

3.8 RATES OF CHANGE IN APPLICATIONS

In this section, we round out our exposition of the derivative by presenting a collection of applications selected from a variety of fields. It has often been said that mathematics is the language of nature. Today, the concepts of calculus are being applied in virtually every field of human endeavor. The applications in this section represent but a small sampling of some elementary uses of the derivative. These are not all of the uses of the derivative nor are they necessarily the most important uses. Our intent is simply to present some interesting applications in a variety of settings.

Recall that the derivative of a function gives the instantaneous rate of change of that function. So, when you see the word *rate*, you should be thinking of a *derivative*. You can hardly pick up a newspaper without finding reference to some rates (e.g., inflation rate or interest rate). These can be thought of as derivatives. There are also many quantities with which you are familiar, but that you might not recognize as rates of change. Our first example is of this type.

Suppose that $Q(t)$ represents the electrical charge in a wire at time t . Then, the derivative $Q'(t)$ gives the **current** flowing through the wire. To see this, consider the cross section of a wire as shown in Figure 3.86. Between times t_1 and t_2 , the net charge passing through such a cross section is $Q(t_2) - Q(t_1)$. The **average current** (charge per unit time) over this time interval is then defined as

$$\frac{Q(t_2) - Q(t_1)}{t_2 - t_1}.$$

The **instantaneous current** $I(t)$ at any time t_1 can then be found by computing the limit

$$I(t_1) = \lim_{t \rightarrow t_1} \frac{Q(t) - Q(t_1)}{t - t_1}. \quad (8.1)$$

Notice that (8.1) is simply the alternative definition of derivative discussed in section 2.2. Thus, we have that $I(t) = Q'(t)$.

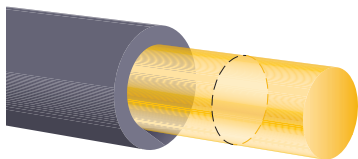


Figure 3.86
An electrical wire.

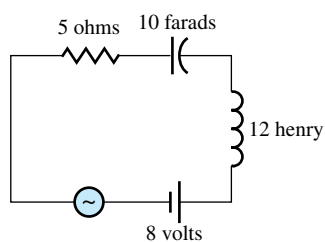


Figure 3.87
A simple electrical circuit.

Example 8.1 Modeling Electrical Current in a Wire

The electrical circuit shown in Figure 3.87 includes a 5-ohm resistor, a 12-henry inductor, a 10-farad capacitor and a battery supplying 8 volts of AC current modeled by the oscillating function $8 \sin 2t$, where t is measured in seconds. Find the current in the circuit at any time t .

Solution It can be shown (using the elementary laws of electricity) that the charge in this circuit is given by

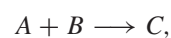
$$Q(t) = 12 \sin(4t - \pi/3) + 4 \sin 2t \text{ coulombs.}$$

The current is then

$$Q'(t) = 48 \cos(4t - \pi/3) + 8 \cos 2t \text{ amps (coulombs per second).}$$

The next example we offer comes from chemistry. It is very important for chemists to have a handle on the rate at which a given reaction proceeds. Reaction rates give chemists information about the nature of the chemical bonds being formed and broken, as well as information about the type and quantity of product to expect. A simple situation is depicted

in the schematic



which indicates that chemicals A and B (the *reactants*) combine to form chemical C (the *product*). Let $[C](t)$ denote the concentration (in moles per liter) of the product. The average reaction rate between times t_1 and t_2 is

$$\frac{[C](t_2) - [C](t_1)}{t_2 - t_1}.$$

The instantaneous reaction rate at any given time t_1 is then given by

$$\lim_{t \rightarrow t_1} \frac{[C](t) - [C](t_1)}{t - t_1} = \frac{d[C]}{dt}(t_1).$$

Depending on the details of the reaction, it is often possible to write down an equation relating the reaction rate $\frac{d[C]}{dt}$ to the concentrations of the reactants, $[A]$ and $[B]$.

Example 8.2 Modeling the Rate of a Chemical Reaction

In an **autocatalytic** chemical reaction, the reactant and the product are the same. The reaction continues until some saturation level is reached. From experimental evidence, chemists know that the reaction rate is jointly proportional to the amount of the product present and the difference between the saturation level and the amount of the product. If the initial concentration of the chemical is 0 and the saturation level is 1 (corresponding to 100%), then the concentration $x(t)$ of the chemical satisfies the equation

$$x'(t) = rx(t)[1 - x(t)],$$

where $r > 0$ is a constant.

Find the concentration of chemical for which the reaction rate $x'(t)$ is a maximum.

Solution To clarify the problem, we write the reaction rate as

$$f(x) = rx(1 - x).$$

Our aim is then to find $x \geq 0$ that maximizes $f(x)$. We have

$$\begin{aligned} f'(x) &= r(1)(1 - x) + rx(-1) \\ &= r(1 - 2x) \end{aligned}$$

and so, the only critical number is $x = \frac{1}{2}$. Notice that the graph of $y = f(x)$ is a parabola opening downward and hence, the critical number must correspond to the absolute maximum. (Draw your own graph to see for yourself.) Although the mathematical problem here was easy to solve, the result gives a chemist some precise information. At the time the reaction rate reaches a maximum, the concentration of chemical equals exactly half of the saturation level.

Calculus and elementary physics are quite closely connected historically. It should come as no surprise, then, that physics provides us with such a large number of important applications of the calculus. We have already explored the concepts of velocity and acceleration. Another important application in physics where the derivative plays a role involves density. There are many different kinds of densities that we could consider. For example, we could study population density (number of people per unit area) or color density (depth of color per unit area) used in the study of radiographs. However, the most common type

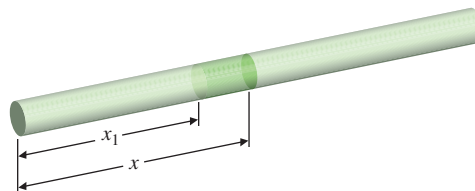


Figure 3.88

A thin rod.

of density discussed is **mass density** (mass per unit volume). You probably already have some idea of what we mean by mass density, but how would you define it? If the object of interest is made of some homogeneous material (i.e., the mass of any portion of the object of a given volume is the same), then the mass density is simply

$$\text{mass density} = \frac{\text{mass}}{\text{volume}}$$

and this quantity is constant throughout the object. However, if the mass of a given volume varies in different parts of the object, then this formula only represents the *average density* of the object. In the next example we find a means of computing the mass density at a specific point in a nonhomogeneous object.

Suppose that the function $f(x)$ gives us the mass (in kilograms) of the first x meters of a thin rod (see Figure 3.88).

The total mass between marks x and x_1 ($x > x_1$) is given by $f(x) - f(x_1)$ kg. The **average linear density** (i.e., mass per unit length) between x and x_1 is then defined as

$$\frac{f(x) - f(x_1)}{x - x_1}.$$

Finally, the **linear density at $x = x_1$** is defined as

$$\rho(x_1) = \lim_{x \rightarrow x_1} \frac{f(x) - f(x_1)}{x - x_1} = f'(x_1), \quad (8.2)$$

where we have again recognized the alternative definition of derivative.

Example 8.3 Density of a Thin Rod

Suppose that the mass in a thin rod is given by $f(x) = \sqrt{2x}$. Compute the linear density at $x = 2$ and at $x = 8$ and compare the densities at the two points.

Solution From (8.2), we have

$$\rho(x) = f'(x) = \frac{1}{2\sqrt{2x}}(2) = \frac{1}{\sqrt{2x}}.$$

Thus, $\rho(2) = 1/\sqrt{4} = 1/2$ and $\rho(8) = 1/\sqrt{16} = 1/4$. Notice that this says that the rod is *inhomogeneous* (i.e., the mass density in the rod is not constant). Specifically, we have that the rod is less dense at $x = 8$ than at $x = 2$.

Our next example comes from medicine (cardiology, to be precise). The rate of change in this application is not actually a derivative. However, heart rate is one of the most familiar and



Figure 3.89
A normal electrocardiogram (EKG).

important rates of change in our lives. The language and concepts of calculus enable us to understand some of the exciting research currently being conducted in this field.

Cardiologists have long used graphs to help them identify heart problems. You are probably familiar with the term *electrocardiogram* (ECG or more commonly, EKG), a graph depicting some of the electrical activity of the heart. A small section of a normal EKG is shown in Figure 3.89.

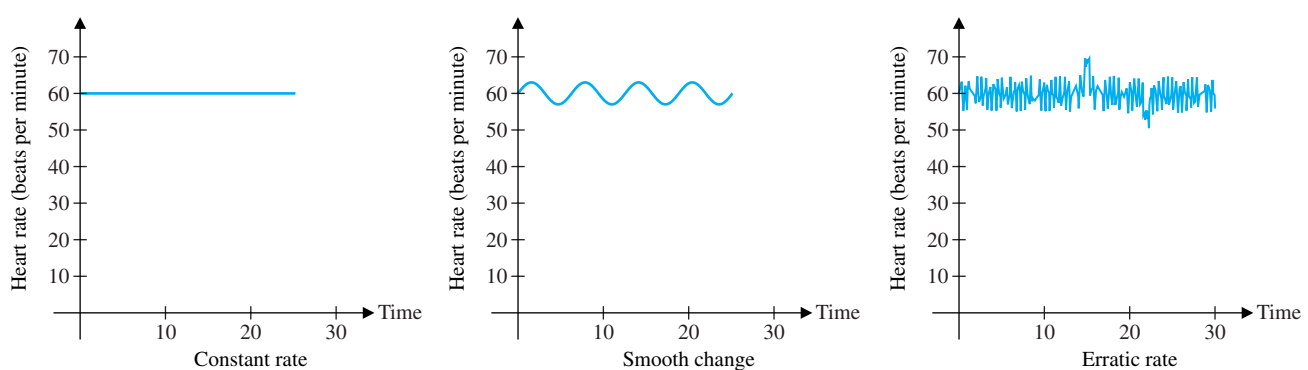
A well-trained cardiologist can determine an amazing amount of information from an EKG strip. Each section of the EKG (containing one peak) corresponds to a primary phase of heart activity, and the particular shape of the curve can indicate various problems with the heart. You should notice that the plot appears to be periodic (that is, the same shape is repeated over and over). It is a simple matter to measure the length of the period from peak to peak (each peak corresponds to a contraction of the ventricles of the heart, and the distance between peaks gives the heart rate). Specifically, if successive peaks occur at times t_1 and t_2 seconds, the **heart rate** equals

$$\frac{1 \text{ beat}}{t_2 - t_1 \text{ seconds}} = \frac{60}{t_2 - t_1} \text{ beats per minute.}$$

Note that heart rate is **not** a derivative. To get a derivative, we would need to have a function representing heart beats as a function of time and would then take the limit as $t_2 \rightarrow t_1$. From an EKG, all we have is a number of observations at specific times and therefore cannot compute a limit. Having defined heart rate, what can we learn about the human body?

Example 8.4 Heart Rates and Electrocardiograms

The following graphs depict heart **rate** over time. Which one would you say corresponds to a healthy individual at rest?



Solution Before answering, we should clarify a couple of points. First, since heart rate is only measured at discrete time intervals (the intervals separating the peaks of the EKG), a graph of heart rate over time is like a computer graph: a finite set of points. If the points are connected and there are many points, the plot may appear to be smooth like the idealized plots we show. Second, be sure to read the graphs carefully! Each graph shows a person resting with a heart rate of about 60 beats per minute (bpm). The difference is in the variation from this average. The first person has a constant heart rate of exactly 60 bpm. The graph depicts a steady 1 beat per second, and is not the “flat-line” of a person whose heart has stopped beating. The second person shows a smooth rise and fall in heart rate, and the third person has a somewhat erratic heart rate.

Which heart rate do you think is healthy? Research by cardiologists in the 1980s and 1990s indicate that healthy hearts generally have a very erratic heart rate, even at rest (You’re right if you picked plot 3!). Highly periodic heart rates like those in plot 2 have been observed in patients who were resting comfortably prior to experiencing a heart attack. The constant heart rate in plot 1 is typical of a person experiencing a heart attack. We should point out that the variations in the healthy third plot are too small to discern without sophisticated equipment. These observations have led to the design of a new type of heart monitor that can be worn by at-risk patients. Current research is investigating the possibility of designing “smart” heart pacemakers that can identify and avoid an impending crisis.



The following example comes from economics. Much as with heart rate, the rate of change discussed here is not precisely a derivative, but the derivative has proved to be a useful tool in economic modeling. In economics, the term **marginal** is used to indicate a rate. Thus, **marginal cost** is the derivative of the cost function, **marginal profit** is the derivative of the profit function, and so on.

Example 8.5

Analyzing the Marginal Cost of Producing a Commercial Product

Suppose that

$$C(x) = 0.02x^2 + 2x + 4000$$

is the total cost (in dollars) for a company to produce x units of a certain product. Compute the marginal cost at $x = 100$ and compare this to the actual cost of producing the 100th unit.

Solution You might think that it is an unfair assumption to start with a function that purports to represent cost. After all, cost is determined by accountants after a product is produced. That’s true, but in order to project what your cost would be for quantities you haven’t actually produced, it is helpful to develop a mathematical model of cost. In practice, this means that you make some observations of the cost of producing a number of different quantities and then try to fit that data to the graph of a known function, which you can then analyze using the tools of calculus. (This is one way in which the calculus is brought to bear on real-world problems.) The marginal cost function is

$$C'(x) = 0.04x + 2$$

and so, the marginal cost at $x = 100$ is $C'(100) = 4 + 2 = 6$ dollars per unit. On the other hand, the actual cost of producing item number 100 would be $C(100) - C(99)$.

(Why?) We have

$$\begin{aligned} C(100) - C(99) &= 200 + 200 + 4000 - (196.02 + 198 + 4000) \\ &= 4400 - 4394.02 = 5.98 \text{ dollars.} \end{aligned}$$

Note that this is very close to the marginal cost of \$6.

Our final example comes from psychology. You have probably heard references to the “learning curve” for a piece of computer software or other technical product. The phrase comes from research attempting to quantify and understand the dynamics of learning something new.

Example 8.6 Analyzing a Learning Curve

Suppose that the percentage of problems a person works correctly on a test is approximately

$$f(t) = \frac{80}{1 + 3e^{-0.4t}}$$

after t hours of training. Graph the learning curve $y = f(t)$ and compute and interpret $f'(2)$ and $f'(10)$.

Solution The graph of $y = f(t)$ looks like the one shown in Figure 3.90.

Note that with no training the person gets 20% correct and after some steady improvement seems to have trouble getting beyond the 80% mark. To compute the derivative, note that we do not need to use the quotient rule, since the numerator is simply a constant. We rewrite the function as

$$f(t) = 80(1 + 3e^{-0.4t})^{-1}$$

and obtain

$$\begin{aligned} f'(t) &= -80(1 + 3e^{-0.4t})^{-2}(-1.2e^{-0.4t}) \\ &= 96e^{-0.4t}(1 + 3e^{-0.4t})^{-2}. \end{aligned}$$

After 2 hours, we then have

$$f'(2) = 96e^{-0.8}(1 + 3e^{-0.8})^{-2} \approx 7.8243 \text{ percent per hour.}$$

That is, after 2 hours, a person can expect to add about 7.8 points to the test score by training for an extra hour. At the 10-hour mark,

$$f'(10) = 96e^{-4}(1 + 3e^{-4})^{-2} \approx 1.5799 \text{ percent per hour.}$$

Thus, the next hour of training only increases the test score by about 1.6 points.

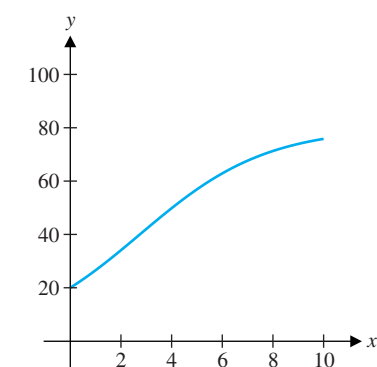






Figure 3.90

$$y = 80/(1 + 3e^{-0.4t}).$$

We have now discussed examples of six rates of change drawn from engineering and the sciences. Add these to the examples we have seen in previous sections (velocity, population growth, etc.) and we have an impressive list of applications of the derivative. Even so, we have barely begun to scratch the surface. In any field where it is possible to quantify and analyze the properties of a function, calculus and the derivative are powerful tools. This list includes at least some portion of every major field of study. The continued study of calculus will give you the ability to read (and understand) technical studies in a wide variety of fields and to see (as we have in this section) the underlying unity that mathematics brings to human endeavors.

EXERCISES 3.8

-  A variable that is defined only at isolated points is called a **discrete** variable. For example, the size of a calculus class is discrete, since it must be an integer. A variable that can assume an interval of values is called **continuous**. In the six applications discussed in the text, in which cases is time a discrete variable and in which cases is it continuous? Which functions are discrete and which are continuous? Among the six examples given in the text, in which cases were the rates of change truly derivatives? Explain this answer in terms of discrete and continuous variables.
-  An important model of population growth is the so-called **logistic equation** $x'(t) = x(t)[1 - x(t)]$. Here, $x(t)$ represents not the actual population size but the proportion of sustainable capacity: for instance, $x(t) = 0.5$ means that the population is half of the total number of organisms that the environment can support and $x(t) = 1.1$ means that there are 10% more organisms than the available resources can support. Note that the differential equation here is the same as was used to describe an autocatalytic chemical reaction. The equation has two competing contributions to the rate of change $x'(t)$. The term $x(t)$ by itself would mean that the larger $x(t)$ is, the faster the population (or concentration of chemical) grows. This is balanced by the term $1 - x(t)$, which indicates that the closer $x(t)$ gets to 1, the slower the population growth is. With these two terms together, the model has the property that for small $x(t)$, slightly larger $x(t)$ means greater growth, but as $x(t)$ approaches 1, the growth tails off. Explain in terms of population growth and the concentration of chemical why the model is reasonable.
-  Corporate deficits and debt are frequently in the news, but the terms are often confused with each other. To take an example, suppose a company finishes a fiscal year owing \$5,000. That is their **debt**. Suppose that in the following year the company has revenues of \$106,000 and expenses of \$109,000. The company's **deficit** for the year is \$3,000 and the company's debt has increased to \$8,000. Briefly explain why deficit is the derivative of debt.
-  Many people find the healthy heart example (example 8.4) surprising. To make it more believable, explain how your level of activity affects your heart rate. Also, explain how breathing rate and emotional status can affect heart rate. Given these and many other factors, discuss whether you should expect your heart rate to be exactly constant.
- Suppose that the charge in an electrical circuit is $Q(t) = e^{-2t}(\cos 3t - 2 \sin 3t)$ coulombs. Find the current.
- Suppose that the charge in an electrical circuit is $Q(t) = e^t(3 \cos 2t + \sin 2t)$ coulombs. Find the current.
- Suppose that the charge at a particular location in an electrical circuit is $Q(t) = e^{-3t} \cos 2t + 4 \sin 3t$ coulombs. What happens to this function as $t \rightarrow \infty$? Explain why the term $e^{-3t} \cos 2t$ is called a **transient** term and $4 \sin 3t$ is known as the **steady-state** or **asymptotic** value of the charge function. Find the transient and steady-state values of the current function.
- As in exercise 7, find the steady-state and transient values of the current function if the charge function is given by $Q(t) = e^{-2t}(\cos t - 2 \sin t) + te^{-3t} + 2 \cos 4t$.
- If the concentration of a chemical changes according to the equation $x'(t) = 2x(t)[4 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.
- If the concentration of a chemical changes according to the equation $x'(t) = 0.5x(t)[5 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.
- Show that in exercise 9, the maximum concentration is 4 if $0 < x(0) < 4$. Find the maximum concentration in exercise 10.
- Find the equation for an autocatalytic reaction in which the maximum concentration is $x(t) = 16$ and the reaction rate equals 12 when $x(t) = 8$.
- Mathematicians often study equations of the form $x'(t) = rx(t)[1 - x(t)]$ instead of the more complicated $x'(t) = cx(t)[K - x(t)]$, justifying the simplification with the statement that the second equation “reduces to” the first equation. Starting with $y'(t) = cy(t)[K - y(t)]$, substitute $y(t) = Kx(t)$ and show that the equation reduces to the form $x'(t) = rx(t)[1 - x(t)]$. How does the constant r relate to the constants c and K ?
- Suppose a chemical reaction follows the equation $x'(t) = cx(t)[K - x(t)]$. Suppose that at time $t = 4$ the concentration is $x(4) = 2$ and the reaction rate is $x'(4) = 3$. At time $t = 6$, suppose that the concentration is $x(6) = 4$ and the reaction rate is $x'(6) = 4$. Find the values of c and K for this chemical reaction.
- In a general **second-order chemical reaction**, chemicals A and B (the **reactants**) combine to form chemical C (the **product**). If the initial concentrations of the reactants A and B are a and b , respectively, then the concentration $x(t)$ of the product satisfies the equation $x'(t) = [a - x(t)][b - x(t)]$. What is the rate of change of the product when $x(t) = a$? At this value, is the concentration of product increasing, decreasing or staying the same? Assuming that $a < b$ and there is no product present when the reaction starts, explain why the maximum concentration of product is $x(t) = a$.
- For the second-order reaction defined in exercise 15, find the (mathematical) value of $x(t)$ that minimizes the reaction rate.

