VIII. POTENTIAL FLOW AND COMPUTATIONAL FLUID DYNAMICS

Review of Velocity-Potential Concepts

This chapter presents examples of problems and their solutions for which the assumption of potential flow is appropriate.

For low speed flows where viscous effects are neglected, the flow is irrotational and

$$\nabla \times V = 0 \qquad V = \nabla \phi \qquad u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

The continuity equation , $\nabla \cdot V = 0$, now reduces to

$$\nabla^2 V = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The momentum equation reduces to Bernoulli's equation:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2} V^2 + g z = \text{const}.$$

Review of Stream Function Concepts

For plane incompressible flow in x-y coordinates, a stream function exists such that

$$u = \frac{\partial \Psi}{\partial y} \quad \textit{and} \quad v = -\frac{\partial \Psi}{\partial x}$$

The condition of irrotationality reduces to Laplace's equation for Ψ with

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

and for a solid surface - Ψ_{solid} = const.

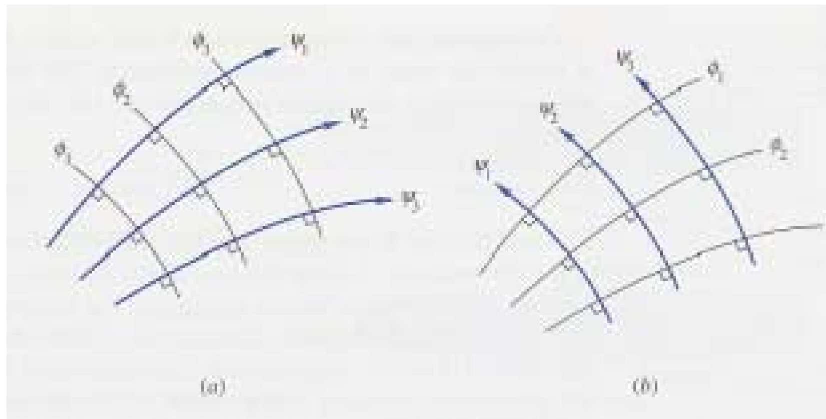


Fig. 8.2 Streamlines and potential lines are orthogonal and may reverse roles if results are useful: (a) typical inviscid-flow pattern: (b) same as (a) with roles reversed

Plane Polar Coordinates

Equations for plane polar velocity components are given below in term of polar coordinates (r, θ) and the polar coordinate velocity potential, ϕ, and stream function, Ψ.

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \Psi}{\partial r}$$

Laplace's equation now has the form in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Elementary Plane-Flow Solutions

Three plane-flow solutions that are very useful in developing more complex potential flow solutions are:

Uniform stream, **i**U, in the x direction: $\Psi = \mathrm{U\,y}$ $\quad \phi = \mathrm{U\,x}$

Line source or sink: $\Psi = \mathrm{m}\,\theta$ $\quad \phi = m \ln \mathrm{r}$

Line vortex: $\Psi = -\mathrm{K} \ln \mathrm{r}$ $\quad \phi = \mathrm{K}\,\theta$

In these expressions, the source strength, ‘m’, and vortex strength, ‘K’, have the dimensions of velocity times length, or [L^2/t].

If the uniform stream is written in plane polar coordinates, we have

Uniform stream, **i**U: $\Psi = \mathrm{U\,r} \sin \theta$ $\quad \phi = \mathrm{U\,r} \cos \theta$

For a uniform stream moving at an angle, a , relative to the x-axis, we can write

$$u = \mathrm{U} \cos \alpha = \frac{\partial \Psi}{\partial \mathrm{y}} = \frac{\partial \phi}{\partial \mathrm{x}} \qquad \mathrm{v} = \mathrm{U} \sin \alpha = -\frac{\partial \Psi}{\partial \mathrm{x}} = \frac{\partial \phi}{\partial \mathrm{y}}$$

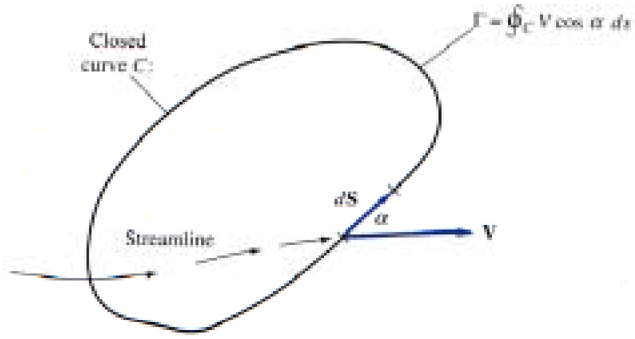
After integration, we obtain the following expressions for the stream function and velocity potential:

$$\Psi = U\left(y \cos \alpha - x \sin \alpha\right) \qquad \phi = U\left(x \cos \alpha + y \sin \alpha\right)$$

Circulation

The concept of fluid circulation is very useful in the analysis of certain potential flows, in particular those useful in aerodynamics analyses. Consider Figure 8.3 shown below:

We define the circulation, Γ , as the counterclockwise line integral of the arc length, dS times the velocity component tangent to the closed curve, C, e.g.



$$\Gamma = \oint_c V \cos\alpha \, d s = \int_c V \cdot ds$$

$$\Gamma = \int_c \left(u\, dx + \mathrm{v}\, \mathrm{dy} + \mathrm{w}\, \mathrm{dz} \right)$$

For most flows, this line integral around a closed path, starting and stopping at the same point, yields $\Gamma = 0$. However,

for a vortex flow for which $\phi = K\,\theta$

the integral yields $\Gamma = 2\,\pi\,K$

An equivalent calculation can by made by defining a circular path of radius r around the vortex center to yield

$$\Gamma = \int_c \mathrm{v}_\theta \, d s = \int_0^{2\pi} \frac{K}{r} r \, d\phi = 2\pi\, K$$

Key Point: A source or sink does not produce circulation. Without the presence of vortices, the circulation will be zero for any closed path around any number of sources or sinks.

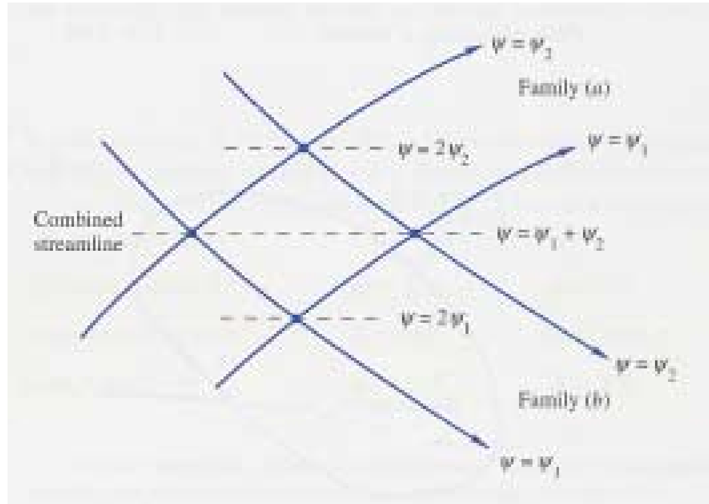
Superposition of Potential Flows

Due to the mathematical character of the equations governing potential flows, the principle of superposition can be used to determine the solution for the flow which results from summing the velocity potential and stream functions two individual potential flow solutions.

This is shown graphically in Fig. 8.4 on the following page.

Note that the value of the stream function at each intersection equals the sum of the values of the stream functions crossing at the point.

This would also be true for the velocity potential at any point in a combined flow.



Several classic examples of superposition of flows are presented as follows:

1. Source m at (-a, 0) added to an equal sink at (+a, 0).

$$\psi = -m\tan^{-1}\frac{2ay}{x^2+y^2-a^2} \qquad \phi = \frac{1}{2}m\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

The streamlines and potential lines are two families of orthogonal circles (Fig. 4.13,, Ch 4, text).

2. Sink m plus a vortex K, both at the origin.

$$\psi = m\theta - K\ln r \qquad \phi = m\ln r + K\theta$$

The streamlines are logarithmic spirals swirling into the origin (Fig. 4.14, text). They resemble a tornado or a bathtub vortex.

3. Uniform steam **i** U_∞ plus a source m at the origin (Fig. 4.15), the Ranking half body. If the origin contains a source, a plane half-body is formed with its nose to the left as shown below. If the origin contains a sink, m < 0, the half-body nose is to the right.. For both cases, the stagnation point is at a position $a = m / U_\infty$ away from the origin.

(a) a uniform stream plus a source yields a half-body with stagnation point at x = -a = -m/U_∞.

(c) a uniform stream plus a sink with a stagnation point at x = a = m/U_∞.

(b & d) variation of free stream velocity and thus free stream pressure with location.

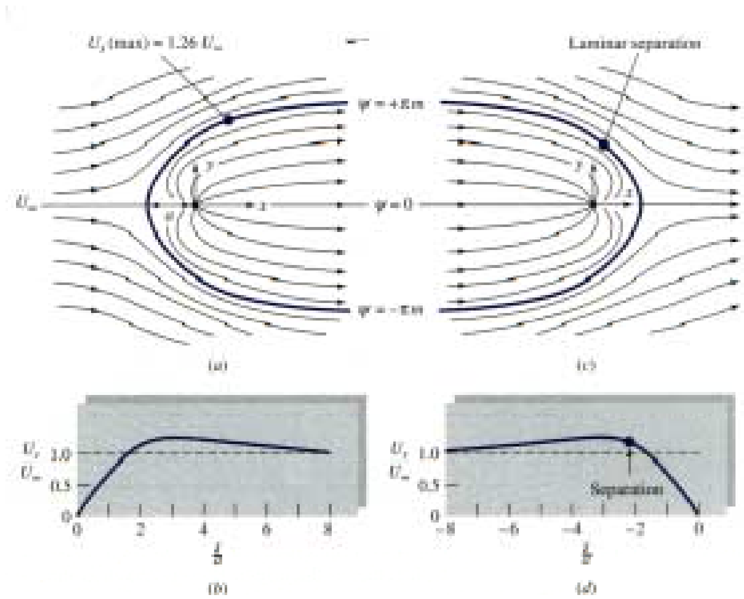
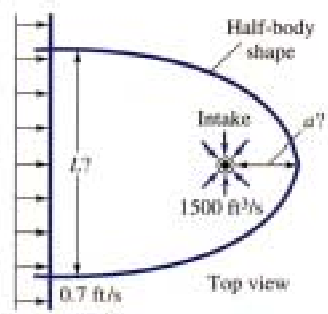


Fig. 8.5 The Rankine half-body

Example 8.1

An offshore power plant cooling water intake has a flow rate of 1500 ft^3/s in water 30 ft deep as in Fig. E8.1. If the tidal velocity approaching the intake is 0.7 ft/s, (a) how far downstream does the intake effect extend and (b) how much width of tidal flow is entrained into the intake?

The sink strength is related to the volume flow, Q and water depth by



E8.1

$$m = \frac{Q}{2\pi b} = \frac{1500\ ft^3/s}{2\pi 30\ ft} = 7.96\ ft^2/s$$

The length, a, of the downstream effect:

$$a = \frac{m}{U_\infty} = \frac{7.96\ ft^2/s}{0.7\ ft/s} = 11.4\ ft$$

The width, L, of flow entrained into the intake:

$$L = 2\pi a = 2\pi 11.4\ ft = 71\ ft$$