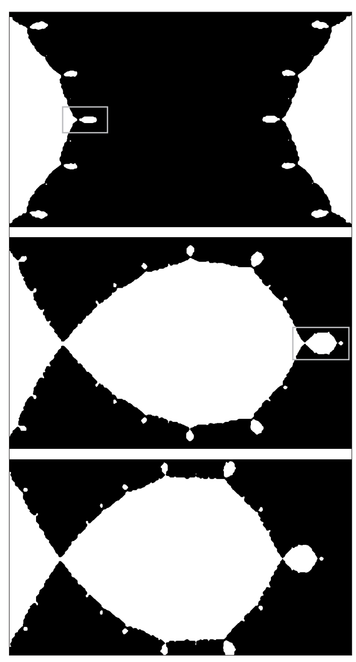

that $x_2 = 1 + \frac{1}{7}i$ and $x_3 = 1 + \frac{1}{182}i$. It certainly appears that the iterates are converging to the zero $x = 1$. Now, for some programming: set up a double loop with the parameter a running from 0 to 2 in steps of 0.02 and b running from -1 to 1 in steps of 0.02. Within the double loop, set $x_0 = a + bi$ and compute 10 Newton's method iterates. If x_{10} is close to 0, say $|x_{10} - 0| < 0.1$, then we can conjecture that the iterates converge to 0. (Note: For complex numbers, $|a + bi| = \sqrt{a^2 + b^2}$.) Color the pixel at the point (a, b) black if the iterates converge to 0 and white if not. You can change the ranges of a and b and the step size to "zoom in" on interesting regions. The pictures to the left show the basin of attraction (in black) for $x = 1$. In the first figure, we display the region with $-1.5 \le x \le 3.5$. In the second figure, we have zoomed in to the portion for $0.26 \le x \le 0.56$. The third shows an even tighter zoom: $0.5 \le x \le 0.555$.

50. Another important question involving Newton's method is how fast it converges to a given zero. Intuitively, we can distinguish between the rate of convergence for $f(x) = x^2 - 1$ (with $x_0 = 1.1$) and that for $g(x) = x^2 - 2x + 1$ (with $x_0 = 1.1$). But how can we measure this? One method is to take successive approximations x_{n-1} and x_n and compute the difference $\Delta_n = x_n - x_{n-1}$. To discover the importance of this quantity, run Newton's method with $x_0 = 1.5$ and then compute the ratios Δ_3/Δ_2, Δ_4/Δ_3, Δ_5/Δ_4 and so on, for each of the following functions

$$F_1(x) = (x-1)(x+2)^3 = x^4 + 5x^3 + 6x^2 - 4x - 8,$$
$$F_2(x) = (x-1)^2(x+2)^2 = x^4 + 2x^3 - 3x^2 - 4x + 4,$$
$$F_3(x) = (x-1)^3(x+2) = x^4 - x^3 - 3x^2 + 5x - 2,$$
$$F_4(x) = (x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1.$$

In each case, conjecture a value for the limit $r = \lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n}$. If the limit exists and is nonzero, we say that Newton's method **converges linearly.** How does r relate to your intuitive sense of how fast the method converges? For $f(x) = (x-1)^4$, we say that the zero $x = 1$ has **multiplicity** 4. For $f(x) = (x-1)^3(x+2)$, $x = 1$ has multiplicity 3, and so on. How does r relate to the multiplicity of the zero? Based on this analysis, why did Newton's method converge faster for $f(x) = x^2 - 1$ than for $g(x) = x^2 - 2x + 1$? Finally, use Newton's method to compute the rate r and hypothesize the multiplicity of the zero $x = 0$ for $f(x) = x \sin x$ and $g(x) = x \sin x^2$.

3.3 MAXIMUM AND MINIMUM VALUES

It seems that no matter where we turn today, we hear about the need to maximize this or minimize that. In order to remain competitive in a global economy, businesses need to minimize waste and maximize the return on their investment. Managers for massively complex projects like the International Space Station must constantly readjust their programs to squeeze the most out of dwindling resources. In the extremely competitive personal computer industry, companies must continually evaluate how low they can afford to set their prices and still earn a profit adequate to survive. With this backdrop, it should be apparent that one of the main thrusts of our increasingly mathematical society is to use mathematical methods to maximize and minimize various quantities of interest. In this section, we investigate the notion of maximum and minimum from a purely mathematical standpoint. In section 3.7, we examine how to apply these notions to problems of an applied nature.

First, we give careful mathematical definitions of some familiar terms.

Definition 3.1

For a function f defined on a set S of real numbers and a number $c \in S$,

(i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$ and

(ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an **absolute extremum.** If a function has more than one extremum, we refer to these as **extrema** (the plural form of extremum).

The first question you might ask is whether every function has an absolute maximum and an absolute minimum. The answer is no, as we can see from Figures 3.17a and 3.17b.

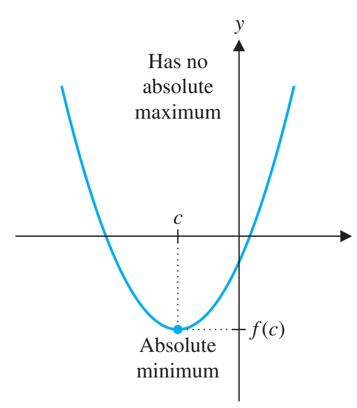

Figure 3.17a

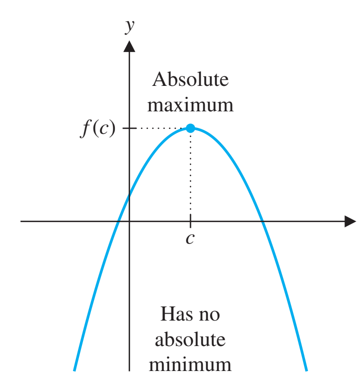

Figure 3.17b

Example 3.1 Absolute Maximum and Minimum Values

(a) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-\infty, \infty)$. (b) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-3, 3)$. (c) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $[-3, 3]$.

Solution (a) From the graph in Figure 3.18, notice that f has an absolute minimum value of $f(0) = -9$, but has no absolute maximum value.

(b) From the graph in Figure 3.19a, we see that f has an absolute minimum value of $f(0) = -9$, but still has no absolute maximum value. Your initial reaction might be to say that f has an absolute maximum of 0, but $f(x) \neq 0$ for any $x \in (-3, 3)$, since this is an open interval and hence, does not include the endpoints -3 and 3.

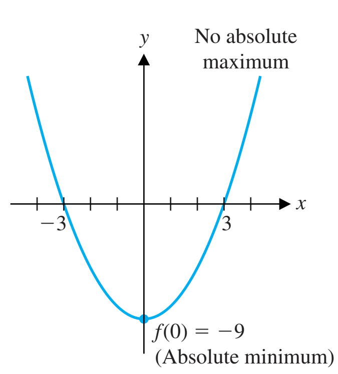

Figure 3.18

$y = x^2 - 9$ on $(-\infty, \infty)$.

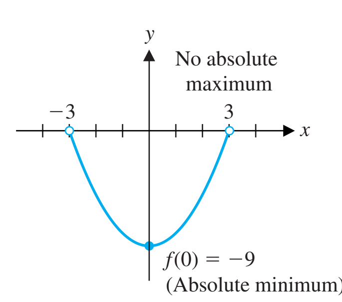

Figure 3.19a

$y = x^2 - 9$ $(-3, 3)$.

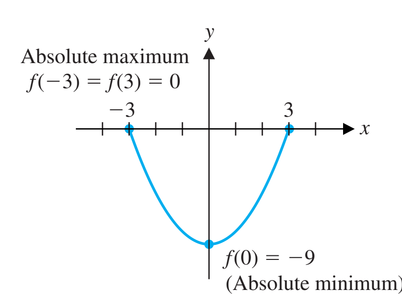

Figure 3.19b

$y = x^2 - 9$ $[-3, 3]$.

(c) In this case, the endpoints 3 and −3 are in the interval [−3, 3]. Here, f assumes its absolute maximum at two points: $f(3) = f(-3) = 0$ (see Figure 3.19b).

From example 3.1, we see that even nice, continuous functions may fail to have absolute extrema, depending on the interval on which we're looking. In example 3.1, the function failed to have an absolute maximum, except on the closed, bounded interval, [−3, 3]. This provides some clues, but the question remains as to when a function is **guaranteed** to have an absolute maximum and an absolute minimum on a given interval. The following example provides one more piece of the puzzle.

Example 3.2 A Function with No Absolute Maximum or Minimum

Locate any absolute extrema of $f(x) = 1/x$, on the interval [−3, 3].

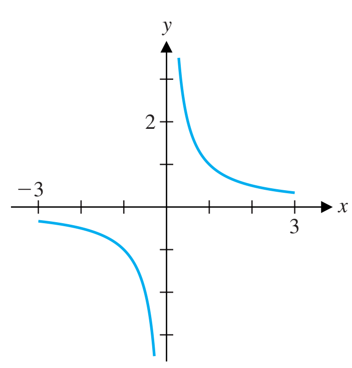

Figure 3.20
$y = 1/x$.

Solution From the graph in Figure 3.20, f clearly fails to have either an absolute maximum or an absolute minimum on [−3, 3]. The following table of values for $f(x)$ for x close to 0 suggests the same conclusion.

x	$1/x$	x	$1/x$
1	1	−1	−1
0.1	10	−0.1	−10
0.01	100	−0.01	−100
0.001	1,000	−0.001	−1,000
0.0001	10,000	−0.0001	−10,000
0.00001	100,000	−0.00001	−100,000
0.000001	1,000,000	−0.000001	−1,000,000

The most obvious difference between the functions in examples 3.1 and 3.2 is that $f(x) = 1/x$ is discontinuous at a point in the interval [−3, 3]. We offer the following theorem without proof.

Theorem 3.1 (Extreme Value Theorem)
A continuous function f defined on a **closed, bounded** interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.

While you do not need to have a continuous function or a closed interval to have an absolute extremum, the theorem says that continuous functions are **guaranteed** to have an absolute maximum and an absolute minimum on a closed, bounded interval.

In the following example, we return to the function from example 3.2, but look on a different interval.

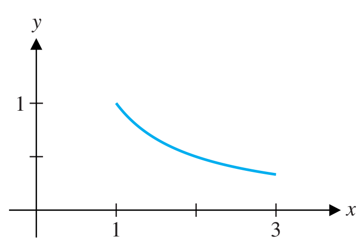

Figure 3.21
$y = 1/x$ on $[1, 3]$.

Example 3.3 Finding Absolute Extrema of a Continuous Function

Find the absolute extrema of $f(x) = 1/x$ on the interval $[1, 3]$.

Solution Notice that on the interval $[1, 3]$, f is continuous. Consequently, the Extreme Value Theorem guarantees that f has both an absolute maximum and an absolute minimum on $[1, 3]$. Judging from the graph in Figure 3.21, it appears that $f(x)$ reaches its maximum at $x = 1$ and its minimum at $x = 3$.

Our objective is to determine how to locate the absolute extrema of a given function. Before we do this, we need to consider one additional type of extremum.

Definition 3.2

(i) $f(c)$ is a **local maximum** of f if $f(c) \geq f(x)$ for all x in some **open** interval containing c.

(ii) $f(c)$ is a **local minimum** of f if $f(c) \leq f(x)$ for all x in some **open** interval containing c.

In either case, we call $f(c)$ a **local extremum** of f.

Local maxima and minima (the plural forms of maximum and minimum, respectively) are sometimes referred to as **relative** maxima and minima, respectively. In Figure 3.22, we see a function with several local extrema.

You should notice from Figure 3.22 that each local extremum seems to occur either at a point where the tangent line is horizontal [i.e., $f'(x) = 0$], at a point where the tangent line is vertical [where $f'(x)$ is undefined] or at a corner [again, where $f'(x)$ is undefined]. We can see this behavior quite clearly in the following two examples.

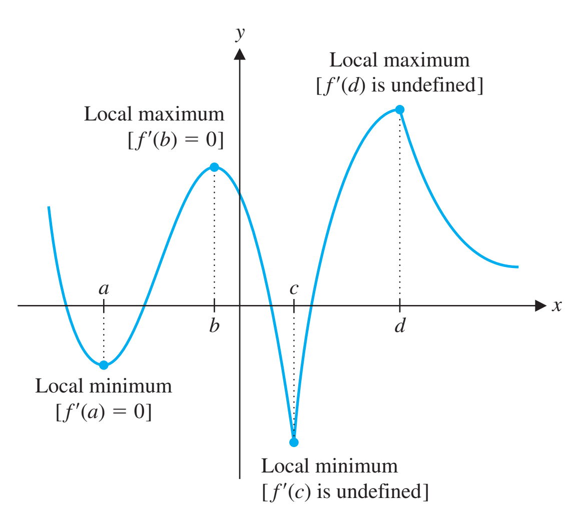

Figure 3.22
Local extrema.

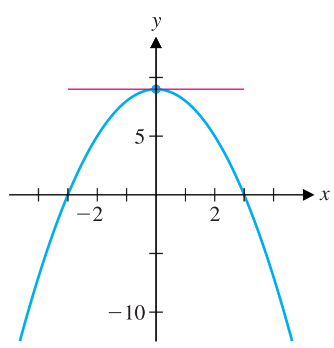

Figure 3.23
$y = 9 - x^2$ and the tangent line at $x = 0$.

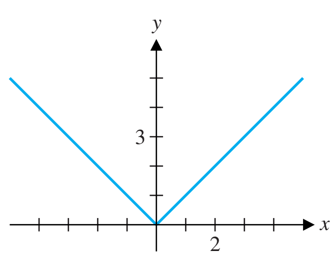

Figure 3.24
$y = |x|$.

Historical Notes

Pierre de Fermat (1601–1665) A French mathematician who discovered many important results, including the theorem named for him. Fermat was a lawyer and member of the Toulouse supreme court, with mathematics as a hobby. The "Prince of Amateurs" left an unusual legacy by writing in the margin of a book that he had discovered a wonderful proof of a clever result, but that the margin of the book was too small to hold the proof. Fermat's Last Theorem confounded many of the world's best mathematicians for more than 300 years before being proved by Andrew Wiles in 1995.

Example 3.4 A Function with a Zero Derivative at a Local Maximum

Locate any local extrema for $f(x) = 9 - x^2$ and describe the behavior of the derivative at the local extremum.

Solution We can see from Figure 3.23 that there is a local maximum at $x = 0$. Further, note that $f'(x) = -2x$ and so, $f'(0) = 0$. Note that this says that the tangent line to $y = f(x)$ at $x = 0$ is horizontal, as indicated in Figure 3.23.

Example 3.5 A Function with an Undefined Derivative at a Local Minimum

Locate any local extrema for $f(x) = |x|$ and describe the behavior of the derivative at the local extremum.

Solution We can see from Figure 3.24 that there is a local minimum at $x = 0$. As we have noted in Chapter 2, the graph has a corner at $x = 0$ and hence, $f'(0)$ is undefined. [Recall that all the tangent lines to the graph for $x < 0$ have slope -1, while all the tangent lines for $x > 0$ have slope 1. Since the limit of the slopes of secant lines from the right and from the left are not the same, we know that $f'(0)$ does not exist.]

The graphs shown in Figures 3.22–3.24 are not unusual. Here is a small challenge: Spend a little time now drawing graphs of functions with local extrema. It should not take long to convince yourself that local extrema occur only at points where the derivative is either zero or undefined. Because of this, we give these points a special name.

Definition 3.3
A number c in the domain of a function f is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

It turns out that our earlier observation regarding the location of extrema is correct. That is, local extrema occur *only* at points where the derivative is zero or undefined. We state this formally in the following theorem.

Theorem 3.2 (Fermat's Theorem)
Suppose that $f(c)$ is a local extremum (local maximum or local minimum). Then c must be a critical number of f.

Proof

Suppose that f is differentiable at $x = c$. (If not, c is a critical number of f and we are done.) Suppose further that $f'(c) \neq 0$. Then, either $f'(c) > 0$ or $f'(c) < 0$.

If $f'(c) > 0$, we have by the definition of derivative that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0.$$

So, for all h sufficiently small,

$$\frac{f(c+h)-f(c)}{h} > 0. \tag{3.1}$$

For $h > 0$, (3.1) says that

$$f(c+h)-f(c) > 0$$

and so,

$$f(c+h) > f(c).$$

Thus, $f(c)$ is not a local maximum.

For $h < 0$, (3.1) says that

$$f(c+h)-f(c) < 0$$

and so,

$$f(c+h) < f(c).$$

Thus, $f(c)$ is not a local minimum.

Since we had assumed that $f(c)$ was a local extremum, this is a contradiction. Consequently, $f'(c) \le 0$.

Similarly, if $f'(c) < 0$, we obtain the same contradiction. This is left as an exercise. The only remaining possibility is to have $f'(c) = 0$ and this proves the theorem. ■

We can use Fermat's Theorem and calculator- or computer-generated graphs to find local extrema, as in the following two examples.

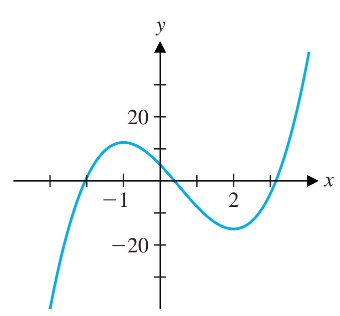

Figure 3.25
$y = 2x^3 - 3x^2 - 12x + 5.$

Example 3.6 **Finding Local Extrema of a Polynomial**

Find the critical numbers and local extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$.

Solution Here,

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1).$$

Thus, f has two critical numbers, $x = -1$ and $x = 2$. Notice from the graph in Figure 3.25 that these correspond to the locations of a local maximum and a local minimum, respectively. ■

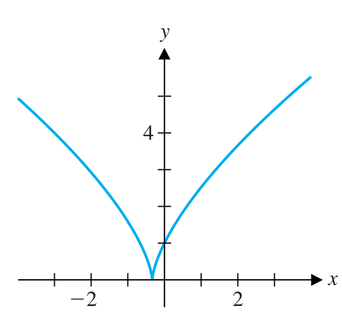

Figure 3.26
$y = (3x+1)^{2/3}.$

Example 3.7 **An Extremum at a Point where the Derivative Is Undefined**

Find the critical numbers and local extrema of $f(x) = (3x+1)^{2/3}$.

Solution Here, we have

$$f'(x) = \frac{2}{3}(3x+1)^{-1/3}(3) = \frac{2}{(3x+1)^{1/3}}.$$

Of course, $f'(x) \neq 0$ for all x, but $f'(x)$ is undefined at $x = -\frac{1}{3}$. Be sure to note that $-\frac{1}{3}$ is in the domain of f. Thus, $x = -\frac{1}{3}$ is the only critical number of f. From the graph in Figure 3.26, we see that this corresponds to the location of a local minimum (also the

Remark 3.2

When we use the terms maximum, minimum or extremum without specifying absolute or local, we will *always* be referring to absolute extrema.

So far in this section, we have been dancing all around the question of how to locate extrema. We have said that local extrema occur only at critical numbers and that continuous functions must have an absolute maximum and an absolute minimum on a closed, bounded interval. But, so far, we haven't really been able to say how to find these extrema. The following theorem is particularly useful.

Theorem 3.3

Suppose that f is continuous on the closed interval $[a, b]$. Then, the absolute extrema of f must occur at an endpoint (a or b) or at a critical number.

Proof

First, recall that by the Extreme Value Theorem, f will attain its maximum and minimum values on $[a, b]$, since f is continuous. Let $f(c)$ be an absolute extremum. If c is *not* an endpoint (i.e., $c \neq a$ and $c \neq b$), then c must be in the open interval (a, b). Thus, $f(c)$ must be a local extremum, also. Finally, by Fermat's Theorem, c must be a critical number, since local extrema occur only at critical numbers.

Remark 3.3

Notice that Theorem 3.3 says that in order to find the absolute extrema of a continuous function on a closed, bounded interval, we need only compare the values of the function at the endpoints and at the critical numbers. The largest of these will be the absolute maximum and the smallest will be the absolute minimum.

We illustrate Theorem 3.3 for the case of a polynomial function in the following example.

Example 3.11 Finding Absolute Extrema on a Closed Interval

Find the absolute extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$ on the interval $[-2, 4]$.

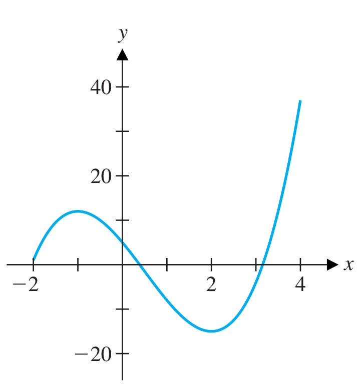

Figure 3.29
$y = 2x^3 - 3x^2 - 12x + 5.$

Solution From the graph in Figure 3.29, the maximum appears to be at the endpoint $x = 4$, while the minimum appears to be at a local minimum near $x = 2$. In example 3.6, we found that the critical numbers of f are $x = -1$ and $x = 2$. Further, both of these are in the interval $[-2, 4]$. So, we compare the values at the endpoints:

$$f(-2) = 1 \quad \text{and} \quad f(4) = 37,$$

and the values at the critical numbers:

$$f(-1) = 12 \quad \text{and} \quad f(2) = -15.$$

Theorem 3.3 says that the absolute extrema must be among these four values. Thus, $f(4) = 37$ is the absolute maximum and $f(2) = -15$ is the absolute minimum. You should take care to note that these values are consistent with what we saw in the graph in Figure 3.29.

Of course, most real problems of interest are unlikely to result in derivatives that are quadratic polynomials with integer zeros. Consider the following somewhat less user-friendly example.

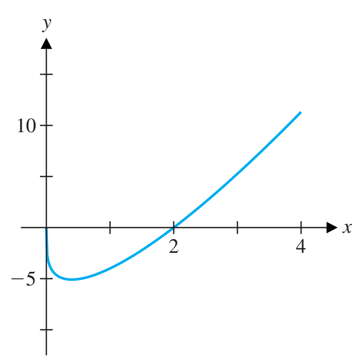

Figure 3.30
$y = 4x^{5/4} - 8x^{1/4}$.

Example 3.12 **Finding Extrema for a Function with Fractional Exponents**

Find the absolute extrema of $f(x) = 4x^{5/4} - 8x^{1/4}$ on the interval [0, 4].

Solution First, we draw a graph of the function to get an idea of where the extrema are located (see Figure 3.30). From the graph, it appears that the maximum occurs at the endpoint $x = 4$ and the minimum near $x = \frac{1}{2}$. Next, observe that

$$f'(x) = 5x^{1/4} - 2x^{-3/4} = \frac{5x - 2}{x^{3/4}}.$$

Thus, the critical numbers are $x = \frac{2}{5}$ [since $f'(\frac{2}{5}) = 0$] and $x = 0$ (since f' is undefined at $x = 0$ and 0 is in the domain of f). We now need only compare

$$f(0) = 0, \quad f(4) \approx 11.3137 \quad \text{and} \quad f\left(\tfrac{2}{5}\right) \approx -5.0897.$$

So, the absolute maximum is $f(4) \approx 11.3137$ and the absolute minimum is $f(\frac{2}{5}) \approx -5.0897$, which is consistent with what we expected from Figure 3.30.

In practice, the critical numbers are not always as easy to find as they were in examples 3.11 and 3.12. In the following example, it is not even known *how many* critical numbers there are. We can, however, estimate the number and locations of these from a careful analysis of computer-generated graphs.

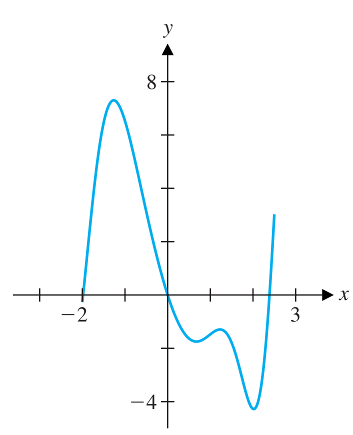

Figure 3.31
$y = f(x) = x^3 - 5x + 3\sin x^2$.

Example 3.13 **Finding Absolute Extrema Approximately**

Find the absolute extrema of $f(x) = x^3 - 5x + 3\sin x^2$ on the interval [−2, 2.5].

Solution Once again, we first draw a graph to get an idea of where the extrema will be located (see Figure 3.31). From the graph, we can see that the maximum seems to occur near $x = -1$, while the minimum seems to occur near $x = 2$. Next, we compute

$$f'(x) = 3x^2 - 5 + 6x\cos x^2.$$

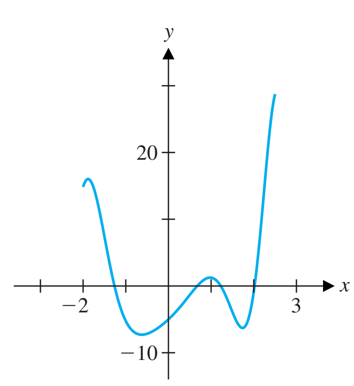

Figure 3.32
$y = f'(x) = 3x^2 - 5 + 6x\cos x^2$.

Unlike examples 3.11 and 3.12, there is no algebra we can use to find the zeros of f'. Our only alternative is to find the zeros approximately. You can do this by using Newton's method to solve $f'(x) = 0$. (You can also use any other rootfinding method built into your calculator or computer.) First, we'll need adequate initial guesses. We obtain these from the graph of $y = f'(x)$ found in Figure 3.32. From the graph, it appears that there are four zeros of $f'(x)$ on the interval in question, located near $x = -1.3$, 0.7, 1.2 and 2.0. Further, referring back to Figure 3.31, these four zeros correspond with the four local extrema seen in the graph of $y = f(x)$. We now apply Newton's method to solve $f'(x) = 0$, using the preceding four values as our initial guesses. This leads us to four approximate critical numbers of f on the interval [−2, 2.5]. We have

$$a \approx -1.26410884789, \quad b \approx 0.674471354085,$$
$$c \approx 1.2266828947 \quad \text{and} \quad d \approx 2.01830371473.$$

While not exact, these are very accurate approximations of the critical numbers. We now need only compare the values of f at the endpoints and the approximate critical numbers:

$$f(a) \approx 7.3, \quad f(b) \approx -1.7, \quad f(c) \approx -1.3$$
$$f(d) \approx -4.3, \quad f(-2) \approx -0.3 \quad \text{and} \quad f(2.5) \approx 3.0.$$

Thus, the absolute maximum is approximately $f(-1.26410884789) \approx 7.3$ and the absolute minimum is approximately $f(2.01830371473) \approx -4.3$.

It is important (especially in light of how much of our work here was approximate and graphical) to verify that the approximate extrema correspond with what we expect from the graph of $y = f(x)$. Since these correspond closely, we have great confidence in their accuracy.

We have now seen how to locate the absolute extrema of a continuous function on a closed interval. In section 3.4, we see how to find local extrema.

EXERCISES 3.3

1. Using one or more graphs, explain why the Extreme Value Theorem is true. Is the conclusion true if we drop the hypothesis that $f(x)$ is a continuous function? Is the conclusion true if we drop the hypothesis that the interval is closed?

2. Using one or more graphs, argue that Fermat's Theorem is true. Discuss how Fermat's Theorem is used. Restate the theorem in your own words to make its use more clear.

3. Suppose that $f(t)$ represents your elevation after t hours on a mountain hike. If you stop to rest, explain why $f'(t) = 0$. Discuss the circumstances under which you would be at a local maximum, local minimum or neither.

4. Mathematically, an if/then statement is usually strictly one-directional. When we say "If A then B" it is generally **not** the case that "If B then A" is also true: when both are true, we say "A if and only if B" which is abbreviated to "A iff B." Unfortunately, common English usage is not always this precise. This occasionally causes a problem interpreting a mathematical theorem. To get this straight, consider the statement, "**If** you wrote a best-selling book, **then** you made a lot of money." Is this true? How does this differ from its **converse,** "**If** you made a lot of money, **then** you wrote a best-selling book." Is the converse always true? Sometimes true? Apply this logic to both the Extreme Value Theorem and Fermat's Theorem: state the converse and decide if it is sometimes true or always true.

In exercises 5–32, find all critical numbers and determine whether each represents a local maximum, local minimum or neither.

5. $f(x) = x^2 + 5x - 1$
6. $f(x) = -x^2 + 4x + 2$
7. $f(x) = x^3 - 3x + 1$
8. $f(x) = x^3 + 3x + 1$
9. $f(x) = x^3 - 3x^2 + 3x$
10. $f(x) = x^3 - 3x^2 + 6x$
11. $f(x) = x^4 - 3x^3 + 2$
12. $f(x) = x^4 + 6x^2 - 2$
13. $f(x) = x^{3/4} - 4x^{1/4}$
14. $f(x) = (x^{2/5} - 3x^{1/5})^2$
15. $f(x) = x^3 - 2x^2 - 4x$
16. $f(x) = x^5 - 20x^2 + 1$
17. $f(x) = \sin x \cos x$, $[0, 2\pi]$
18. $f(x) = \sin x + \cos x$, $[0, 2\pi]$
19. $f(x) = \dfrac{x+1}{x-1}$
20. $f(x) = \dfrac{x^2 - x + 4}{x - 1}$
21. $f(x) = \dfrac{x}{x^2 + 1}$
22. $f(x) = \dfrac{3x}{x^2 - 1}$

23. $f(x) = \frac{1}{2}(e^x + e^{-x})$

24. $f(x) = \frac{1}{2}(e^x - e^{-x})$

25. $f(x) = x^{4/3} + 4x^{1/3} + 4x^{-2/3}$

26. $f(x) = x^{7/3} - 28x^{1/3}$

27. $f(x) = 2x\sqrt{x+1}$

28. $f(x) = x/\sqrt{x^2+1}$

29. $f(x) = e^{-x^2}$

30. $f(x) = xe^{-x}$

31. $f(x) = \sin x^2$, $[0, \pi]$

32. $f(x) = \sin^2 x$, $[0, 2\pi]$

In exercises 33–42, find the absolute extrema of the given function on the indicated interval.

33. $f(x) = x^3 - 3x + 1$, $[0, 2]$

34. $f(x) = x^3 - 3x + 1$, $[-3, 2]$

35. $f(x) = x^4 - 8x^2 + 2$, $[-3, 1]$

36. $f(x) = x^4 - 8x^2 + 2$, $[-1, 3]$

37. $f(x) = x^{2/3}$, $[-4, -2]$

38. $f(x) = x^{2/3}$, $[-1, 3]$

39. $f(x) = \sin x + \cos x$, $[0, 2\pi]$

40. $f(x) = \sin x + \cos x$, $[\pi/2, \pi]$

41. $f(x) = x \sin x + 3$, $[0, 2\pi]$

42. $f(x) = x^2 + e^x$, $[-2, 2]$

In exercises 43–46, numerically estimate the absolute extrema of the given function on the indicated interval.

43. $f(x) = x^4 - 3x^2 + 2x + 1$ on (a) $[-1, 1]$ and (b) $[-3, 2]$

44. $f(x) = x^6 - 3x^4 - 2x + 1$ on (a) $[-1, 1]$ and (b) $[-2, 2]$

45. $f(x) = x^2 - 3x \cos x$ on (a) $[-2, 1]$ and (b) $[-5, 0]$

46. $f(x) = xe^{\cos 2x}$ on (a) $[-2, 2]$ and (b) $[2, 5]$

47. Repeat exercises 33–38, except instead of finding extrema on the closed interval, find the extrema on the open interval, if they exist.

48. Briefly outline a procedure for finding extrema on an open interval (a, b), a procedure for the half-open interval $(a, b]$ and a procedure for the half-open interval $[a, b)$.

49. Sketch a graph of a function $f(x)$ such that the absolute maximum of $f(x)$ on the interval $[-2, 2]$ equals 3 and the absolute minimum does not exist.

50. Sketch a graph of a continuous function $f(x)$ such that the absolute maximum of $f(x)$ on the interval $(-2, 2)$ does not exist and the absolute minimum equals 2.

51. Sketch a graph of a continuous function $f(x)$ such that the absolute maximum of $f(x)$ on the interval $(-2, 2)$ equals 4 and the absolute minimum equals 2.

52. Sketch a graph of a function $f(x)$ such that the absolute maximum of $f(x)$ on the interval $[-2, 2]$ does not exist and the absolute minimum does not exist.

53. Sketch a graph of $f(x) = \dfrac{x^2}{x^2+1}$ for $x > 0$ and determine where the graph is steepest (that is, find where the slope is a maximum).

54. Sketch a graph of $f(x) = e^{-x^2}$ and determine where the graph is steepest. (Note: This is an important problem in probability theory.)

55. Sketch a graph showing that $y = f(x) = x^2 + 1$ and $y = g(x) = \ln x$ do not intersect. Find x to minimize $f(x) - g(x)$. At this value of x, show that the tangent lines to $y = f(x)$ and $y = g(x)$ are parallel.

56. Explain graphically why it makes sense that the tangent lines in exercise 55 are parallel, given that at this point the vertical distance between the graphs is smallest.

57. Give an example showing that the following statement is false (not always true): between any two local minima of $f(x)$ there is a local maximum.

58. Is the statement in exercise 57 true if $f(x)$ is continuous?

59. A section of roller coaster is in the shape of $y = x^5 - 4x^3 - x + 10$, where x is between -2 and 2. Find all relative extrema and explain what portions of the roller coaster they represent. Find the location of the steepest part of the roller coaster.

60. Suppose a large computer file is sent over the Internet. If the probability that it reaches its destination without any errors is x, then the probability that an error is made is $1 - x$. The field of information theory studies such situations. An important quantity is **entropy** (a measure of unpredictability), defined by $H = -x \ln x - (1 - x)\ln(1 - x)$, for $0 < x < 1$. Find the value of x that maximizes this quantity. Explain why this value makes sense as the probability that maximizes entropy.

61. Researchers in a number of fields (including population biology, economics and the study of animal tumors) make use of the Gompertz growth curve, $W(t) = ae^{-be^{-t}}$. As $t \to \infty$, show that $W(t) \to a$ and $W'(t) \to 0$. Find the maximum growth rate.

62. In this exercise, we will explore the family of functions $f(x) = x^3 + cx + 1$, where c is a constant. How many and what types of relative extrema are there? (Your answer will depend on the value of c.) Assuming that this family is indicative of all cubic functions, list all types of cubic functions. Without looking at specific examples, try to list all types of fourth-order polynomials, sketching a graph of each.

63. Explore the graphs of e^{-x}, xe^{-x}, x^2e^{-x} and x^3e^{-x}. Find all local extrema and determine the behavior as $x \to \infty$. You can think of the graph of $x^n e^{-x}$ as showing the results of a tug-of-war: $x^n \to \infty$ as $x \to \infty$ but $e^{-x} \to 0$ as $x \to \infty$. Describe the graph of $x^n e^{-x}$ in terms of this tug-of-war.

64. Johannes Kepler (1571–1630) is best known as an astronomer, especially for his three laws of planetary motion. However, his discoveries were primarily due to his brilliance as a mathematician. While serving in Austrian Emperor Matthew I's court, Kepler observed the ability of Austrian vintners to quickly and mysteriously compute the capacities of a variety of wine casks. Each cask (barrel) had a hole in the middle of its side (see Figure a). The vintner would insert a rod in the hole until it hit the far corner and then announce the volume. Kepler first analyzed the problem for a cylindrical barrel (see Figure b). The volume of a cylinder is $V = \pi r^2 h$. In Figure b, $r = y$ and $h = 2x$ so $V = 2\pi y^2 x$. Call the rod measurement z. By the Pythagorean Theorem, $x^2 + (2y)^2 = z^2$. Kepler's mystery was how to compute V given only z. The key observation made by Kepler was that Austrian wine casks were made with the same height-to-diameter ratio (for us, x/y). Let $t = x/y$ and show that $z^2/y^2 = t^2 + 4$. Use this to replace y^2 in the volume formula. Then replace x with $\sqrt{z^2 - 4y^2}$. Show that $V = \dfrac{2\pi z^3 t}{(4 + t^2)^{3/2}}$. In this formula, t is a constant so the vintner could measure z and quickly estimate the volume. We haven't told you yet what t equals. Kepler assumed that the vintners would have made a smart choice for this ratio. Find the value of t that maximizes the volume for a given z. This is, in fact, the ratio used in the construction of Austrian wine casks!

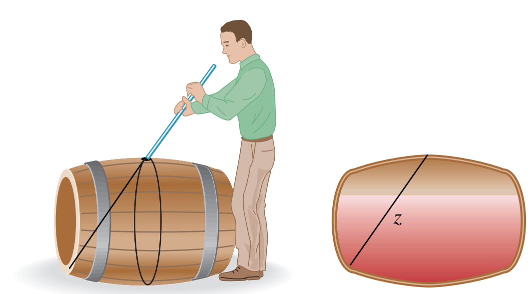

Figure a

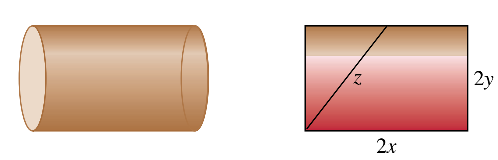

Figure b

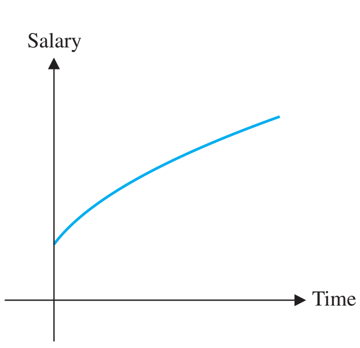

Figure 3.33
Increasing salary.

3.4 INCREASING AND DECREASING FUNCTIONS

One of the questions from section 3.3 that we have yet to answer is how to determine where a function has a local maximum or local minimum. We have already determined that local extrema occur only at critical numbers. Unfortunately, not all critical numbers correspond to local extrema. In this section, we develop a means of deciding which critical numbers correspond to local extrema. At the same time, we'll learn more about the connection between the derivative and graphing. We begin with a very simple notion.

We are all familiar with the terms **increasing** and **decreasing.** If your employer informs you that your salary will be increasing steadily over the term of your employment, you have in mind that as time goes on, your salary will rise. If you plotted your salary against time, the graph might look something like Figure 3.33. If you take out a loan to purchase a car or a home or to pay for your college education, once you start paying back the