61. Researchers in a number of fields (including population biology, economics and the study of animal tumors) make use of the Gompertz growth curve, $W(t) = ae^{-be^{-t}}$. As $t \to \infty$, show that $W(t) \to a$ and $W'(t) \to 0$. Find the maximum growth rate.

62. In this exercise, we will explore the family of functions $f(x) = x^3 + cx + 1$, where c is a constant. How many and what types of relative extrema are there? (Your answer will depend on the value of c.) Assuming that this family is indicative of all cubic functions, list all types of cubic functions. Without looking at specific examples, try to list all types of fourth-order polynomials, sketching a graph of each.

63. Explore the graphs of e^{-x}, xe^{-x}, x^2e^{-x} and x^3e^{-x}. Find all local extrema and determine the behavior as $x \to \infty$. You can think of the graph of x^ne^{-x} as showing the results of a tug-of-war: $x^n \to \infty$ as $x \to \infty$ but $e^{-x} \to 0$ as $x \to \infty$. Describe the graph of x^ne^{-x} in terms of this tug-of-war.

64. Johannes Kepler (1571–1630) is best known as an astronomer, especially for his three laws of planetary motion. However, his discoveries were primarily due to his brilliance as a mathematician. While serving in Austrian Emperor Matthew I's court, Kepler observed the ability of Austrian vintners to quickly and mysteriously compute the capacities of a variety of wine casks. Each cask (barrel) had a hole in the middle of its side (see Figure a). The vintner would insert a rod in the hole until it hit the far corner and then announce the volume. Kepler first analyzed the problem for a cylindrical barrel (see Figure b). The volume of a cylinder is $V = \pi r^2 h$. In Figure b, $r = y$ and $h = 2x$ so $V = 2\pi y^2 x$. Call the rod measurement z. By the Pythagorean Theorem, $x^2 + (2y)^2 = z^2$. Kepler's mystery was how to compute V given only z. The key observation made by Kepler was that Austrian wine casks were made with the same height-to-diameter ratio (for us, x/y). Let $t = x/y$ and show that $z^2/y^2 = t^2 + 4$. Use this to replace y^2 in the volume formula. Then replace x with $\sqrt{z^2 - 4y^2}$. Show that $V = \frac{2\pi z^3 t}{(4+t^2)^{3/2}}$. In this formula, t is a constant so the vintner could measure z and quickly estimate the volume. We haven't told you yet what t equals. Kepler assumed that the vintners would have made a smart choice for this ratio. Find the value of t that maximizes the volume for a given z. This is, in fact, the ratio used in the construction of Austrian wine casks!

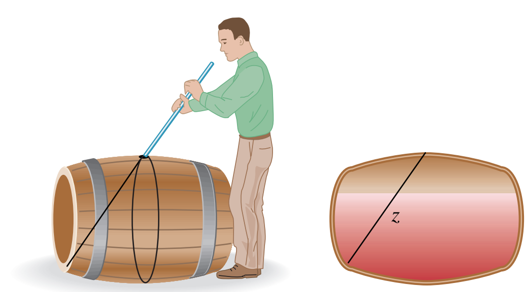


Figure a

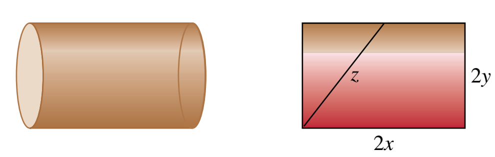


Figure b

3.4 INCREASING AND DECREASING FUNCTIONS

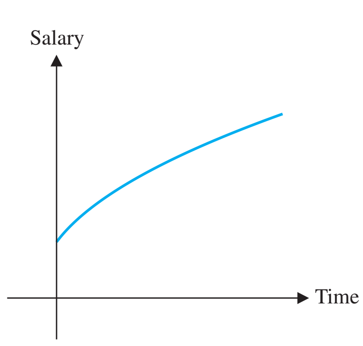


Figure 3.33
Increasing salary.

One of the questions from section 3.3 that we have yet to answer is how to determine where a function has a local maximum or local minimum. We have already determined that local extrema occur only at critical numbers. Unfortunately, not all critical numbers correspond to local extrema. In this section, we develop a means of deciding which critical numbers correspond to local extrema. At the same time, we'll learn more about the connection between the derivative and graphing. We begin with a very simple notion.

We are all familiar with the terms **increasing** and **decreasing.** If your employer informs you that your salary will be increasing steadily over the term of your employment, you have in mind that as time goes on, your salary will rise. If you plotted your salary against time, the graph might look something like Figure 3.33. If you take out a loan to purchase a car or a home or to pay for your college education, once you start paying back the

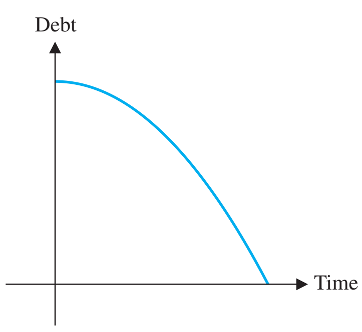


Figure 3.34
Decreasing debt.

loan, your indebtedness will decrease over time. If you plotted your debt against time, the graph might look something like Figure 3.34.

We now carefully define these notions. Notice that the following definition is merely a formal statement of something you already understand.

Definition 4.1

A function f is **(strictly) increasing** on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$ [i.e., $f(x)$ gets larger as x gets larger].

A function f is **(strictly) decreasing** on the interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) > f(x_2)$ [i.e., $f(x)$ gets smaller as x gets larger].

Why do we bother with such an obvious definition? Of course, anyone can look at a graph of a function and immediately see where that function is increasing and decreasing. The real challenge is to determine where a function is increasing and decreasing, given only a mathematical formula for the function. For example, can you determine where $f(x) = x^2 \sin x$ is increasing and decreasing, **without** looking at a graph? Even with a graph, can you determine where this happens precisely? Look carefully at Figure 3.35 to see if you can notice what happens at every point at which the function is increasing or decreasing.

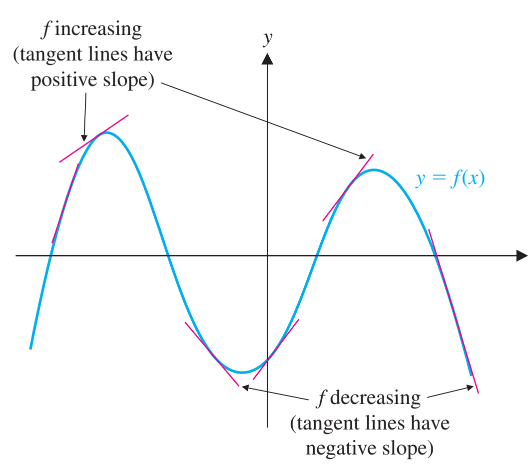


Figure 3.35
Increasing and decreasing.

You should observe that on intervals where f is increasing, the tangent lines all have positive slope, while on intervals where f is decreasing, the tangent lines all have negative slope. Of course, we already know that the slope of the tangent line at a point is given by the value of the derivative at that point. So, whether a function is increasing or decreasing on an interval seems to be connected to the sign of its derivative on that interval. This conjecture, although it's based only on a single picture, sounds like a theorem and it is.

Theorem 4.1

Suppose that f is differentiable on an interval I.

(i) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.

(ii) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

Proof

(i) Pick any two points $x_1, x_2 \in I$, with $x_1 < x_2$. Applying the Mean Value Theorem (Theorem 9.4 in section 2.9) to f on the interval (x_1, x_2), we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \tag{4.1}$$

for some number $c \in (x_1, x_2)$. (Why can we apply the Mean Value Theorem here?) By hypothesis, $f'(c) > 0$ and since $x_1 < x_2$ (so that $x_2 - x_1 > 0$), we have from (4.1) that

$$0 < f(x_2) - f(x_1)$$

or

$$f(x_1) < f(x_2). \tag{4.2}$$

Since (4.2) holds for all $x_1 < x_2$, f is increasing on I.

The proof of (ii) is nearly identical and is left as an exercise. ■

What You See May Not Be What You Get

One aim in the next few sections is to learn how to draw fairly representative graphs of functions (i.e., graphs that display all of the significant features of a function: where it is increasing or decreasing, any extrema, asymptotes and two features we'll introduce in section 3.5: concavity and inflection points). You might think that there is little to be concerned about, given the ease with which you can draw graphs by machine, but there is a significant issue here. When we draw a graph, we are drawing in a particular viewing window (i.e., a particular range of x- and y-values). When we use a computer or calculator to draw graphs, the window is often chosen by the machine. So, when is the window properly adjusted to show enough of the graph to be representative of the behavior of the function? We need to know when significant features are hidden outside of a given window and how to determine the precise locations of features that we can see in a given window. As we'll see, the only way we can resolve these questions is with a healthy dose of calculus.

Example 4.1 Drawing a Graph

Draw a graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$ showing all local extrema.

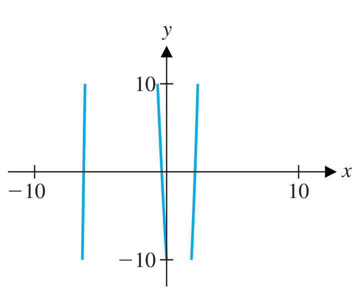
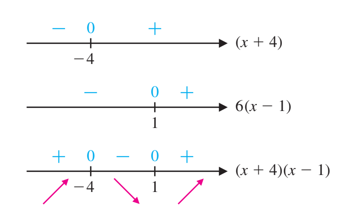


Figure 3.36
$y = 2x^3 + 9x^2 - 24x - 10.$

Solution The most popular graphing calculators use the window defined by $-10 \le x \le 10$ and $-10 \le y \le 10$ as their default. Using this window, the graph of $y = f(x)$ looks like that displayed in Figure 3.36. The three segments seen in Figure 3.36 are not particularly revealing. Instead of blindly manipulating the window in the hope that a reasonable graph will magically appear, we stop briefly to determine where the function is increasing and decreasing. First, we need the derivative

$$f'(x) = 6x^2 + 18x - 24 = 6(x^2 + 3x - 4)$$
$$= 6(x - 1)(x + 4).$$

Note that the critical numbers are 1 and -4, so these are the only candidates for local extrema. We can see where the two factors and consequently the derivative are positive



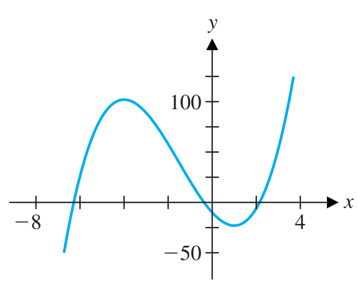


Figure 3.37a
$y = 2x^3 + 9x^2 - 24x - 10$.

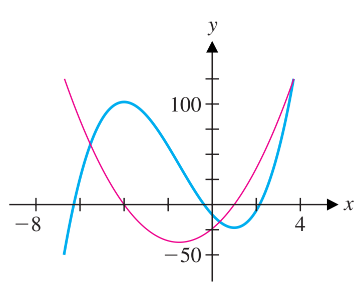


Figure 3.37b
$y = f(x)$ and $y = f'(x)$.

and negative from the number lines displayed in the margin. From this, note that

$$f'(x) > 0 \text{ on } (-\infty, -4) \cup (1, \infty) \qquad f \text{ increasing.}$$

and

$$f'(x) < 0 \text{ on } (-4, 1). \qquad f \text{ decreasing.}$$

So that you can conveniently see this information at a glance, we have placed arrows indicating where the function is increasing and decreasing beneath the last number line. In particular, notice that this suggests that we are not seeing enough of the graph in the window in Figure 3.36. In Figure 3.37a, we redraw the graph in the window defined by $-8 \le x \le 4$ and $-25 \le y \le 105$. Here, we have selected the y-range so that the critical points $(-4, 102)$ and $(1, -23)$ (and every point in between) are displayed. Since we know that f is increasing on all of $(-\infty, -4)$, we know that the function is still increasing to the left of the portion displayed in Figure 3.37a. Likewise, since we know that f is increasing on all of $(1, \infty)$, we know that the function continues to increase to the right of the displayed portion. The graph in Figure 3.37a conveys all of this significant behavior of the function. In Figure 3.37b, we have plotted both $y = f(x)$ (shown in blue) and $y = f'(x)$ (shown in red). Notice the connection between the two graphs. When $f'(x) > 0$, f is increasing; when $f'(x) < 0$, f is decreasing and also notice what happens to $f'(x)$ at the local extrema of f. (We'll say more about this shortly.)

You may be tempted to think that you can dispense with using any calculus to draw a graph. After all, it's easy to get lured into thinking that your graphing calculator or computer can draw a much better graph than you can in only a fraction of the time and with virtually no effort. To some extent, this is true. You can draw graphs by machine and with a little fiddling with the graphing window, get a reasonable looking graph. Unfortunately, this frequently isn't enough. For instance, while it's clear that the graph in Figure 3.36 is incomplete, the initial graph in the following example has a familiar shape and may look reasonable, but it is incorrect. The calculus tells you what features you should expect to see in a graph. Without it, you're simply fooling around and hoping you get something reasonable.

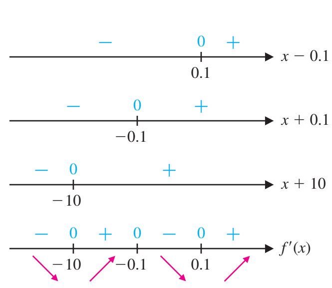
Example 4.2 Uncovering Hidden Behavior in a Graph

Graph $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$ showing all local extrema.

Solution We first show the default graph drawn by our computer algebra system (see Figure 3.38a). We show a common default graphing calculator graph in Figure 3.38b. You can certainly make Figure 3.38b look more like Figure 3.38a by fooling around with the window some. But with some calculus, you can discover features of the graph that would otherwise remain hidden.

First, notice that

$$\begin{aligned} f'(x) &= 12x^3 + 120x^2 - 0.12x - 1.2 \\ &= 12(x^2 - 0.01)(x + 10) \\ &= 12(x - 0.1)(x + 0.1)(x + 10). \end{aligned}$$



We show number lines for the three factors in the margin. Observe that

$$f'(x) \begin{cases} > 0 \text{ on } (-10, -0.1) \cup (0.1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-\infty, -10) \cup (-0.1, 0.1). & f \text{ decreasing.} \end{cases}$$

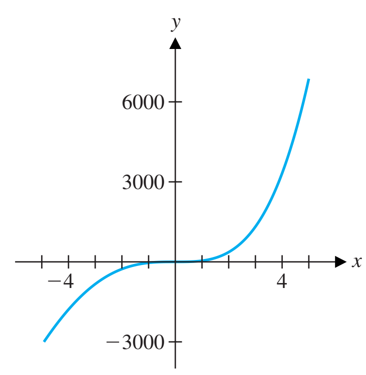


Figure 3.38a
Default CAS graph of $y = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$.

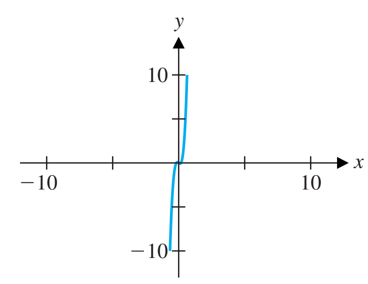


Figure 3.38b
Default calculator graph of $y = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$.

This says that neither of the machine-generated graphs seen in Figures 3.38a or 3.38b are adequate, as the behavior on $(-\infty, -10) \cup (-0.1, 0.1)$ cannot be seen in either graph. As it turns out, no single graph captures all of the behavior of this function. However, by increasing the range of x-values to the interval $[-15, 5]$, we get the graph seen in Figure 3.39a. This shows the *big picture,* what we refer to as the **global** behavior of the function. Here, you can see the local minimum at $x = -10$, which was missing in our earlier graphs, but the behavior for values of x close to zero is not clear. To see this, we need a separate graph, restricted to a smaller range of x-values, as seen in Figure 3.39b.

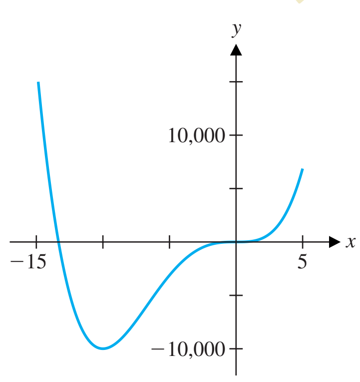


Figure 3.39a
The global behavior of $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$.

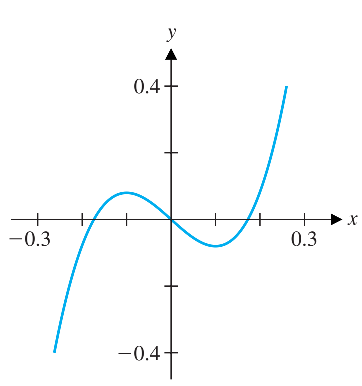


Figure 3.39b
Local behavior of $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$.

Notice that here, we can see the behavior of the function for x close to zero quite clearly. In particular, the local maximum at $x = -0.1$ and the local minimum at $x = 0.1$ are clearly visible. We often say that a graph such as Figure 3.39b shows the **local** behavior of the function. In Figures 3.40a and 3.40b, we show graphs indicating the global and local behavior of $f(x)$ (in blue) and $f'(x)$ (in red) on the same set of axes. Pay particular attention to the behavior of $f'(x)$ in the vicinity of local extrema of $f(x)$.

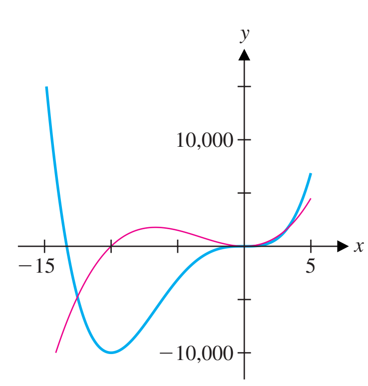


Figure 3.40a
$y = f(x)$ and $y = f'(x)$ (global behavior).

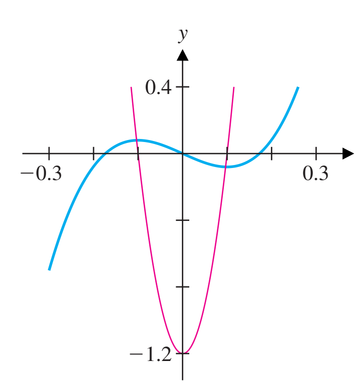


Figure 3.40b
$y = f(x)$ and $y = f'(x)$ (local behavior).

You may have already noticed a connection between local extrema and the intervals on which a function is increasing and decreasing. We state this in the following theorem.

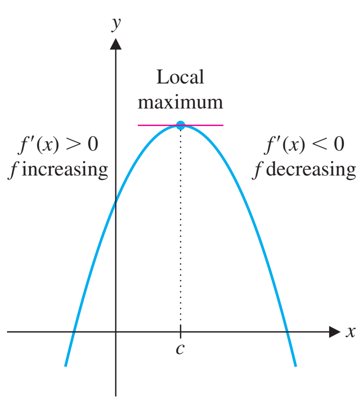


Figure 3.41a
Local maximum.

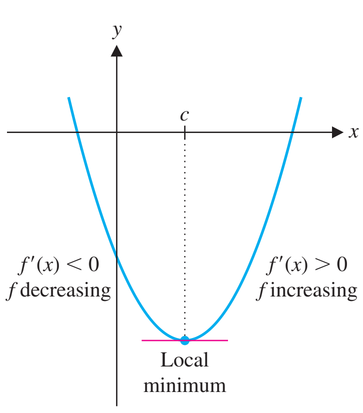


Figure 3.41b
Local minimum.

Theorem 4.2 (First Derivative Test)

Suppose that f is continuous on the interval $[a, b]$ and $c \in (a, b)$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$ (i.e., f changes from increasing to decreasing at c), then $f(c)$ is a local maximum.

(ii) If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$ (i.e., f changes from decreasing to increasing at c), then $f(c)$ is a local minimum.

(iii) If $f'(x)$ has the **same** sign on (a, c) and (c, b), then $f(c)$ is **not** a local extremum.

It's easiest to think of this result graphically. If f is increasing to the left of a critical number and decreasing to the right, then there must be a local maximum at the critical number (see Figure 3.41a). Likewise, if f is decreasing to the left of a critical number and increasing to the right, then there must be a local minimum at the critical number (see Figure 3.41b). By the way, the preceding argument suggests a proof of the theorem. The job of writing out all of the details is left as an exercise.

Example 4.3 Finding Local Extrema Using the First Derivative Test

Find the local extrema of the function from example 4.1, $f(x) = 2x^3 + 9x^2 - 24x - 10$.

Solution We had found in example 4.1 that

$$f'(x) \begin{cases} > 0 \text{ on } (-\infty, -4) \cup (1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-4, 1). & f \text{ decreasing.} \end{cases}$$

It now follows from the first derivative test that f has a local maximum located at $x = -4$ and a local minimum located at $x = 1$.

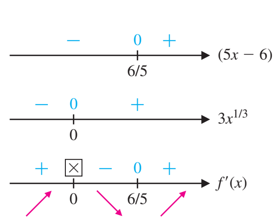
Theorem 4.2 works equally well for a function with critical points where the derivative is undefined.

Example 4.4 Finding Local Extrema of a Function with Fractional Exponents

Find the local extrema of $f(x) = x^{5/3} - 3x^{2/3}$.

Solution We have

$$f'(x) = \frac{5}{3}x^{2/3} - 3\left(\frac{2}{3}\right)x^{-1/3} = \frac{5x - 6}{3x^{1/3}},$$



so that the critical numbers are $\frac{6}{5}$ $[f'(\frac{6}{5}) = 0]$ and 0 $[f'(0)$ is undefined$]$. Again drawing number lines for the factors, we determine where f is increasing and decreasing. Here, we have placed an ⊠ above the 0 on the number line for $f'(x)$ to indicate that $f'(x)$ is not defined at $x = 0$. From this, we can see at a glance where f' is positive and negative:

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, 0) \cup (\frac{6}{5}, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (0, \frac{6}{5}). & f \text{ decreasing.} \end{cases}$$

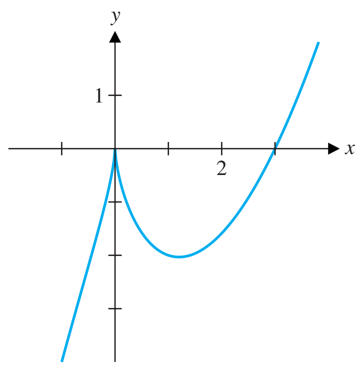


Figure 3.42
$y = x^{5/3} - 3x^{2/3}$.

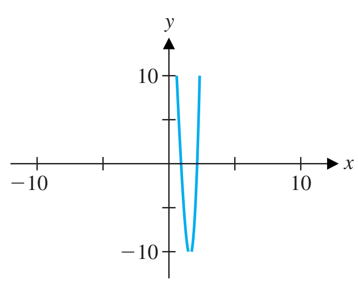


Figure 3.43
$f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$.

Consequently, f has a local maximum at $x = 0$ and a local minimum at $x = \frac{6}{5}$. These local extrema are both clearly visible in the graph in Figure 3.42.

Example 4.5 Finding Local Extrema Approximately

Find the local extrema of $f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$ and draw a graph.

Solution A graph of $y = f(x)$ using the most common graphing calculator default window appears in Figure 3.43. Without further analysis, we do not know if this graph shows all of the significant behavior of the function. [Note that some fourth-degree polynomials (e.g., $f(x) = x^4$) have graphs that look very much like the one in Figure 3.43.] First, we compute

$$f'(x) = 4x^3 + 12x^2 - 10x - 31.$$

Unlike the last several examples, this derivative does not easily factor. A graph of $y = f'(x)$ (see Figure 3.44) reveals three zeros, one near each of $x = -3, -1.5$, and 1.5. Since a cubic polynomial has at most three zeros, there are no others. Using Newton's method or some other root-finding method [applied to $f'(x)$], we can find approximations to the three zeros of f'. We get $a \approx -2.96008$, $b \approx -1.63816$ and $c \approx 1.59824$. From Figure 3.44, we can see that $f'(x)$ is positive for $a < x < b$ and for $x > c$ and is negative elsewhere. That is,

$$f'(x) > 0 \text{ on } (a, b) \cup (c, \infty) \qquad f \text{ increasing.}$$

and

$$f'(x) < 0 \text{ on } (-\infty, a) \cup (b, c). \qquad f \text{ decreasing.}$$

You can quickly read off the local extrema: a local minimum at $a \approx -2.96008$, a local maximum at $b \approx -1.63816$ and a local minimum at $c \approx 1.59824$. Since only the local minimum at $x = c$ is visible in the graph in Figure 3.43, this graph is clearly not representative of the behavior of the function. By narrowing the range of displayed x-values and widening the range of displayed y-values, we obtain the far more useful graph seen in Figure 3.45. You should convince yourself, using the preceding analysis, that the local minimum at $x = c \approx 1.59824$ is also the **absolute** minimum.

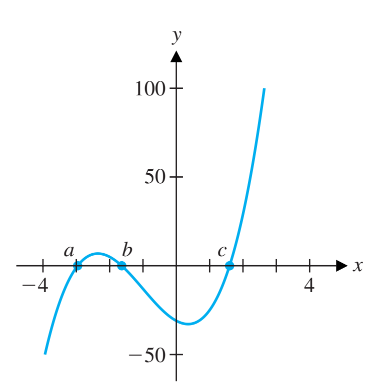


Figure 3.44
$f'(x) = 4x^3 + 12x^2 - 10x - 31$.

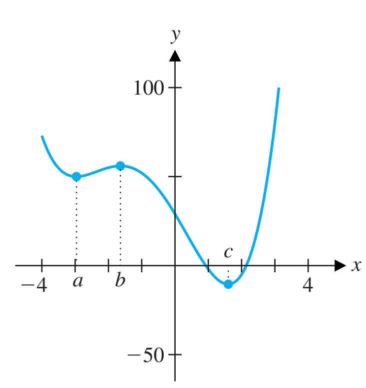


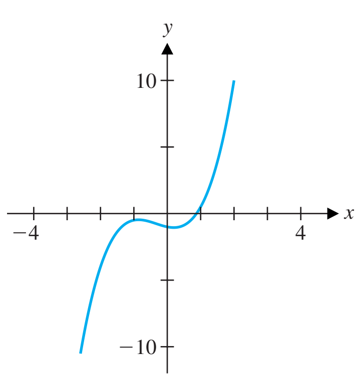
Figure 3.45
$f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$.

EXERCISES 3.4

1. Suppose that $f(0) = 2$ and $f(x)$ is an increasing function. To sketch the graph of $y = f(x)$, you could start by plotting the point (0, 2). Filling in the graph to the left, would you move your pencil up or down? How does this fit with the definition of increasing?

2. Suppose you travel east on an east-west interstate highway. You reach your destination, stay a while and then return home. Explain the First Derivative Test in terms of your velocities (positive and negative) on this trip.

3. Suppose that you have a differentiable function $f(x)$ with two critical numbers. Your computer has shown you a graph that looks like the one below.



Discuss the possibility that this is a representative graph: that is, is it possible that there are any important points not shown in this window?

4. Suppose that the function in exercise 3 has three critical numbers. Explain why the graph is not a representative graph. Explain how you would change the graphing window to show the rest of the graph.

In exercises 5–14, find (by hand) the intervals where the function is increasing and decreasing. Verify your answers by graphing both $f(x)$ and $f'(x)$.

5. $y = x^3 - 3x + 2$

6. $y = x^3 + 2x^2 + 1$

7. $y = x^4 - 8x^2 + 1$

8. $y = x^3 - 3x^2 - 9x + 1$

9. $y = (x + 1)^{2/3}$

10. $y = (x - 1)^{1/3}$

11. $y = \sin 3x$

12. $y = \sin^2 x$

13. $y = e^{x^2-1}$

14. $y = \ln(x^2 - 1)$

In exercises 15–34, find the x-coordinates of all extrema and sketch a graph.

15. $y = x^3 + 2x^2 - x - 1$

16. $y = x^3 + 4x - 2$

17. $y = x^4 + 2x^2 - x + 2$

18. $y = x^5 + 2x^4 - x^2 + 1$

19. $y = x\sqrt{x^2 + 1}$

20. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

21. $y = xe^{-2x}$

22. $y = x^2e^{-x}$

23. $y = \ln x^2$

24. $y = e^{-x^2}$

25. $y = \dfrac{x}{x^2 - 1}$

26. $y = \dfrac{x^2}{x^2 - 1}$

27. $y = \dfrac{x^3}{x^2 - 1}$

28. $y = \dfrac{x^2}{x^2 + 1}$

29. $y = \sin x + \cos x$

30. $y = \cos x - x$

31. $y = \sqrt{x^3 + 3x^2}$

32. $y = 2x^{1/2} - 4x^{-1/2}$

33. $y = x^{2/3} - 2x^{-1/3}$

34. $y = x^{4/3} + 4x^{1/3}$

In exercises 35–42, find the x-coordinates of all extrema and sketch graphs showing global and local behavior of the function.

35. $y = x^3 - 13x^2 - 10x + 1$

36. $y = x^3 + 15x^2 - 70x + 2$

37. $y = x^4 - 15x^3 - 2x^2 + 40x - 2$

38. $y = x^4 - 16x^3 - 0.1x^2 + 0.5x - 1$

39. $y = x^5 - 200x^3 + 605x - 2$

40. $y = x^4 - 0.5x^3 - 0.02x^2 + 0.02x + 1$

41. $y = (x^2 + x + 0.45)e^{-2x}$

42. $y = x^5 \ln 8x^2$

In exercises 43–46, sketch a graph of a function with the given properties.

43. $f(0) = 1$, $f(2) = 5$, $f'(x) < 0$ for $x < 0$ and $x > 2$, $f'(x) > 0$ for $0 < x < 2$.

44. $f(-1) = 1$, $f(2) = 5$, $f'(x) < 0$ for $x < -1$ and $x > 2$, $f'(x) > 0$ for $-1 < x < 2$, $f'(-1) = 0$, $f'(2)$ does not exist.

45. $f(3) = 0$, $f'(x) < 0$ for $x < 0$ and $x > 3$, $f'(x) > 0$ for $0 < x < 3$, $f'(3) = 0$, $f(0)$ and $f'(0)$ do not exist.

46. $f(1) = 0$, $\lim_{x\to\infty} f(x) = 2$, $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, $f'(1) = 0$.

47. Suppose an object has position function $s(t)$, velocity function $v(t) = s'(t)$ and acceleration function $a(t) = v'(t)$. If $a(t) > 0$, then the velocity $v(t)$ is increasing. Sketch two possible velocity functions, one with velocity getting less negative and one with velocity getting more positive. In both cases, sketch possible position graphs. The position graph should be curved (nonlinear). Does it look more like part of an upward-curving parabola or a downward-curving parabola? We look more closely at curving in section 3.5.

48. Repeat exercise 47 for the case where $a(t) < 0$.

49. Prove Theorem 4.2 (The First Derivative Test).

50. Give a graphical argument that if $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$. Use the Mean Value Theorem to prove it.

In exercises 51–54, use the result of exercise 50 to verify the inequality.

51. $2\sqrt{x} > 3 - \frac{1}{x}$ for $x > 1$

52. $x > \sin x$ for $x > 0$

53. $e^x > x + 1$ for $x > 0$

54. $x - 1 > \ln x$ for $x > 1$

55. Give an example showing that the following statement is false. If $f(0) = 4$ and $f(x)$ is a decreasing function, then the equation $f(x) = 0$ has exactly one solution.

56. Determine if the following statement is true or false: If $f(0) = 4$ and $f(x)$ is an increasing function, then the equation $f(x) = 0$ has no solutions.

57. Suppose that the total sales of a product after t months is given by $s(t) = \sqrt{t + 4}$ thousand dollars. Compute and interpret $s'(t)$.

58. In exercise 57, show that $s'(t) > 0$ for all $t > 0$. Explain in business terms why it is impossible to have $s'(t) < 0$.

59. In this exercise, you will play the role of professor and construct a tricky graphing exercise. The first goal is to find a function with local extrema so close together that they're difficult to see. For instance, suppose you want local extrema at $x = -0.1$ and $x = 0.1$. Explain why you could start with $f'(x) = (x - 0.1)(x + 0.1) = x^2 - 0.01$. Look for a function whose derivative is as given. Graph your function to see if the extrema are "hidden." Next, construct a polynomial of degree 4 with two extrema very near $x = 1$ and another near $x = 0$.

60. In this exercise, we look at the ability of fireflies to synchronize their flashes. (To see a remarkable demonstration of this ability, see David Attenborough's video series *Trials of Life*.) Let the function $f(t)$ represent an individual firefly's rhythm, so that the firefly flashes whenever $f(t)$ equals an integer. Let $e(t)$ represent the rhythm of a neighboring firefly, where again $e(t) = n$, for some integer n, whenever the neighbor flashes. One model of the interaction between fireflies is $f'(t) = \omega + A \sin[e(t) - f(t)]$ for constants ω and A. If the fireflies are synchronized ($e(t) = f(t)$), then $f'(t) = \omega$, so the fireflies flash every $1/\omega$ time units. Assume that the difference between $e(t)$ and $f(t)$ is less than π. Show that if $f(t) < e(t)$, then $f'(t) > \omega$. Explain why this means that the individual firefly is speeding up its flash to match its neighbor. Similarly, discuss what happens if $f(t) > e(t)$.

61. The HIV virus attacks specialized T cells that trigger the human immune system response to a foreign substance. If $T(t)$ is the population of uninfected T cells at time t (days) and $V(t)$ is the population of infectious HIV in the bloodstream, a model that has been used to study AIDS is given by the following **differential equation** that describes the rate at which the population of T cells changes.

$$T'(t) = 10\left[1 + \frac{1}{1 + V(t)}\right] - 0.02T(t) + 0.01\frac{T(t)V(t)}{100 + V(t)} - 0.000024T(t)V(t).$$

If there is no HIV present [that is, $V(t) = 0$] and $T(t) = 1000$, show that $T'(t) = 0$. Explain why this means that the T-cell count will remain constant at 1000 (cells per cubic mm). Now, suppose that $V(t) = 100$. Show that if $T(t)$ is small enough, then $T'(t) > 0$ and the T-cell population will increase. On the other hand, if $T(t)$ is large enough, then $T'(t) < 0$ and the T-cell population will decrease. For what value of $T(t)$ is $T'(t) = 0$? Even though this population would remain stable, explain why this would be bad news for the infected human.