3.5 CONCAVITY

In section 3.4, we saw how to determine where a function is increasing and decreasing and how this relates to drawing a graph of the function. At this point, we need to see how to refine the shape of a graph. First, you must realize that simply knowing where a function increases and decreases is not sufficient to draw good graphs. In Figures 3.46a and 3.46b, we show two very different shapes of increasing functions joining the same two points.

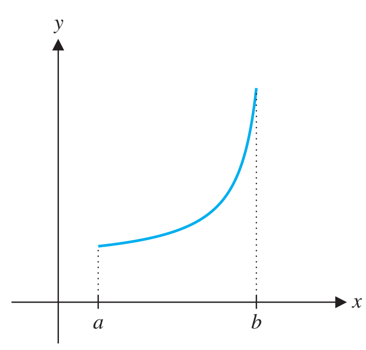


Figure 3.46a
Increasing function.

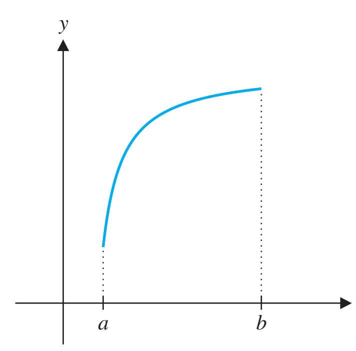


Figure 3.46b
Increasing function.

So, given that a graph joins two particular points and is increasing, which of the two shapes shown do we draw? Without further information, there's no way to tell. Realize that this is an important distinction to make. For example, suppose that Figure 3.46a or 3.46b depicts the balance in your bank account. Both indicate a balance that is growing. Notice that the rate of growth in Figure 3.46a, is increasing, while the rate of growth depicted in Figure 3.46b is decreasing. Which would you want to have describe your bank balance? Why?

Figures 3.47a and 3.47b are the same as Figures 3.46a and 3.46b, respectively, but with a few tangent lines drawn in.

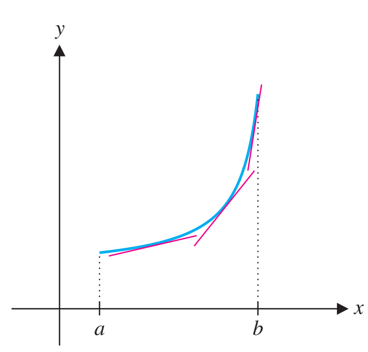


Figure 3.47a
Concave up increasing.

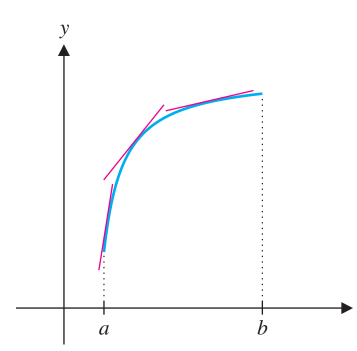


Figure 3.47b
Concave down increasing.

Notice that although all of the tangent lines have positive slope [since $f'(x) > 0$], the slopes of the tangent lines in Figure 3.47a are increasing, while those in Figure 3.47b are decreasing. We call the graph in Figure 3.47a **concave up** and the graph in Figure 3.47b **concave down.** The situation is similar for decreasing functions. In Figures 3.48a and 3.48b, we show two different shapes of decreasing function. Again, although both functions are decreasing, the one shown in Figure 3.48a is concave up (slopes of tangent lines increasing) and the one shown in Figure 3.48b is concave down (slopes of tangent lines decreasing). We have the following definition.

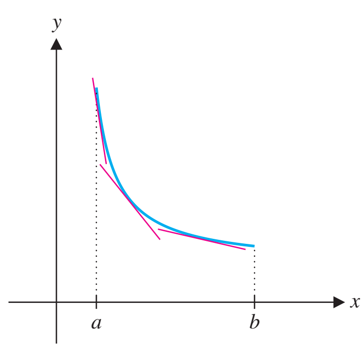


Figure 3.48a
Concave up.

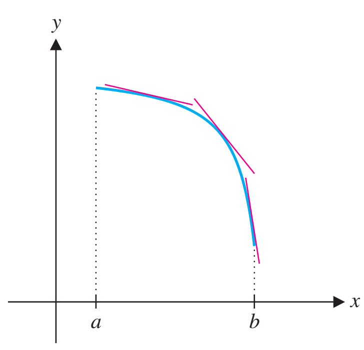


Figure 3.48b
Concave down.

Definition 5.1

For a function f that is differentiable on an interval I, the graph of f is
(i) **concave up** on I, if f' is increasing on I or
(ii) **concave down** on I, if f' is decreasing on I.

How can you tell when f' is increasing or decreasing? The derivative of f' (i.e., f'') yields that information. The following theorem connects this definition with what we already know about increasing and decreasing functions. The proof of the theorem is a straightforward application of Theorem 4.1 to Definition 5.1.

Theorem 5.1

Suppose that f'' exists on an interval I.
(i) If $f''(x) > 0$ on I, then the graph of f is concave up on I.
(ii) If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Example 5.1 Determining Concavity

Determine where the graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$ is concave up and concave down and draw a graph showing all significant behavior of the function.

Solution Here, we have

$$f'(x) = 6x^2 + 18x - 24$$

and from our work in example 4.3, we have

$$f'(x) \begin{cases} > 0 \text{ on } (-\infty, -4) \cup (1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-4, 1). & f \text{ decreasing.} \end{cases}$$

We now have

$$f''(x) = 12x + 18 \begin{cases} > 0, \text{ for } x > -\frac{3}{2} & \text{Concave up.} \\ < 0, \text{ for } x < -\frac{3}{2}. & \text{Concave down.} \end{cases}$$

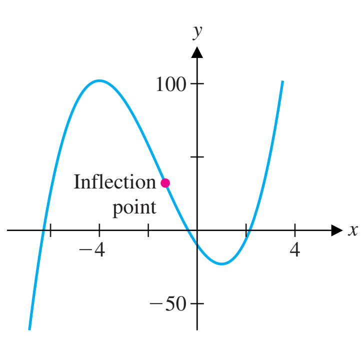


Figure 3.49
$y = 2x^3 + 9x^2 - 24x - 10.$

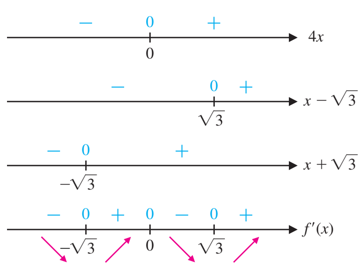
Using all of this information, we are able to draw the graph shown in Figure 3.49. Notice that at the point $\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right)$, the graph changes from concave down to concave up. Such points are called **inflection points.** More precisely, we have the following definition.

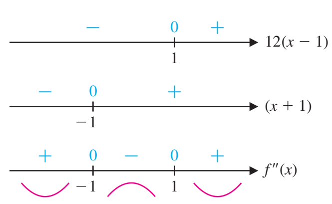
Notes

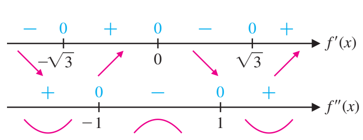
If $(c, f(c))$ is an inflection point, then either $f''(c) = 0$ or $f''(c)$ is undefined. So, finding all points where $f''(x)$ is zero or is undefined gives you all possible candidates for inflection points. But beware: not all points where $f''(x)$ is zero or undefined correspond to inflection points.

Definition 5.2

Suppose that f is continuous on the interval (a, b) and that the graph changes concavity at a point $c \in (a, b)$ (i.e., the graph is concave down on one side of c and concave up on the other). Then, the point $(c, f(c))$ is called an **inflection point** of f.







Example 5.2 Determining Concavity and Locating Inflection Points

Determine where the graph of $f(x) = x^4 - 6x^2 + 1$ is concave up and concave down, find any inflection points and draw a graph showing all significant features.

Solution Here, we have

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$$
$$= 4x(x - \sqrt{3})(x + \sqrt{3}).$$

We have drawn number lines for the factors of $f'(x)$ in the margin. From this, notice that

$$f'(x) \begin{cases} > 0, \text{ on } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}). & f \text{ decreasing.} \end{cases}$$

Next, we have

$$f''(x) = 12x^2 - 12 = 12(x - 1)(x + 1).$$

We have drawn number lines for the two factors in the margin. From this, we can see that

$$f''(x) \begin{cases} > 0, \text{ on } (-\infty, -1) \cup (1, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-1, 1). & \text{Concave down.} \end{cases}$$

So that you can see this information at a glance, we have indicated the concavity below the bottom number line, with small concave up and concave down segments. Finally, observe that since the graph changes concavity at $x = -1$ and $x = 1$, there are inflection points located at $(-1, -4)$ and $(1, -4)$. Using all of this information, we are able to draw the graph shown in Figure 3.50. For your convenience, we have reproduced the number lines for $f'(x)$ and $f''(x)$ together above the figure.

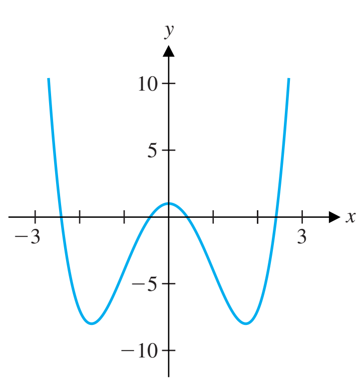


Figure 3.50
$y = x^4 - 6x^2 + 1.$

As we see in the following example, having $f''(x) = 0$ does not imply the existence of an inflection point.

Example 5.3 A Graph with No Inflection Points

Determine the concavity of $f(x) = x^4$ and locate any inflection points.

Solution There's nothing tricky about this function. We have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Since $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, we know that f is increasing for $x > 0$ and decreasing for $x < 0$. Further, $f''(x) > 0$ for all $x \neq 0$, while $f''(0) = 0$. So, the graph is concave up for $x \neq 0$. Further, even though $f''(0) = 0$, there is **no** inflection point at $x = 0$. We show a graph of the function in Figure 3.51.

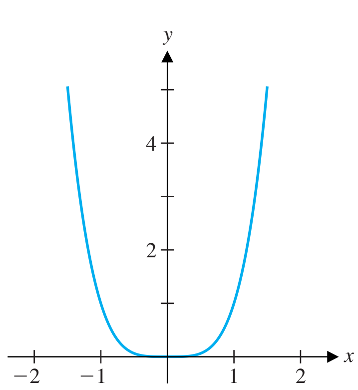


Figure 3.51
$y = x^4.$

There is also a connection between second derivatives and extrema. Suppose that $f'(c) = 0$ and that the graph of f is concave down in some open interval containing c. Then, nearby $x = c$, the graph looks like that in Figure 3.52a and hence, $f(c)$ is a local maximum.

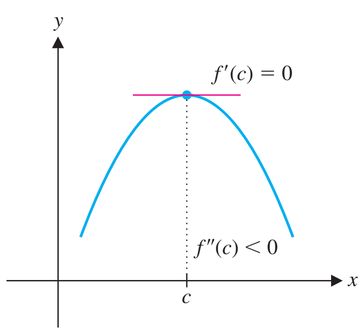


Figure 3.52a
Local maximum.

Likewise, if $f'(c) = 0$ and the graph of f is concave up in some open interval containing c, then nearby $x = c$, the graph looks like that in Figure 3.52b and hence, $f(c)$ is a local minimum.

More precisely, we have the following theorem.

Theorem 5.2 (Second Derivative Test)
Suppose that f is continuous on the interval (a, b) and $f'(c) = 0$, for some number $c \in (a, b)$.
(i) If $f''(c) < 0$, then $f(c)$ is a local maximum and
(ii) if $f''(c) > 0$, then $f(c)$ is a local minimum.

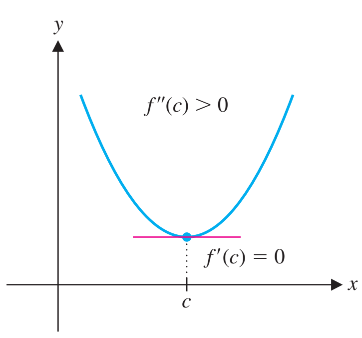


Figure 3.52b
Local minimum.

We leave a formal proof of this theorem as an exercise. When applying the theorem, simply think about Figures 3.52a and 3.52b.

Example 5.4 Using the Second Derivative Test to Find Extrema

Use the second derivative test to find the local extrema of $f(x) = x^4 - 8x^2 + 10$.

Solution Here,

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x - 2)(x + 2).$$

Thus, the critical numbers are $x = 0, 2$ and -2. We can test these using the second derivative test as follows. We have

$$f''(x) = 12x^2 - 16$$

and so,

$$f''(0) = -16 < 0,$$
$$f''(-2) = 32 > 0$$

and

$$f''(2) = 32 > 0.$$

Thus, $f(0)$ is a local maximum and $f(-2)$ and $f(2)$ are local minima. We show a graph of $y = f(x)$ in Figure 3.53.

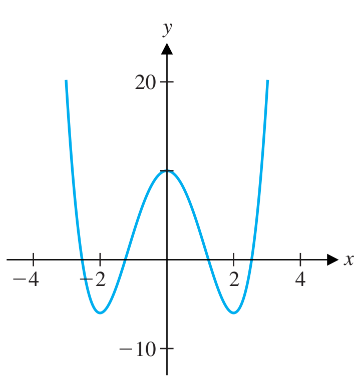


Figure 3.53
$y = x^4 - 8x^2 + 10.$

Remark 5.1

If $f''(c) = 0$ or $f''(c)$ is undefined, the second derivative test yields no conclusion. That is, $f(c)$ may be a local maximum, a local minimum or neither. In this event, we must rely solely on first derivative information (i.e., the first derivative test) to determine if $f(c)$ is a local extremum or not. We illustrate this with the following example.

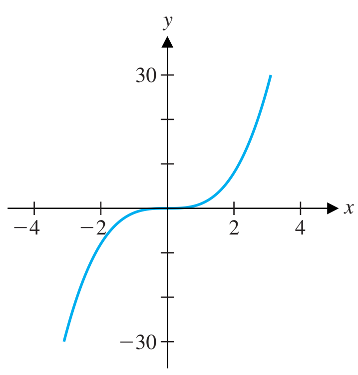


Figure 3.54a
$y = x^3$.

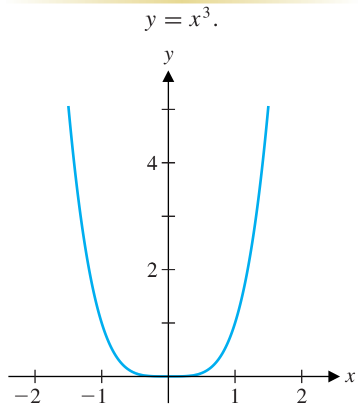


Figure 3.54b
$y = x^4$.

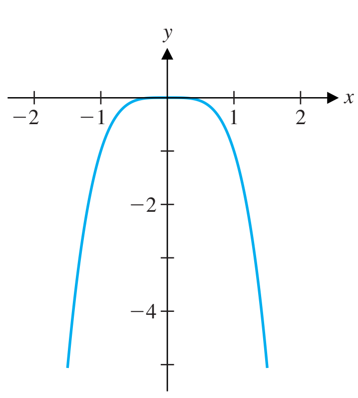
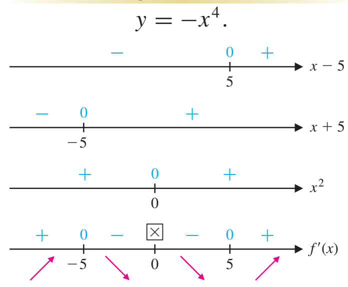


Figure 3.54c
$y = -x^4$.



Example 5.5 **Functions for Which the Second Derivative Test Is Inconclusive**

Use the second derivative test to try to classify any local extrema for (a) $f(x) = x^3$, (b) $g(x) = x^4$ and (c) $h(x) = -x^4$.

Solution (a) Note that $f'(x) = 3x^2$ and $f''(x) = 6x$. So, the only critical number is $x = 0$ and $f''(0) = 0$, also. We leave it as an exercise to show that the point (0, 0) is not a local extremum (see Figure 3.54a).
(b) We have $g'(x) = 4x^3$ and $g''(x) = 12x^2$. Again, the only critical number is $x = 0$ and $g''(0) = 0$. In this case, though, $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$. So, by the first derivative test, (0, 0) is a local minimum (see Figure 3.54b).
(c) Finally, we have $h'(x) = -4x^3$ and $h''(x) = -12x^2$. Once again, the only critical number is $x = 0$, $h''(0) = 0$ and we leave it as an exercise to show that (0, 0) is a local maximum for h (see Figure 3.54c).

We can use first and second derivative information to help produce a meaningful graph of a function, as in the following example.

Example 5.6 **Drawing a Graph of a Rational Function**

Draw a graph of $f(x) = x + \dfrac{25}{x}$, showing all significant features.

Solution The first thing that you should notice here is that the domain of f consists of all real numbers other than $x = 0$. Next, we have

$$f'(x) = 1 - \frac{25}{x^2} = \frac{x^2 - 25}{x^2} \qquad \text{Add the fractions.}$$

$$= \frac{(x-5)(x+5)}{x^2}.$$

So, the only two critical numbers are $x = -5, 5$. (Why is $x = 0$ **not** a critical number?)

Looking at the three factors in $f'(x)$, we get the number lines shown in the margin. Thus,

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -5) \cup (5, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (-5, 0) \cup (0, 5). & f \text{ decreasing.} \end{cases}$$

Further,

$$f''(x) = \frac{50}{x^3} \begin{cases} > 0, \text{ on } (0, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-\infty, 0). & \text{Concave down.} \end{cases}$$

Be careful here. There is **no** inflection point on the graph, even though the graph is concave up on one side of $x = 0$ and concave down on the other. (Why not?) We can now use either the first derivative test or the second derivative test to determine the local extrema. By the second derivative test,

$$f''(5) = \frac{50}{125} > 0$$

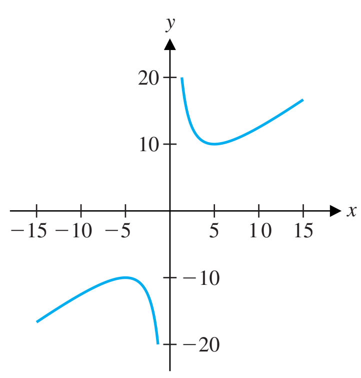


Figure 3.55
$y = x + \frac{25}{x}$.

and

$$f''(-5) = -\frac{50}{125} < 0,$$

so that there is a local minimum at $x = 5$ and a local maximum at $x = -5$. Finally , be-

near $x = 0$, since 0 is not in the domain of f. We have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x + \frac{25}{x}\right) = \infty$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x + \frac{25}{x}\right) = -\infty.$$

So, there is a vertical asymptote at $x = 0$. Putting together all of this information, we get the graph shown in Figure 3.55.

In example 5.6, we computed $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ to uncover the behavior of the function near $x = 0$. We needed to consider this since $x = 0$ was not in the domain of f. In the following example, we'll see that since $x = -2$ is not in the domain of f' (although it **is** in the domain of f), we must compute $\lim_{x\to -2^+} f'(x)$ and $\lim_{x\to -2^-} f'(x)$. This will tell us about the behavior of the tangent lines near $x = -2$.

Example 5.7 A Function with a Vertical Tangent Line at an Inflection Point

Draw a graph of $f(x) = (x + 2)^{1/5} + 4$, showing all significant features.

Solution First, notice that the domain of f is the entire real line. We also have

$$f'(x) = \frac{1}{5}(x + 2)^{-4/5} > 0, \text{ for } x \neq -2.$$

So, f is increasing everywhere, except at $x = -2$ [the only critical number , where $f'(-2)$ is undefined]. This also says that f has no local extrema. Further ,

$$f''(x) = -\frac{4}{25}(x + 2)^{-9/5} \begin{cases} > 0, \text{ on } (-\infty, -2) & \text{Concave up} \\ < 0, \text{ on } (-2, \infty). & \text{Concave down} \end{cases}$$

So, there is an inflection point at $x = -2$. In this case, $f'(x)$ is undefined at $x = -2$. Since -2 is in the domain of f, but not in the domain of f', we consider

$$\lim_{x\to -2^-} f'(x) = \lim_{x\to -2^-} \frac{1}{5}(x + 2)^{-4/5} = \infty$$

and

$$\lim_{x\to -2^+} f'(x) = \lim_{x\to -2^+} \frac{1}{5}(x + 2)^{-4/5} = \infty.$$

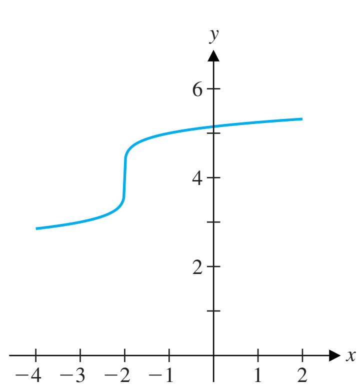


Figure 3.56
$y = (x + 2)^{1/5} + 4$.

This says that the graph has a vertical tangent line at $x = -2$. We obtain the graph shown in Figure 3.56.

EXERCISES 3.5

1. It is often said that a graph is concave up if it “holds water.” This is certainly true for parabolas like $y = x^2$, but is it true for graphs like $y = 1/x^2$? It is always helpful to put a difficult concept into everyday language, but the danger is in oversimplification. Do you think that “holds water” is helpful or can it be confusing? Give your own description of concave up, using everyday language. (Hint: One popular image involves smiles and frowns.)

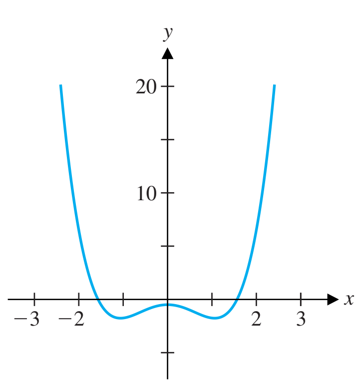
2. Find a reference book with the population of the United States since 1800. From 1800 to 1900, the numerical increase by decade increased. Argue that this means that the population curve is concave up. From 1960 to 1990, the numerical increase by decade has been approximately constant. Argue that this means that the population curve is near a point of zero concavity. Why does this not necessarily mean that we are at an inflection point? Argue that we should hope, in order to avoid overpopulation, that it is indeed an inflection point.

3. The goal of investing in the stock market is to buy low and sell high. But, how can you tell whether a price has peaked or not? Once a stock price goes down, you can see that it *was* at a peak but then it’s too late to do anything about it! Concavity can help. Suppose a stock price is increasing and the price curve is concave up. Why would you suspect that it will continue to rise? Is this a good time to buy? Now, suppose the price is increasing but the curve is concave down. Why should you be preparing to sell? Finally, suppose the price is decreasing. If the curve is concave up, should you buy or sell? What if the curve is concave down?

4. Suppose that $f(t)$ is the amount of money in your bank account at time t. Explain in terms of spending and saving what would cause $f(t)$ to be decreasing and concave down; increasing and concave up; decreasing and concave up.

In exercises 5–8, estimate the intervals where the function is concave up and concave down. (Hint: Estimate where the slope is increasing and decreasing.)

5.



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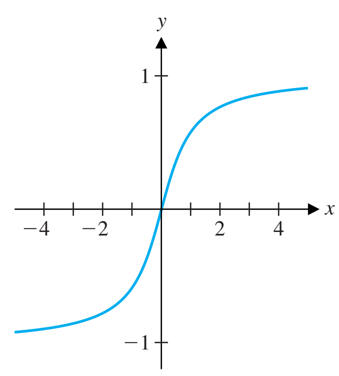

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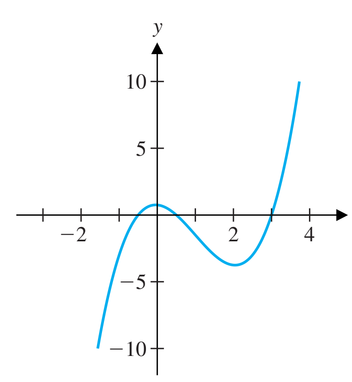
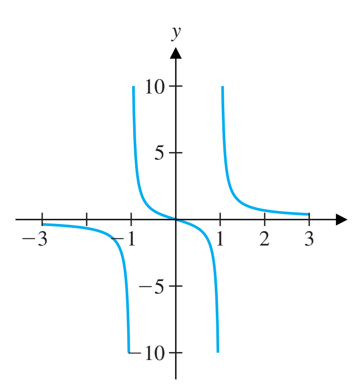


8.



In exercises 9–40, find the intervals of increase and decrease, all local extrema, the intervals of concavity, all inflection points and sketch a graph.

9. $f(x) = x^3 - 3x^2 + 4$

10. $f(x) = x^3 + 3x^2 - 6x$

11. $f(x) = x^4 - 2x^2 + 1$

12. $f(x) = x^4 + 4x - 2$

13. $f(x) = x + 1/x$

14. $f(x) = x^2 - 16/x$

15. $f(x) = x^3 - 6x + 1$

16. $f(x) = x^3 + 3x - 1$

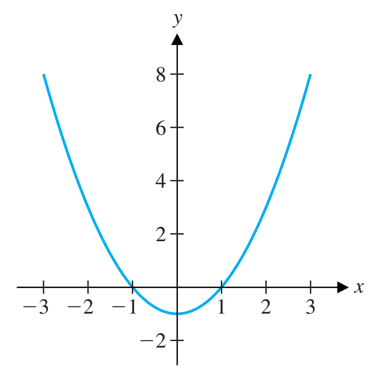
17. $f(x) = x^4 + 4x^3 - 1$
18. $f(x) = x^4 + 4x^2 + 1$
19. $f(x) = xe^{-x}$
20. $f(x) = e^{-x^2}$
21. $f(x) = x^2\sqrt{x^2 - 9}$
22. $f(x) = x\sqrt{x^2 - 9}$
23. $f(x) = (x^2 + 1)^{2/3}$
24. $f(x) = x \ln x$
25. $f(x) = \dfrac{x^2}{x^2 - 9}$
26. $f(x) = \dfrac{x}{x + 2}$
27. $f(x) = \sin x + \cos x$
28. $f(x) = x + \cos x$
29. $f(x) = e^{-x} \sin x$
30. $f(x) = e^{-2x} \cos x$
31. $f(x) = x^{3/4} - 4x^{1/4}$
32. $f(x) = x^{2/3} - 4x^{1/3}$
33. $f(x) = \sqrt[3]{2x^2 - 1}$
34. $f(x) = \sqrt{x^3 + 1}$
35. $f(x) = x^4 - 26x^3 + x$
36. $f(x) = 2x^4 - 11x^3 + 17x^2$
37. $f(x) = \dfrac{x^2 - 5x + 4}{x}$
38. $f(x) = \dfrac{x^2 - 1}{x}$
39. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
40. $f(x) = x^4 + 32x^3 - 0.02x^2 - 0.8x$

In exercises 41–46, sketch a graph with the given properties.

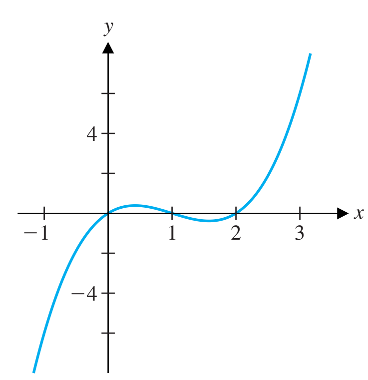
41. $f(0) = 2$, $f'(x) > 0$ for all x, $f'(0) = 1$, $f''(x) > 0$ for $x > 0$, $f''(x) < 0$ for $x < 0$, $f''(0) = 0$
42. $f(0) = 1$, $f'(x) \geq 0$ for all x, $f'(0) = 0$, $f''(x) > 0$ for $x > 0$, $f''(x) < 0$ for $x < 0$, $f''(0) = 0$
43. $f(0) = 0$, $f'(x) > 0$ for $x < -1$ and $-1 < x < 1$, $f'(x) < 0$ for $x > 1$, $f''(x) > 0$ for $x < -1, 0 < x < 1$ and $x > 1$, $f''(x) < 0$ for $-1 < x < 0$
44. $f(0) = 2$, $f'(x) > 0$ for all x, $f'(0) = 1$, $f''(x) > 0$ for $x < 0$, $f''(x) < 0$ for $x > 0$
45. $f(0) = 0$, $f(-1) = -1$, $f(1) = 1$, $f'(x) > 0$ for $x < -1$ and $0 < x < 1$, $f'(x) < 0$ for $-1 < x < 0$ and $x > 1$, $f''(x) < 0$ for $x < 0$ and $x > 0$
46. $f(1) = 0$, $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, $f''(x) < 0$ for $x < 1$ and $x > 1$

In exercises 47 and 48, estimate the intervals of increase and decrease, the locations of local extrema, intervals of concavity and locations of inflection points.

47.



48.



49. Repeat exercise 47 if the given graph is of $f'(x)$ instead of $f(x)$.
50. Repeat exercise 48 if the given graph is of $f'(x)$ instead of $f(x)$.
51. Suppose that $w(t)$ is the depth of water in a city's water reservoir. Which would be better news at time $t = 0$, $w''(0) = 0.05$ or $w''(0) = -0.05$ or would you need to know the value of $w'(0)$ to determine which is better?
52. Suppose that $T(t)$ is a sick person's temperature at time t. Which would be better news at time t, $T''(0) = 2$ or $T''(0) = -2$ or would you need to know the value of $T'(0)$ and $T(0)$ to determine which is better?
53. Suppose that a company that spends \$$x$ thousand on advertising sells \$$s(x)$ of merchandise, where $s(x) = -3x^3 + 270x^2 - 3600x + 18{,}000$. Find the value of x that maximizes the rate of change of sales. (Hint: Read the question carefully!)
54. For the sales function in exercise 53, find the inflection point and explain why in advertising terms this is the "point of diminishing returns."

55. Suppose that it costs a company $C(x) = 0.01x^2 + 40x + 3600$ dollars to manufacture x units of a product. For this **cost function,** the **average cost function** is $\bar{C}(x) = \frac{C(x)}{x}$. Find the value of x that minimizes the average cost.

56. In exercise 55, the cost function can be related to the efficiency of the production process. Explain why a cost function that is concave down indicates better efficiency than a cost function that is concave up.

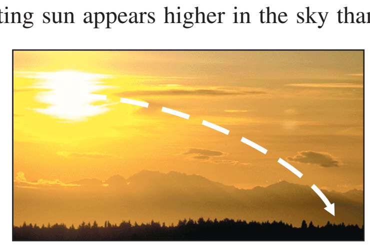
57. Show that there is an inflection point at $(0, 0)$ for any function of the form $f(x) = x^4 + cx^3$, where c is a nonzero constant. What role(s) does c play in the graph of $y = f(x)$?

58. The following examples show that there is not a perfect match between inflection points and places where $f''(x) = 0$. First, for $f(x) = x^6$, show that $f''(0) = 0$, but there is no inflection point at $x = 0$. Then, for $g(x) = x|x|$, show that there is an inflection point at $x = 0$, but that $g''(0)$ does not exist.

59. Give an example of a function showing that the following statement is false. If the graph of $y = f(x)$ is concave down for all x, the equation $f(x) = 0$ has at least one solution.

60. Determine if the following statement is true or false. If $f(0) = 1$, $f''(x)$ exists for all x and the graph of $y = f(x)$ is concave down for all x, the equation $f(x) = 0$ has at least one solution.

61. One basic principle of physics is that light follows the path of minimum time. Assuming that the speed of light in the earth's atmosphere decreases as altitude decreases, argue that the path that light follows is concave down. Explain why this means that the setting sun appears higher in the sky than it really is.



62. Prove Theorem 5.2 (the Second Derivative Test). (Hint: Think about what the definition of $f''(c)$ says when $f''(c) > 0$ or $f''(c) < 0$.)

63. The linear approximation that we defined in section 3.1 is the line having the same location and the same slope as the function being approximated. Since two points determine a line, two requirements (point, slope) are all that a linear function can satisfy. However, a quadratic function can satisfy three requirements since three points determine a parabola (and there are three constants in a general quadratic function $ax^2 + bx + c$). Suppose we want to define a **quadratic approximation** to $f(x)$ at $x = a$. Building on the linear approximation, the general form is $g(x) = f(a) + f'(a)(x - a) + c(x - a)^2$ for some constant c to be determined. In this way, show that $g(a) = f(a)$ and $g'(a) = f'(a)$. That is, $g(x)$ has the right position and slope at $x = a$. The third requirement is that $g(x)$ have the right concavity at $x = a$, so that $g''(a) = f''(a)$. Find the constant c that makes this true. Then, find such a quadratic approximation for each of the functions $\sin x$, $\cos x$ and e^x at $x = 0$. In each case, graph the original function, linear approximation and quadratic approximation and describe how close the approximations are to the original functions.

64. In this exercise, we explore a basic problem in genetics. Suppose that a species reproduces according to the following probabilities: p_0 is the probability of having no children, p_1 is the probability of having one offspring, p_2 is the probability of having two offspring, ..., p_n is the probability of having n offspring and n is the largest number of offspring possible. Explain why for each i, we have $0 \le p_i \le 1$ and $p_0 + p_1 + p_2 + \cdots + p_n = 1$. We define the function $F(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$. The smallest nonnegative solution of the equation $F(x) = x$ for $0 \le x \le 1$ represents the probability that the species becomes extinct. Show graphically that if $p_0 > 0$ and $F'(1) > 1$, then there is a solution of $F(x) = x$ with $0 < x < 1$. Thus, there is a positive probability of survival. However, if $p_0 > 0$ and $F'(1) < 1$, show that there are no solutions of $F(x) = x$ with $0 < x < 1$. (Hint: First show that F is increasing and concave up.)

3.6 OVERVIEW OF CURVE SKETCHING

You might be wondering why you need to spend any more time on curve sketching. We have already drawn numerous graphs over the last three sections. Besides, with a graphing calculator or computer algebra system at your disposal, why must you even consider drawing graphs by hand?

Of course, graphing calculators or computer algebra systems are powerful tools today in the study or application of mathematics. As the authors of this text, we admit it. We have