

# *Mathematica* Technology Manual

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to accompany

*Differential Equations and Mathematical Modeling by Glenn Ledder*

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## **Preface**

This manual provides a straightforward and practical introduction to the computer algebra system *Mathematica* for students and instructors who wish to use it in the study of ordinary differential equations. Readers will find what they need to get a quick start with the program in the context of simple numerical calculations, fundamental calculus computations, and elementary graphing techniques. Applications are interwoven with explanations of how *Mathematica* works and examples illustrating the important features of this important software tool.

Once a sufficient level of confidence has been attained, the reader is guided through *Mathematica* procedures designed to solve differential equations, analyze the behavior of solutions, and use solutions, both exact and approximate, in the mathematical modeling process. The exercises can be used to clarify the reader's understanding of how the procedures work. Annotated examples guide the reader through the more challenging problems.

## **Prerequisites and Corequisites**

Instructions in the manual are written with the assumption that the reader knows how to start a computer and make a document using a word processor. The ability to use a mouse: point and click, select and drag, pull down a menu, etc. will be taken for granted. The only mathematics prerequisite is the successful completion of a university level calculus course or its equivalent. As a corequisite it is hoped that the reader is actively engaged in the learning of the fundamentals of ordinary differential equations. Please note that this is not a text in differential equations. Indeed, the reader will often be referred to a differential equations textbook (Ledder) for statements of definitions, algorithms, solution techniques, and solution formulas, as well as background material for some of the examples and exercises.

## **Part I. The *Mathematica* Notebook**

This is a brief overview of *Mathematica*'s notebook interface. The essential components of a notebook are described and simple examples illustrate how to enter and process mathematics and text.

## **Part II. Calculations and Calculus with *Mathematica***

The second part contains a more detailed introduction to *Mathematica* and how to use it to "do mathematics." The input/output paradigm is stressed as the reader learns how to use *Mathematica* as a calculator. The very useful `Table` function is discussed. *Mathematica* output can be used as subsequent input by assigning the output a name, and then referring to it by that name just as one might do using paper and pencil. The notion of assignments are important for the successful use of any mathematics software to a problem that requires more than one calculation. This idea is illustrated several times using familiar examples from calculus. Graphing examples introduce plotting procedures.

## **Part III. First Order Ordinary Differential Equations**

The discussion of differential equations in *Mathematica* starts here in the context of mathematical models requiring first order equations in their analysis. Readers who are already familiar with *Mathematica* are encouraged to skim over Parts I and II and begin in Part III. Direction fields and exact solution curves are plotted. Implementation of Euler's method provides the reader with an opportunity to learn how to write a *Mathematica* program that can easily be converted into a user-defined procedure.

## **Part IV. Linear Differential Equations**

Linear differential equations, linear systems, and Laplace transforms are featured in this part of the manual. Procedures in the `LinearAlgebra` package are introduced to handle vectors and matrices gracefully.

## **Exercises**

Exercise sets are divided by Part and Section. Solutions to asterisked exercises appear in the solution section.

## **Appendix**

The appendix, which itself is divided into parts 1 - 5 contains miscellaneous items that did not fit nicely into Part I, II, III, or IV: Power series and special functions, Picard iterates, Partial differential equations, and some Sound advice and encouraging words.

## **Solutions to Selected Exercises**

A list of solutions to the exercises that appear with an asterisk.

The Table of Contents that follows is detailed enough to be used as a complete outline of the topics that are treated in the manual.

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Selected Solutions