3.7 OPTIMIZATION

One of the most frequently heard questions in industry today is "Is this design optimal?" We are finally in a position to bring the power of the calculus to bear on questions of this sort. The tools that we've been developing over the last six sections are exactly what we need. Everywhere we look in business and industry today, we see people struggling to minimize waste and maximize productivity. Certainly some of this is done by using common-sense methods to reduce unnecessary expenditures, but much of this is accomplished by using a careful mathematical analysis of the problem. You might also wonder whether you can solve such problems simply by reading the answer from a calculator- or computer-generated graph. The answer to this is that you can do this sometimes. Remember that functions (even reasonably familiar ones) sometimes have behavior that is hidden in the standard graph you get from a computer or calculator. Further, the calculus can be used to solve more general questions, such as the one following example 7.1, which you cannot solve by using a machine alone.

This section contains a number of illustrations of how to apply the methods of calculus to problems requiring you to find a maximum or a minimum. Pay close attention to how we solve the problems. We start by giving a few general guidelines. (Notice that we said *guidelines* and not *rules*.) Do not memorize them, but rather keep these in mind as you work through the section.

- If there's a picture to draw, draw it! Don't try to envision how things look in your head. Put a picture down on paper and label it.
- Determine what the variables are and how they are related.
- Decide what quantity needs to be maximized or minimized.
- Write an expression for the quantity to be maximized or minimized in terms of only **one** variable. To do this, you may need to solve for any other variables in terms of this one variable.
- Determine the minimum and maximum allowable values (if any) of the variable you're using.
- Solve the problem. (Be sure to answer the question that is asked!)

We begin with a simple example where the goal is to accomplish what businesses face every day: getting the most from limited resources.

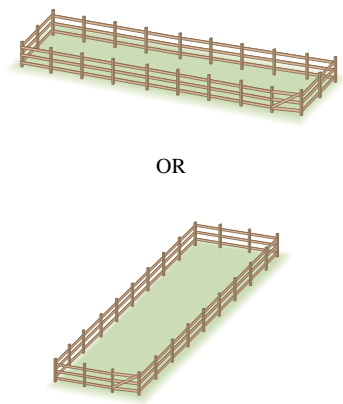


Figure 3.71
Possible plots.

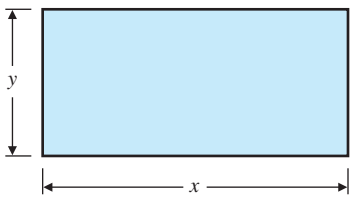


Figure 3.72
Rectangular plot.

Example 7.1 **Constructing a Rectangular Garden of Maximum Area**

You have 40 (linear) feet of fencing with which to enclose a rectangular space for a garden. Find the **largest** area that can be enclosed with this much fencing and the dimensions of the corresponding garden.

Solution First, note that there are lots of possibilities. We could enclose a plot that is very long but narrow, or one that is very wide, but not very long (see Figure 3.71). How are we to decide which configuration is optimal? We first draw a picture and label it appropriately (see Figure 3.72). The variables for a rectangular plot are length and width, which we name x and y, respectively.

We want to find the dimensions of the largest possible area, that is, maximize

$$A = xy.$$

You might immediately notice that this function has two variables and so, cannot be dealt with via the means we have available. However, there is another piece of

information that we can use here. If we want the maximum area, then **all** of the fencing must be used. This says that the perimeter of the resulting fence must be $40'$ and hence,

$$40 = \textit{perimeter} = 2x + 2y. \tag{7.1}$$

Notice that we can use (7.1) to solve for one variable (either one) in terms of the other. We have

$$2y = 40 - 2x$$

and hence,

$$y = 20 - x.$$

Substituting for y, we get that

$$A = xy = x(20 - x).$$

So, our job is to find the maximum value of the function

$$A(x) = x(20 - x).$$

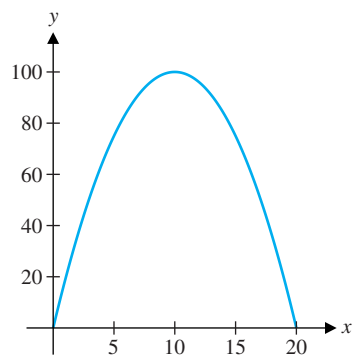


Figure 3.73
$y = x(20 - x)$.

Hold it! We know what you're thinking. You want to multiply out this expression. While it doesn't matter very much here (this was to be a very simple problem, after all), this is, in practice, a very bad idea. In short, don't do algebra just for the sake of doing it. Unless you have a clear picture of how it will affect your later work, leave the function alone!

Before we attempt to maximize $A(x)$, we need to determine if there is an interval in which x must lie. First, notice that since x is a distance, we must have $0 \le x$. Further, since the perimeter is $40'$, we must have $x \le 20$. (Why don't we have $x \le 40$?) So, we want to find the maximum value of $A(x)$ on the closed interval $[0, 20]$. This is now a simple problem. As a check on what a reasonable answer should be, we draw a graph of $y = A(x)$ (see Figure 3.73). Here, we have only drawn that portion of the graph on the interval $[0, 20]$ and simply adjusted the y-range to display all of the plotted values. The maximum value appears to occur around $x = 10$. Now, let's analyze the problem carefully. We have

$$\begin{aligned} A'(x) &= 1(20 - x) + x(-1) \\ &= 20 - 2x \\ &= 2(10 - x). \end{aligned}$$

So, the only critical number is $x = 10$ and this is in the interval under consideration. Recall that the maximum and minimum values of a continuous function on a closed and bounded interval must occur at either the endpoints or a critical number. This says that we need only compare the function values

$$A(0) = 0, \quad A(20) = 0 \quad \text{and} \quad A(10) = 100.$$

Thus, the maximum area that can be enclosed with $40'$ of fencing is $100\ \text{ft}^2$. We also want the dimensions of the plot. (This result is only of theoretical value, if we don't know how to construct the rectangle with the maximum area.) We have that $x = 10$ and

$$y = 20 - x = 10.$$

That is, the rectangle of perimeter $40'$ with maximum area is a **square** $10'$ on a side.

A more general problem that you can now solve is to show that (given a fixed perimeter) the rectangle of maximum area is a square. This is virtually identical to example 7.1 and is left as an exercise. It's worth noting here that this more general problem is one that cannot be solved by simply using a calculator to draw a graph. You'll need to use some calculus here.

Manufacturing companies routinely make countless decisions that affect the efficiency of their production processes. One decision that is surprisingly important is how to economically package their products for shipping. The following example is a simple version of this problem.

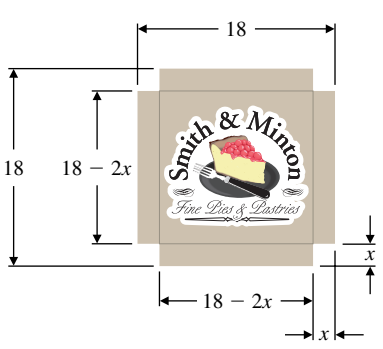


Figure 3.74a
A sheet of cardboard.

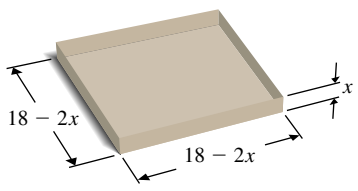


Figure 3.74b
Rectangular box.

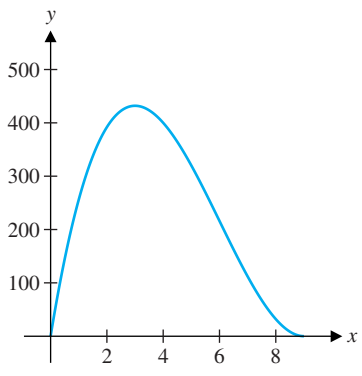


Figure 3.75
$y = 4x(9-x)^2$.

Example 7.2 **Constructing a Box of Maximum Volume**

A square sheet of cardboard 18″ on a side is made into an open box (i.e., there's no top), by cutting squares of equal size out of each corner (see Figure 3.74a) and folding up the sides along the dotted lines (see Figure 3.74b). Find the dimensions of the box with the maximum volume.

Solution Recall that the volume of a rectangular parallelepiped (a box) is given by

$$V = l \times w \times h.$$

From Figure 3.74b, we can see that the height is $h = x$, while the length and width are $l = w = 18 - 2x$. Thus, we can write the volume in terms of the one variable x as

$$V = V(x) = (18-2x)^2(x) = 4x(9-x)^2.$$

Once again, don't multiply this out, just out of habit. Notice that since x is a distance, we have $x \geq 0$. Further, we have $x \leq 9$, since cutting squares of side 9 out of each corner will cut up the entire sheet of cardboard. Thus, we are faced with finding the absolute maximum of the continuous function

$$V(x) = 4x(9-x)^2$$

on the closed interval $0 \leq x \leq 9$.

This should be a simple matter. The graph of $y = V(x)$ on the interval $[0, 9]$ is seen in Figure 3.75. From the graph, the maximum volume seems to be somewhat over 400 and seems to occur around $x = 3$. Now, we solve the problem precisely. We have

$$\begin{aligned} V'(x) &= 4(9-x)^2 + 4x(2)(9-x)(-1) && \text{Product rule and chain rule.} \\ &= 4(9-x)[(9-x) - 2x] && \text{Factor out } 4(9-x). \\ &= 4(9-x)(9-3x). \end{aligned}$$

So, V has two critical numbers: 3 and 9 and these are both in the interval under consideration. We now need only compare the value of the function at the endpoints and the critical numbers. We have

$$V(0) = 0, \quad V(9) = 0 \quad \text{and} \quad V(3) = 432.$$

Obviously, the maximum possible volume is 432 cubic inches. We can achieve this volume if we cut squares of side 3″ out of each corner. You should note that this corresponds with what we expected from the graph of $y = V(x)$ in Figure 3.75. Finally, observe that the dimensions of this optimal box are 12″ long by 12″ wide by 3″ deep.

When a new building is built, it must be connected to existing telephone and power cables, water and sewer lines and paved roads. If the cables, water or sewer lines or road are curved, then it may not be obvious how to make the shortest (i.e., least expensive) connection possible. The next two examples consider the general problem of finding the shortest distance from a point to a curve.

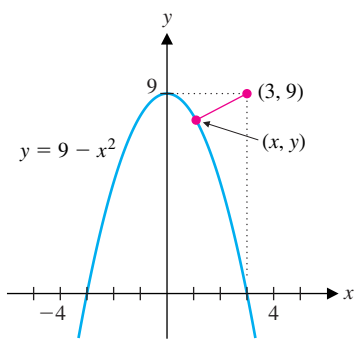


Figure 3.76
$y = 9 - x^2$.

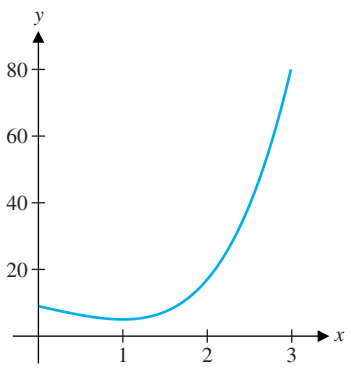


Figure 3.77
$y = (x - 3)^2 + x^4$.

Example 7.3 Finding the Closest Point to a Parabola

Find the point on the parabola $y = 9 - x^2$ closest to the point $(3, 9)$ (see Figure 3.76).

Solution Using the usual distance formula, we find that the distance between the point $(3, 9)$ and any point (x, y) is

$$d = \sqrt{(x - 3)^2 + (y - 9)^2}.$$

If the point (x, y) is on the parabola, note that its coordinates satisfy the equation $y = 9 - x^2$ and so, we can write the distance in terms of the single variable x as follows

$$\begin{aligned} d(x) &= \sqrt{(x - 3)^2 + [(9 - x^2) - 9]^2} \\ &= \sqrt{(x - 3)^2 + (-x^2)^2} \\ &= \sqrt{(x - 3)^2 + x^4}. \end{aligned}$$

Although we can certainly solve the problem in its present form, we can simplify our work by observing that $d(x)$ is minimized if and only if the quantity under the square root is minimized. (We leave it as an exercise to show why this is true.) So, instead of minimizing $d(x)$ directly, we minimize the **square** of $d(x)$:

$$f(x) = [d(x)]^2 = (x - 3)^2 + x^4$$

instead. Notice from Figure 3.76 that any point on the parabola to the left of the y-axis is farther away from $(3, 9)$ than is the point $(0, 9)$. Likewise, any point on the parabola below the x-axis is farther from $(3, 9)$ than is the point $(3, 0)$. So, it suffices to look for the closest point with

$$0 \le x \le 3.$$

This is the now familiar problem of minimizing a continuous function over a closed and bounded interval. See Figure 3.77 for a graph of $y = f(x)$ over the interval of interest. The minimum value of f (the square of the distance) seems to be around 10 and seems to occur near $x = 1$. We have

$$f'(x) = 2(x - 3)^1 + 4x^3 = 4x^3 + 2x - 6.$$

You might notice that $f'(x)$ factors. [One way to see this is to recognize that $x = 1$ is a zero of $f'(x)$, which makes $(x - 1)$ a factor. You can then get the factorization by dividing $f'(x)$ by $(x - 1)$.] We have

$$f'(x) = 2(x - 1)(2x^2 + 2x + 3).$$

So, $x = 1$ is a critical number. In fact it's the only critical number, since $(2x^2 + 2x + 3)$ has no zeros. (Why not?) We now need only compare the value of f at the endpoints and the critical number. We have

$$f(0) = 9, \quad f(3) = 81 \quad \text{and} \quad f(1) = 5.$$

Thus, the minimum value of $f(x)$ is 5. This says that the minimum distance from the point $(3, 9)$ to the parabola is $\sqrt{5}$ and the closest point on the parabola is $(1, 8)$. Again, notice that this corresponds with what we expected from the graph of $y = f(x)$.

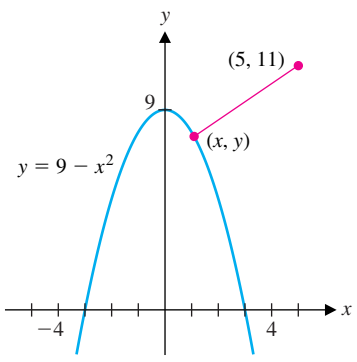


Figure 3.78
$y = 9 - x^2$.

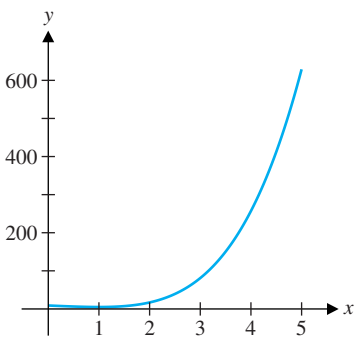


Figure 3.79
$y = f(x) = [d(x)]^2$.

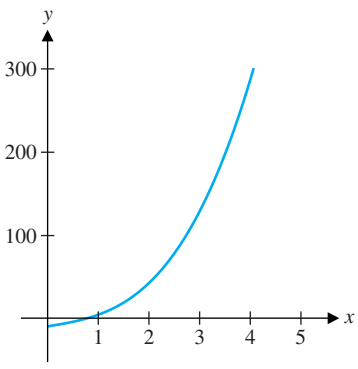


Figure 3.80
$y = f'(x)$.

The following example is very similar to example 7.3, except that we need to use approximate methods to find the critical number.

Example 7.4 Finding Minimum Distance Approximately

Find the point on the parabola $y = 9 - x^2$ closest to the point (5, 11) (see Figure 3.78).

Solution As in example 7.3, we want to minimize the distance from a fixed point [in this case, the point (5, 11)] to a point (x, y) on the parabola. Using the distance formula, the distance from any point (x, y) on the parabola to the point (5, 11) is

$$\begin{aligned} d &= \sqrt{(x-5)^2 + (y-11)^2} \\ &= \sqrt{(x-5)^2 + [(9-x^2)-11]^2} \\ &= \sqrt{(x-5)^2 + (x^2+2)^2}. \end{aligned}$$

Again, it is equivalent (and simpler) to minimize the quantity under the square root:

$$f(x) = [d(x)]^2 = (x-5)^2 + (x^2+2)^2.$$

As in example 7.3, we can see from Figure 3.78 that any point on the parabola to the left of the y-axis is farther from (5, 11) than is (0, 9). Likewise, any point on the parabola to the right of $x = 5$ is farther from (5, 11) than is (5, −16). Thus, we minimize $f(x)$ for $0 \le x \le 5$. In Figure 3.79 we see a graph of $y = f(x)$ on the interval of interest. The minimum value of f seems to be about 25 and seems to occur around $x = 1$. We can make this more precise as follows:

$$\begin{aligned} f'(x) &= 2(x-5) + 2(x^2+2)(2x) \\ &= 4x^3 + 10x - 10. \end{aligned}$$

Unlike in example 7.3, the expression for $f'(x)$ has no obvious factorization. Our only choice then is to find zeros of $f'(x)$ approximately. First, we draw a graph of $y = f'(x)$ on the interval of interest (see Figure 3.80). The only zero appears to be slightly less than 1. Using $x_0 = 1$ as an initial guess in Newton's method (applied to $f'(x) = 0$) or using your calculator's solver, you should get the approximate root $x_c \approx 0.79728$. We now compare function values:

$$f(0) = 29, \quad f(5) = 729 \quad \text{and} \quad f(x_c) \approx 24.6.$$

Thus, the minimum distance from (5, 11) to the parabola is approximately $\sqrt{24.6} \approx 4.96$ and the closest point on the parabola is located at approximately (0.79728, 8.364). Notice that this example was essentially the same as example 7.3, except that we needed to use approximate methods, as there was no algebra available to us.

Notice that in both Figures 3.76 and 3.78 the shortest path appears to be perpendicular to the tangent line to the curve at the point where the path intersects the curve. We leave it as an exercise to prove that this is always the case. This observation is an important geometric principle that you can apply to many problems of this type.

Remark 7.1

At this point you might be tempted to forgo the comparison of function values at the endpoints and at the critical numbers. After all, in all of the examples we have seen so far, the desired maximizer or minimizer (i.e., the point at which the maximum or minimum occurred) was the only critical number

in the interval under consideration. You might just suspect that if there is only one critical number, it will correspond to the maximizer or minimizer for which you are searching. Unfortunately, this is not always the case. In 1945, two prominent aeronautical engineers derived a function to model the range of an aircraft. Their intention was to use this function to discover how to maximize the range. They found a critical number of this function (corresponding to distributing virtually all of the plane's weight in the wings) and reasoned that it gave the **maximum** range. The result was the famous "Flying Wing" aircraft. Some years later, it was argued that they had in fact stumbled onto a critical number corresponding to a local **minimum** of the range function. In the engineers' defense, they did not have easy, accurate computational power at their fingertips, as we do today. Remarkably, this design strongly resembles the modern B-2 Stealth bomber. This story came out as controversy brewed over the production of the B-2 (see *Science*, **244,** pp. 650–651, May 12, 1989, also see the *Monthly* of the Mathematical Association of America, October, 1993, pp. 737–738). The moral should be crystal clear: check the function values at the critical numbers and at the endpoints. Do not simply *assume* (even by virtue of having only one critical number) that a given critical number corresponds to the extremum you are seeking.

Next, we consider an optimization problem that cannot be restricted to a closed interval. We will use the fact that for a continuous function, a single local extremum must be an absolute extremum. (Think about why this is true.)

Example 7.5 **Designing a Soda Can That Uses a Minimum Amount of Material**

A soda can is to hold 12 fluid ounces. Find the dimensions that will minimize the amount of material used in its construction, assuming that the thickness of the material is uniform (i.e., the thickness of the aluminum is the same everywhere in the can).

Figure 3.81
Soda can.

Solution First, we draw and label a picture of a typical soda can (see Figure 3.81). Here we have drawn a right circular cylinder of height h and radius r. Assuming uniform thickness of the aluminum, notice that we can minimize the amount of material by minimizing the surface area of the can. We have

$$\begin{aligned} area &= area\ of\ top + bottom + curved\ surface\ area \\ &= 2\pi r^2 + 2\pi rh. \end{aligned} \tag{7.2}$$

We can eliminate one of the variables by using the fact that the volume (using 1 fluid ounce ≈ 1.80469 in.^{3}) must be

$$12 \text{ fluid ounces} = 12 \text{ fl oz} \times 1.80469 \frac{\text{in.}^3}{\text{fl oz}} = 21.65628 \text{ in.}^3.$$

Further, the volume of a right circular cylinder is

$$vol = \pi r^2 h,$$

and so,

$$h = \frac{vol}{\pi r^2} = \frac{21.65628}{\pi r^2}. \tag{7.3}$$

Thus, from (7.2) and (7.3), the surface area is

$$\begin{aligned} A(r) &= 2\pi r^2 + 2\pi r \frac{21.65628}{\pi r^2} \\ &= 2\pi \left(r^2 + \frac{21.65628}{\pi r} \right). \end{aligned}$$

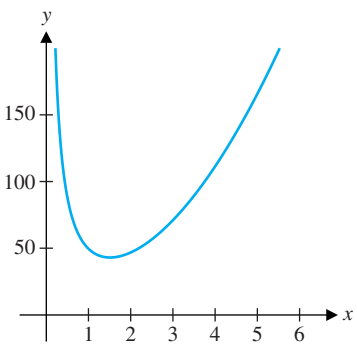


Figure 3.82
$y = A(r)$.

So, our job is to minimize $A(r)$, but here, there is no closed and bounded interval of allowable values. In fact, all we can say is that $r > 0$. We can have r as large or small as you can imagine, simply by taking h to be correspondingly small or large, respectively. That is, we must find the absolute minimum of $A(r)$ on the open and unbounded interval $(0, \infty)$. To get an idea of what a plausible answer might be, we graph $y = A(r)$ (see Figure 3.82). There appears to be a local (and possibly an absolute) minimum located between $r = 1$ and $r = 2$. This minimum value seems to be slightly less than 50. Next, we compute

$$\begin{aligned} A'(r) &= \frac{d}{dr}\left[2\pi\left(r^2 + \frac{21.65628}{\pi r}\right)\right] \\ &= 2\pi\left(2r - \frac{21.65628}{\pi r^2}\right) \\ &= 2\pi\left(\frac{2\pi r^3 - 21.65628}{\pi r^2}\right). \end{aligned}$$

Notice that the only critical numbers are those for which the numerator of the fraction is zero:

$$0 = 2\pi r^3 - 21.65628.$$

This occurs if and only if

$$r^3 = \frac{21.65628}{2\pi}$$

and hence, the only critical number is

$$r = r_c = \sqrt[3]{\frac{21.65628}{2\pi}} \approx 1.510548.$$

Further, notice that for $0 < r < r_c$, $A'(r) < 0$ and for $r_c < r$, $A'(r) > 0$. That is, $A(r)$ is decreasing on the interval $(0, r_c)$ and increasing on the interval (r_c, ∞). Thus, $A(r)$ has not only a local minimum, but also an absolute minimum at $r = r_c$. Notice, too, that this corresponds with what we expected from the graph of $y = A(r)$ in Figure 3.82. This says that the can that uses a minimum of material has radius $r_c \approx 1.510548$ and height

$$h = \frac{21.65628}{\pi r_c^2} \approx 3.0211.$$

Note that the optimal can from example 7.5 is "square," in the sense that the height (h) equals the diameter ($2r$). Also, we should observe that example 7.5 is not completely realistic. A standard 12-ounce soda can has a radius of about 1.156″. You should review example 7.5 to find any unrealistic assumptions we made. We study the problem of designing a soda can further in the exercises.

In our final example, we consider a problem where most of the work must be done numerically and graphically.

Example 7.6 **Minimizing the Cost of Highway Construction**

The state wants to build a new stretch of highway to link an existing bridge with a turnpike interchange, located 8 miles to the east and 8 miles to the south of the bridge. There is a 5-mile-wide stretch of marsh land adjacent to the bridge that must be crossed (see Figure 3.83). Given that the highway costs $10 million per mile to build over the marsh and only $7 million to build over dry land, how far to the east of the bridge should the highway be when it crosses out of the marsh?

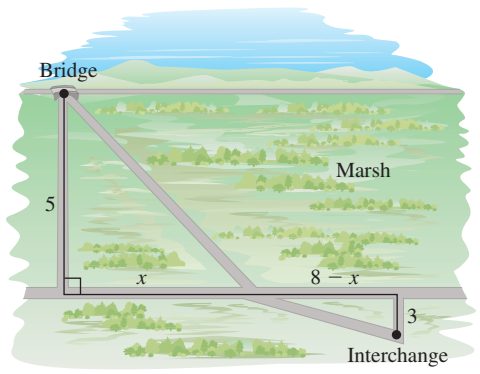


Figure 3.83
A new highway.

Solution You might guess that the highway should cut directly across the marsh, so as to minimize the amount built over marsh land. We will use the calculus to decide this question. We let x represent the distance in question (see Figure 3.83). Then, the interchange lies $(8 - x)$ miles to the east of the point where the highway leaves the marsh. Thus, the total cost (in millions of dollars) is

$$cost = 10\ (distance\ across\ marsh) + 7\ (distance\ across\ dry\ land).$$

Using the Pythagorean Theorem on the two right triangles seen in Figure 3.83, we get the cost function

$$C(x) = 10\sqrt{x^2 + 25} + 7\sqrt{(8 - x)^2 + 9}.$$

From Figure 3.83, you can see that $0 \le x \le 8$. (Making $x < 0$ means going **away** from the interchange and making $x > 8$ means the highway would be beyond the interchange at the point at which it exits the marsh.) So, we have the routine problem of minimizing a continuous function $C(x)$ over the closed and bounded interval $[0, 8]$. Or is it really that routine? First, we draw a graph of $y = C(x)$ on the interval in question to get an idea of a plausible answer (see Figure 3.84). From the graph, the minimum appears to be slightly less than 100 and occurs around $x = 4$. To solve the problem precisely, we need to compute the derivative

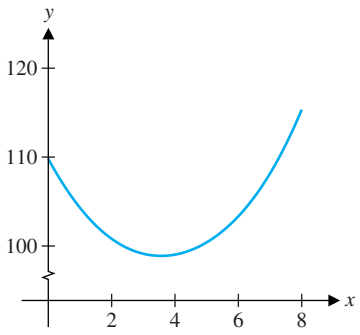


Figure 3.84
$y = C(x)$.

$$\begin{aligned} C'(x) &= \frac{d}{dx}\left[10\sqrt{x^2 + 25} + 7\sqrt{(8 - x)^2 + 9}\right] \\ &= 5(x^2 + 25)^{-1/2}(2x) + \frac{7}{2}[(8 - x)^2 + 9]^{-1/2}(2)(8 - x)^1(-1) \\ &= \frac{10x}{\sqrt{x^2 + 25}} - \frac{7(8 - x)}{\sqrt{(8 - x)^2 + 9}}. \end{aligned}$$

First, note that the only critical numbers are where $C'(x) = 0$. (Why?) The only way to find these is to approximate them. A graph of $y = C'(x)$ is seen in Figure 3.85. The only zero of $C'(x)$ on the interval $[0, 8]$ appears to be between $x = 3$ and $x = 4$. We approximate this zero numerically (e.g., with bisections or your calculator's solver), to obtain the approximate critical number

$$x_c \approx 3.560052.$$

Now, we need only compare the value of $C(x)$ at the endpoints and at this one critical number:

$$C(0) \approx \$109.8 \text{ million},$$
$$C(8) \approx \$115.3 \text{ million}$$

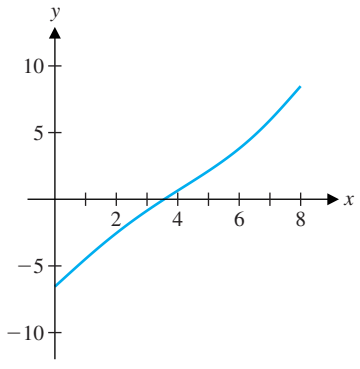


Figure 3.85
$y = C'(x)$.

and

$$C(x_c) \approx \$98.9 \text{ million.}$$

So, by using a little calculus, we can save the taxpayers more than \$10 million over cutting directly across the marsh and more than \$16 million over cutting diagonally across the marsh (not a bad reward for a few minutes of work).

The examples that we've presented in this section together with the exercises should give you the basis for solving a wide range of applied optimization problems. Be careful when solving these problems to draw good pictures, as well as graphs of the functions involved. Make sure that the answer you obtain computationally is consistent with what you expect from the graphs. If not, further analysis is required to see what you have missed. Also, make sure that the solution makes physical sense, when appropriate. All of these multiple checks on your work will reduce the likelihood of error.

EXERCISES 3.7

1. Suppose some friends complain to you that they can't work any of the problems in this section. When you ask to see their work, they say that they couldn't even get started. In the text, we have emphasized sketching a picture and defining variables. Part of the benefit of this is to help you get started writing something (anything) down. Do you think this advice helps? What do you think is the most difficult aspect of these problems? Give your friends the best advice you can.

2. We have neglected one important aspect of optimization problems, an aspect that might be called "common sense." For example, suppose you are finding the optimal dimensions for a fence and the mathematical solution is to build a square fence of length $10\sqrt{5}$ feet on each side. At the meeting with the carpenter who is going to build the fence, what length fence do you order? Why is $10\sqrt{5}$ probably not the best way to express the length? We can approximate $10\sqrt{5} \approx 22.36$. What would you tell the carpenter? Suppose the carpenter accepts measurements down to the inch. Assuming that the building constraint was that the perimeter of the fence could not exceed a certain figure, why should you truncate to $22'4''$ instead of rounding up to $22'5''$?

3. In example 7.3, we stated that $d(x) = \sqrt{f(x)}$ is minimized by exactly the same x-value(s) as $f(x)$. Use the fact that $\sqrt{x}$ is an increasing function to explain why this is true.

4. Suppose that $f(x)$ is a continuous function with a single critical number, and $f(x)$ has a local minimum at that critical number. Explain why $f(x)$ also has an absolute minimum at the critical number.

5. Give an example showing that $f(x)$ and $\sin(f(x))$ need not be minimized by the same x-values.

6. True or false: $e^{f(x)}$ is minimized by exactly the same x-value(s) as $f(x)$.

7. A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. The enclosed area is to equal 1800 ft^2. Find the minimum perimeter and the dimensions of the corresponding enclosure.

8. A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. There is 96 feet of fencing available. Find the maximum enclosed area and the dimensions of the corresponding enclosure.

9. A two-pen corral is to be built. The outline of the corral forms two identical adjoining rectangles. If there is 120 ft of fencing available, what dimensions of the corral will maximize the enclosed area?

10. A showroom for a department store is to be rectangular with walls on three sides, 6-ft door openings on the two facing sides and a 10-ft door opening on the remaining wall. The showroom is to have 800 ft^2 of floor space. What dimensions will minimize the length of wall used?

11. Show that the rectangle of maximum area for a given perimeter P is always a square.

12. Show that the rectangle of minimum perimeter for a given area A is always a square.

13. Find the point on the curve $y = x^2$ closest to the point $(0, 1)$.

14. Find the point on the curve $y = x^2$ closest to the point $(3, 4)$.

15. Find the point on the curve $y = \cos x$ closest to the point $(0, 0)$.

16. Find the point on the curve $y = \cos x$ closest to the point (1, 1).

17. In exercises 13 and 14, find the slope of the line through the given point and the closest point on the given curve. Show that, in each case, this line is perpendicular to the tangent line to the curve at the given point.

18. Sketch the graph of some function $y = f(x)$ and mark a point not on the curve. Explain why the result of exercise 17 is true. (Hint: Pick a point for which the joining line is *not* perpendicular and explain why you can get closer.)

19. A box with no top is to be built by taking a 6″-by-10″ sheet of cardboard and cutting x-in. squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

20. A box with no top is to be built by taking a 12″-by-16″ sheet of cardboard and cutting x-in. squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

21. A water line runs east-west. A town wants to connect two new housing developments to the line by running lines from a single point on the existing line to the two developments. One development is 3 miles south of the existing line, the other development is 4 miles south of the existing line and 5 miles east of the first development. Find the place on the existing line to make the connection to minimize the total length of new line.

22. A company needs to run an oil pipeline from an oil rig 25 miles out to sea to a storage tank that is 5 miles inland. The shoreline runs east-west and the tank is 8 miles east of the rig. Assume it costs \$50 thousand dollars per mile to construct the pipeline underwater and \$20 thousand dollars per mile to construct the pipeline on land. The pipeline will be built in a straight line from the rig to a selected point on the shoreline, then in a straight line to the storage tank. What point on the shoreline should be selected to minimize the total cost of the pipeline?

23. A city wants to build a new section of highway to link an existing bridge with an existing highway interchange, which lies 8 miles to the east and 10 miles to the south of the bridge. The first 4 miles south of the bridge is marsh land. Assume that the highway costs \$5 million dollars per mile over marsh and \$2 million dollars per mile over dry land. The highway will be built in a straight line from the bridge to the edge of the marsh, then in a straight line to the existing interchange. At what point should the highway emerge from the marsh in order to minimize the total cost of the new highway? How much is saved over building the new highway in a straight line from the bridge to the interchange? (Hint: Use similar triangles to find the point on the boundary corresponding to a straight path and evaluate your cost function at that point.)

24. After construction has begun on the highway in exercise 23, the cost per mile over marsh land is reestimated at \$6 million dollars. Find the point on the marsh/dry land boundary that would minimize the total cost of the highway with the new cost function. If the construction is too far along to change paths, how much extra cost is there in using the path from exercise 23?

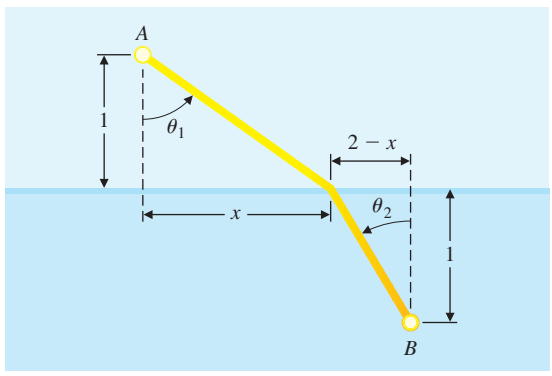
25. After construction has begun on the highway in exercise 23, the cost per mile over dry land is reestimated at \$3 million dollars. Find the point on the marsh/dry land boundary that would minimize the total cost of the highway with the new cost function. If the construction is too far along to change paths, how much extra cost is there in using the path from exercise 23?

26. In an endurance contest, contestants 2 miles at sea need to reach a location 2 miles inland and 3 miles east (the shoreline runs east-west). Assume a contestant can swim 4 mph and run 10 mph. To what point on the shoreline should the person swim to minimize the total time? Compare the amount of time spent in the water and the amount of time spent on land.

27. Suppose that light travels from point A to point B as shown in the figure. (Recall that light always follows the path that minimizes time.) Assume that the velocity of light above the boundary line is v_1 and the velocity of light below the boundary is v_2. Show that the total time to get from point A to point B is

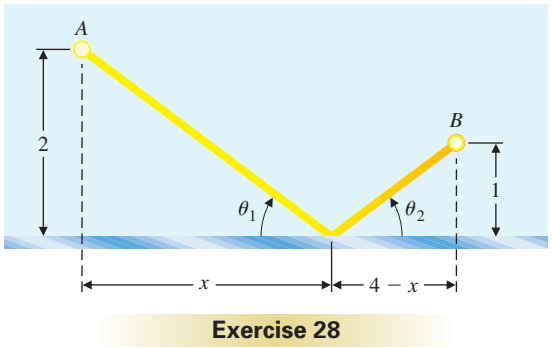
$$T(x) = \frac{\sqrt{1+x^2}}{v_1} + \frac{\sqrt{1+(2-x)^2}}{v_2}.$$

Write out the equation $T'(x) = 0$, replace the square roots using the sines of the angles in the figure and derive Snell's Law $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$.



Exercise 27

28. Suppose that light reflects off a mirror to get from point A to point B as indicated in the figure. Assuming a constant velocity of light, we can minimize time by minimizing the distance traveled. Find the point on the mirror that minimizes the distance traveled. Show that the angles in the figure are equal (the angle of incidence equals the angle of reflection).



Exercise 28

29. A soda can is to hold 12 fluid ounces. Suppose that the bottom and top are twice as thick as the sides. Find the dimensions of the can which minimize the amount of material used. (Hint: Instead of minimizing surface area, minimize the cost, which is proportional to the product of the thickness and the area.)

30. Following example 7.5, we mentioned that real soda cans have a radius of about $1.156''$. Show that this radius minimizes the cost if the top and bottom are 2.23 times as thick as the sides.

31. The human cough is intended to increase the flow of air to the lungs, by dislodging any particles blocking the windpipe and changing the radius of the pipe. Suppose a windpipe under no pressure has radius r_0. The velocity of air through the windpipe at radius r is approximately $V(r) = cr^2(r_0 - r)$ for some constant c. Find the radius that maximizes the velocity of air through the windpipe. Does this mean the windpipe expands or contracts?

32. To supply blood to all parts of the body, the human artery system must branch repeatedly. Suppose an artery of radius r branches off from an artery of radius R ($R > r$) at an angle θ. The energy lost due to friction is approximately

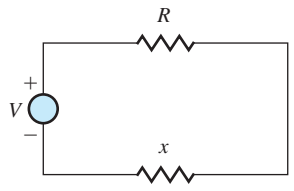
$$E(\theta) = \frac{\csc\theta}{r^4} + \frac{1 - \cot\theta}{R^4}.$$

Find the value of θ that minimizes the energy loss.

33. In an electronic device, individual circuits may serve many purposes. In some cases, the flow of electricity must be controlled by reducing the power instead of amplifying it. In the circuit shown below, a voltage V volts and resistance R ohms are given. We want to determine the size of the remaining resistor (x ohms). The power absorbed by the circuit is

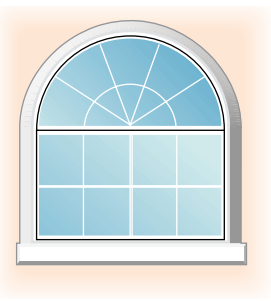
$$p(x) = \frac{V^2 x}{(R + x)^2}.$$

Find the value of x that maximizes the power absorbed.



34. In an AC circuit with voltage $V(t) = v \sin 2\pi ft$, a voltmeter actually shows the average (root-mean-square) voltage of $v/\sqrt{2}$. If the frequency is $f = 60$ (Hz) and the meter registers 115 volts, find the maximum voltage reached. [Hint: this is "obvious" if you determine v and think about the graph of $V(t)$.]

35. A Norman window has the outline of a semicircle on top of a rectangle, as shown below. Suppose there is $8 + \pi$ feet of wood trim available. Discuss why a window designer might want to maximize the area of the window. Find the dimensions of the rectangle (and, hence, the semicircle) that will maximize the area of the window.



36. Suppose a wire 2 ft long is to be cut into two pieces, each of which will be formed into a square. Find the size of each piece to maximize the total area of the two squares.

37. An advertisement consists of a rectangular printed region plus 1-in. margins on the sides and 2-in. margins at top and bottom. If the area of the printed region is to be 92 in.^{2}, find the dimensions of the printed region and overall advertisement that minimize the total area.

38. An advertisement consists of a rectangular printed region plus 1-in. margins on the sides and 1.5-in. margins at top and bottom. If the total area of the advertisement is to be 120 in.^{2}, what dimensions should the advertisement be to maximize the area of the printed region?

39. A farmer relishes growing cucumbers. A government subsidy is available to growers of more than 2 acres of cucumbers. The farmer has 10 acres available to plant. The net income for the farmer planting A acres is $2A^3 - 33A^2 + 108A - 310$ dollars. Determine the best strategy for the farmer.

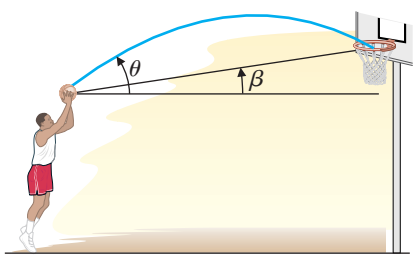
40. The owners of the Big Belly Deli run a special on bologna sandwiches, with a limit of six per customer. The amount of heartburn obtained from n sandwiches is $h(n) = n^3 + 12n - 11$ gasu's (gastronomical units). How many sandwiches gives the worst case of heartburn?

41. In sports where balls are thrown or hit, the ball often finishes at a different height than it starts. Examples include a downhill golf shot and a basketball shot. In the diagram, a ball is released at an angle θ and finishes at an angle β above the horizontal (for downhill trajectories, β would be negative). Neglecting air

resistance and spin, the horizontal range is given by

$$R = \frac{2v^2 \cos^2 \theta}{g}(\tan\theta - \tan\beta)$$

if the initial velocity is v and g is the gravitational constant. In the following cases, find θ to maximize R (treat v and g as constants): (a) $\beta = 10^\circ$, (b) $\beta = 0^\circ$ and (c) $\beta = -10^\circ$. Verify that $\theta = 45^\circ + \beta^\circ/2$ maximizes the range.



42. For your favorite sport in which it is important to throw or hit a ball a long way, explain the result of exercise 41 in the language of your sport.

43. A ball is thrown from $s = b$ to $s = a$ (where $a < b$) with initial speed v_0. Assuming that air resistance is proportional to speed, the time it takes the ball to reach $s = a$ is

$$T = -\frac{1}{c}\ln\left(1 - c\,\frac{b-a}{v_0}\right)$$

where c is a constant of proportionality. A baseball player is 300 ft from home plate and throws a ball directly toward home plate with an initial speed of 125 ft/s. Suppose that $c = 0.1$. How long does it take the ball to reach home plate? Another player standing x feet from home plate has the option of catching the ball and then, after a delay of 0.1s, relaying the ball toward home plate with an initial speed of 125 ft/s. Find x to minimize the total time for the ball to reach home plate. Is the straight throw or the relay faster? What, if anything, changes if the delay is 0.2 s instead of 0.1 s?

44. For the situation in exercise 43, for what length delay is it equally fast to have a relay and not have a relay? Do you think that you could catch and throw a ball in such a short time? Why do you think it is considered important to have a relay option in baseball?

45. Repeat exercises 43 and 44 if the second player throws the ball with initial speed 100 ft/s.

46. For a delay of 0.1 s in exercise 43, find the value of the initial speed of the second player's throw for which it is equally fast to have a relay and not have a relay.

47. In exercise 64 in section 3.3, you did a preliminary investigation of Kepler's wine cask problem. You showed that a height-to-diameter ratio (x/y) of $\sqrt{2}$ for a cylindrical barrel will maximize the volume (see Figure a). However, real wine casks are bowed out (like beer kegs). Kepler continued his investigation of wine cask construction by approximating a cask with the straight-sided barrel in Figure b. It can be shown (we told you Kepler was good!) that the volume of this barrel is $V = \frac{2}{3}\pi[y^2 + (w-y)^2 + y(w-y)]\sqrt{z^2 - w^2}$. Treating w and z as constants, show that $V'(y) = 0$ if $y = w/2$. Recall that such a critical point can correspond to a maximum or minimum of $V(y)$, but it also could correspond to something else (e.g., inflection point). To discover which one we have here, redraw Figure b to scale (show the correct relationship between $2y$ and w). In physical terms (think about increasing and decreasing y), argue that this critical point is neither a maximum nor minimum. Interestingly enough, such a nonextreme critical point would have a definite advantage to the Austrian vintners. Recall that their goal was to convert the measurement z into an estimate of the volume. The vintners would hope that small imperfections in the dimensions of the cask would have little effect on the volume. Explain why $V'(y) = 0$ means that small variations in y would convert to small errors in the volume V.

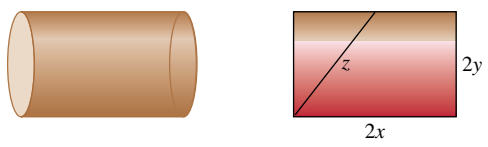


Figure a

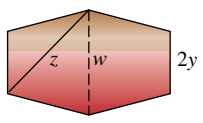


Figure b

48. The following problem is fictitious, but involves the kind of ambiguity that can make technical jobs challenging. The Band Candy Company decides to liquidate one of its candies. The company has 600,000 bags in inventory that it wants to sell. The candy had cost 35 cents per bag to manufacture and originally sold for 90 cents per bag. A marketing study indicates that if the candy is priced at p cents per bag, approximately $Q(p) = -p^2 + 40p + 250$ thousand bags will be sold. Your task as consultant is to recommend the best selling price for the candy. As such, you should do the following: (a) find p to maximize $Q(p)$; (b) find p to maximize $10pQ(p)$, which is the actual revenue brought in by selling the candy. Then form your opinion, based on your evaluation of the relative importance of getting rid of as much candy as possible and making the most money possible.