

3.3

ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

In this section

- Absolute Value Equations
- Absolute Value Inequalities
- Applications

In Chapter 1 we learned that absolute value measures the distance of a number from 0 on the number line. In this section we will learn to solve equations and inequalities involving absolute value.

Absolute Value Equations

Solving equations involving absolute value requires some techniques that are different from those studied in previous sections. For example, the solution set to the equation

$$|x| = 5$$

is $\{-5, 5\}$ because both 5 and -5 are five units from 0 on the number line, as shown in Fig. 3.19. So $|x| = 5$ is equivalent to the compound equation

$$x = 5 \text{ or } x = -5.$$

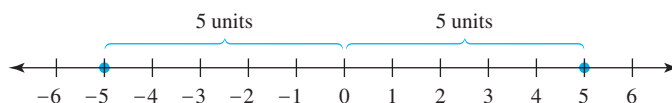


FIGURE 3.19

The equation $|x| = 0$ is equivalent to the equation $x = 0$ because 0 is the only number whose distance from 0 is zero. The solution set to $|x| = 0$ is $\{0\}$.

The equation $|x| = -7$ is inconsistent because absolute value measures distance, and distance is never negative. So the solution set is empty. These ideas are summarized as follows.

Basic Absolute Value Equations

Absolute Value Equation	Equivalent Equation	Solution Set
$ x = k$ ($k > 0$)	$x = k$ or $x = -k$	$\{k, -k\}$
$ x = 0$	$x = 0$	$\{0\}$
$ x = k$ ($k < 0$)		\emptyset

We can use these ideas to solve more complicated absolute value equations.

EXAMPLE 1

Absolute value equal to a positive number

Solve each equation.

a) $|x - 7| = 2$ b) $|3x - 5| = 7$

Solution

a) First rewrite $|x - 7| = 2$ without absolute value:

$$\begin{array}{l} x - 7 = 2 \quad \text{or} \quad x - 7 = -2 \quad \text{Equivalent equation} \\ x = 9 \quad \text{or} \quad x = 5 \end{array}$$

The solution set is $\{5, 9\}$. The distance from 5 to 7 or from 9 to 7 is 2 units.

calculator



close-up

Use $Y=$ to set $y_1 = \text{abs}(x - 7)$. Make a table to see that y_1 has value 2 when $x = 5$ or $x = 9$. The table supports the conclusion of Example 1(a).

X	Y1
5	2
7	0
9	2
10	3
11	4

Y1=abs(X-7)

b) First rewrite $|3x - 5| = 7$ without absolute value:

$$3x - 5 = 7 \quad \text{or} \quad 3x - 5 = -7 \quad \text{Equivalent equation}$$

$$3x = 12 \quad \text{or} \quad 3x = -2$$

$$x = 4 \quad \text{or} \quad x = -\frac{2}{3}$$

The solution set is $\left\{-\frac{2}{3}, 4\right\}$. ■

EXAMPLE 2 Absolute value equal to zero

Solve $|2(x - 6) + 7| = 0$.

Solution

Since 0 is the only number whose absolute value is 0, the expression within the absolute value bars must be 0.

$$2(x - 6) + 7 = 0 \quad \text{Equivalent equation}$$

$$2x - 12 + 7 = 0$$

$$2x - 5 = 0$$

$$2x = 5$$

$$x = \frac{5}{2}$$

The solution set is $\left\{\frac{5}{2}\right\}$. ■

Solving absolute value equations is based on the form $|x| = k$, in which the absolute value expression is isolated on one side. In Examples 1 and 2, the equations were given in that form. In the next example, we must first isolate the absolute value expression to solve the equation.

EXAMPLE 3 Absolute value equal to a negative number

Solve $-5|3x - 7| + 4 = 14$.

Solution

First subtract 4 from each side to isolate the absolute value expression:

$$-5|3x - 7| + 4 = 14 \quad \text{Original equation}$$

$$-5|3x - 7| = 10 \quad \text{Subtract 4 from each side.}$$

$$|3x - 7| = -2 \quad \text{Divide each side by } -5.$$

There is no solution because no number has a negative absolute value. ■

The equation in the next example has an absolute value on both sides.

EXAMPLE 4 Absolute value on both sidesSolve $|2x - 1| = |x + 3|$.**Solution**

Two quantities have the same absolute value only if they are equal or opposites. So we can write an equivalent compound equation:

$$2x - 1 = x + 3 \quad \text{or} \quad 2x - 1 = -(x + 3)$$

$$x - 1 = 3 \quad \text{or} \quad 2x - 1 = -x - 3$$

$$x = 4 \quad \text{or} \quad 3x = -2$$

$$x = 4 \quad \text{or} \quad x = -\frac{2}{3}$$

Check that both 4 and $-\frac{2}{3}$ satisfy the original equation. The solution set is $\{-\frac{2}{3}, 4\}$. ■

study tip

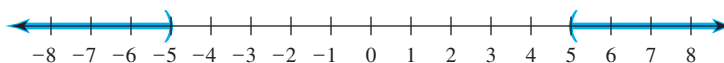
Many commuting students find it difficult to get help. Students are often stuck on a minor point that can be easily resolved over the telephone. So ask your instructor if he or she will answer questions over the telephone and during what hours.

Absolute Value Inequalities

Since absolute value measures distance from 0 on the number line, $|x| > 5$ indicates that x is more than five units from 0. Any number on the number line to the right of 5 or to the left of -5 is more than five units from 0. So $|x| > 5$ is equivalent to

$$x > 5 \quad \text{or} \quad x < -5.$$

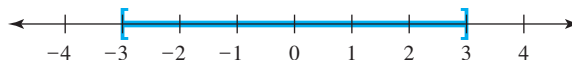
The solution set to this inequality is the union of the solution sets to the two simple inequalities. The solution set is $(-\infty, -5) \cup (5, \infty)$. The graph of $|x| > 5$ is shown in Fig. 3.20.

**FIGURE 3.20**

The inequality $|x| \leq 3$ indicates that x is less than or equal to three units from 0. Any number between -3 and 3 inclusive satisfies that condition. So $|x| \leq 3$ is equivalent to

$$-3 \leq x \leq 3.$$

The graph of $|x| \leq 3$ is shown in Fig. 3.21. These examples illustrate the basic types of absolute value inequalities.

**FIGURE 3.21**

We can solve more complicated inequalities in the same manner as simple ones.

Basic Absolute Value Inequalities ($k > 0$)

Absolute Value Inequality	Equivalent Inequality	Solution Set	Graph of Solution Set
$ x > k$	$x > k$ or $x < -k$	$(-\infty, -k) \cup (k, \infty)$	
$ x \geq k$	$x \geq k$ or $x \leq -k$	$(-\infty, -k] \cup [k, \infty)$	
$ x < k$	$-k < x < k$	$(-k, k)$	
$ x \leq k$	$-k \leq x \leq k$	$[-k, k]$	

EXAMPLE 5

Absolute value inequality

Solve $|x - 9| < 2$ and graph the solution set.

Solution

Because $|x| < k$ is equivalent to $-k < x < k$, we can rewrite $|x - 9| < 2$ as follows:

$$-2 < x - 9 < 2$$

$$-2 + 9 < x - 9 + 9 < 2 + 9 \quad \text{Add 9 to each part of the inequality.}$$

$$7 < x < 11$$

The graph of the solution set $(7, 11)$ is shown in Fig. 3.22. Note that the graph consists of all real numbers that are within two units of 9.

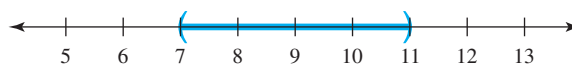


FIGURE 3.22

calculator

close-up

Use Y= to set $y_1 = \text{abs}(x - 9)$.
Make a table to see that $y_1 < 2$
when x is between 7 and 11.

X	Y1
6	3
7	2
8	1
9	0
10	1
11	2
12	3

Y1 abs(X-9)

EXAMPLE 6

Absolute value inequality

Solve $|3x + 5| > 2$ and graph the solution set.

Solution

$$3x + 5 > 2 \quad \text{or} \quad 3x + 5 < -2 \quad \text{Equivalent compound inequality}$$

$$3x > -3 \quad \text{or} \quad 3x < -7$$

$$x > -1 \quad \text{or} \quad x < -\frac{7}{3}$$

The solution set is $(-\infty, -\frac{7}{3}) \cup (-1, \infty)$, and its graph is shown in Fig. 3.23.

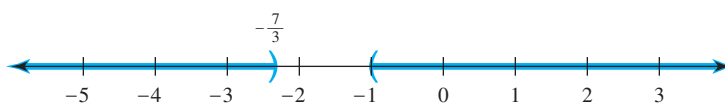


FIGURE 3.23

EXAMPLE 7 Absolute value inequality

Solve $|5 - 3x| \leq 6$ and graph the solution set.

calculator

4 5 6 X

close-up

Use $Y=$ to set $y_1 = \text{abs}(5 - 3x)$. The table supports the conclusion that $y \leq 6$ when x is between $-\frac{1}{3}$ and $\frac{11}{3}$ even though $-\frac{1}{3}$ and $\frac{11}{3}$ do not appear in the table. For more accuracy, make a table in which the change in x is $\frac{1}{3}$.

X	Y1
-1	8
0	5
1	2
2	1
3	2
4	5
5	10

Y1=abs(5-3X)

Solution

$$\begin{aligned}
 -6 &\leq 5 - 3x \leq 6 && \text{Equivalent inequality} \\
 -11 &\leq -3x \leq 1 && \text{Subtract 5 from each part.} \\
 \frac{11}{3} &\geq x \geq -\frac{1}{3} && \text{Divide by } -3 \text{ and reverse each inequality symbol.} \\
 -\frac{1}{3} &\leq x \leq \frac{11}{3} && \text{Write } -\frac{1}{3} \text{ on the left because it is smaller than } \frac{11}{3}.
 \end{aligned}$$

The solution set is $[-\frac{1}{3}, \frac{11}{3}]$ and its graph is shown in Fig. 3.24.

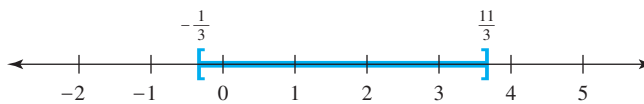


FIGURE 3.24

There are a few absolute value inequalities that do not fit the preceding categories. They are easy to solve using the definition of absolute value.

EXAMPLE 8 Special case

Solve $3 + |7 - 2x| \geq 3$.

Solution

Subtract 3 from each side to isolate the absolute value expression.

$$|7 - 2x| \geq 0$$

Because the absolute value of any real number is greater than or equal to 0, the solution set is R , the set of all real numbers.

EXAMPLE 9 An impossible case

Solve $|5x - 12| < -2$.

Solution

We write an equivalent inequality only when the value of k is positive. With -2 on the right-hand side, we do not write an equivalent inequality. Since the absolute value of any quantity is greater than or equal to 0, no value for x can make this absolute value less than -2 . The solution set is \emptyset , the empty set.

Applications

A simple example will show how absolute value equations and inequalities can be used in applications.

EXAMPLE 10 Controlling water temperature

The water temperature in a certain production process should be 143°F .

- a) If the actual temperature differs by 13° from what it should be, then what is the actual temperature?

- b) The product is defective if the actual temperature is more than 7° away from what it should be. For what temperatures is the product defective?

Solution

- a) If x represents the actual water temperature, then $x - 143$ represents the difference between the actual temperature and the desired temperature. Since $x - 143$ could be positive or negative, we use absolute value:

study tip

When taking a test, try not to spend too much time on a single problem. If a problem is taking a long time, then you might be approaching it incorrectly. Move on to another problem and make sure that you get finished with the test.

$$\begin{aligned} |x - 143| &= 13 \\ x - 143 &= 13 & \text{or} & & x - 143 &= -13 \\ x &= 156 & \text{or} & & x &= 130 \end{aligned}$$

The actual temperature is either 130°F or 156°F .

- b) The product is defective if the absolute value of $x - 143$ is greater than 7:

$$\begin{aligned} |x - 143| &> 7 \\ x - 143 &> 7 & \text{or} & & x - 143 &< -7 \\ x &> 150 & \text{or} & & x &< 136 \end{aligned}$$

The product is defective for temperatures greater than 150°F or less than 136°F . ■

WARM - UPS

True or false? Explain your answer.

- The equation $|x| = 2$ is equivalent to $x = 2$ or $x = -2$.
- All absolute value equations have two solutions.
- The equation $|2x - 3| = 7$ is equivalent to $2x - 3 = 7$ or $2x + 3 = 7$.
- The inequality $|x| > 5$ is equivalent to $x > 5$ or $x < -5$.
- The equation $|x| = -5$ is equivalent to $x = 5$ or $x = -5$.
- There is only one solution to the equation $|3 - x| = 0$.
- We should write the inequality $x > 3$ or $x < -3$ as $3 < x < -3$.
- The inequality $|x| < 7$ is equivalent to $-7 \leq x \leq 7$.
- The equation $|x| + 2 = 5$ is equivalent to $|x| = 3$.
- The inequality $|x| < -2$ is equivalent to $x < 2$ and $x > -2$.

3.3 EXERCISES

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

- What does absolute value measure?
- Why does $|x| = 0$ have only one solution?
- Why does $|x| = 4$ have two solutions?

4. Why is $|x| = -3$ inconsistent?

5. Why do all real numbers satisfy $|x| \geq 0$?

6. Why do no real numbers satisfy $|x| < -3$?

Solve each absolute value equation. See Examples 1-3.

7. $|a| = 5$ 8. $|x| = 2$ 9. $|x - 3| = 1$

10. $|x - 5| = 2$ 11. $|3 - x| = 6$ 12. $|7 - x| = 6$

13. $|3x - 4| = 12$ 14. $|5x + 2| = -3$

15. $\left| \frac{2}{3}x - 8 \right| = 0$ 16. $\left| 3 - \frac{3}{4}x \right| = \frac{1}{4}$

17. $|6 - 0.2x| = 10$


18. $|5 - 0.1x| = 0$


19. $|7(x - 6)| = -3$

20. $|2(a + 3)| = 15$

21. $|2(x - 4) + 3| = 5$

22. $|3(x - 2) + 7| = 6$

 23. $|7.3x - 5.26| = 4.215$

 24. $|5.74 - 2.17x| = 10.28$

Solve each absolute value equation. See Examples 3 and 4.

25. $3 + |x| = 5$

26. $|x| - 10 = -3$

27. $2 - |x + 3| = -6$

28. $4 - 3|x - 2| = -8$

29. $5 - \frac{|3 - 2x|}{3} = 4$

30. $3 - \frac{1}{2} \left| \frac{1}{2}x - 4 \right| = 2$

31. $|x - 5| = |2x + 1|$

32. $|w - 6| = |3 - 2w|$

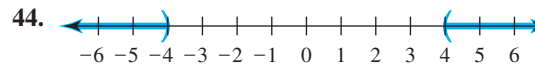
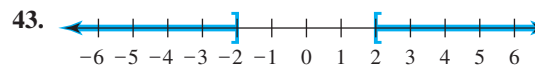
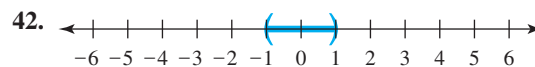
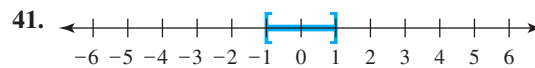
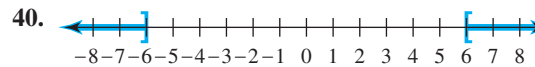
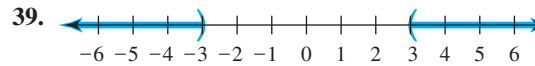
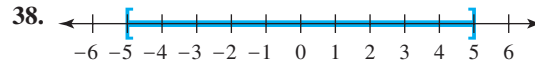
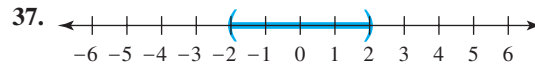
33. $\left| \frac{5}{2} - x \right| = \left| 2 - \frac{x}{2} \right|$

34. $\left| x - \frac{1}{4} \right| = \left| \frac{1}{2}x - \frac{3}{4} \right|$

35. $|x - 3| = |3 - x|$

36. $|a - 6| = |6 - a|$

Write an absolute value inequality whose solution set is shown by the graph. See Examples 5-7.



Determine whether each absolute value inequality is equivalent to the inequality following it. See Examples 5-7.

45. $|x| < 3, x < 3$

46. $|x| > 3, x > 3$

47. $|x - 3| > 1, x - 3 > 1$ or $x - 3 < -1$

48. $|x - 3| \leq 1, -1 \leq x - 3 \leq 1$

49. $|x - 3| \geq 1, x - 3 \geq 1$ or $x - 3 \leq -1$

50. $|x - 3| > 0, x - 3 > 0$

51. $|4 - x| < 1, 4 - x < 1$ and $-(4 - x) < 1$

52. $|4 - x| > 1, 4 - x > 1$ or $-(4 - x) > 1$

Solve each absolute value inequality and graph the solution set. See Examples 5-7.

53. $|x| > 6$

54. $|w| > 3$

55. $|2a| < 6$

56. $|3x| < 21$

57. $|x - 2| \geq 3$

58. $|x - 5| \geq 1$

59. $\frac{1}{5}|2x - 4| < 1$

60. $\frac{1}{3}|2x - 1| < 1$

61. $-2|5 - x| \geq -14$

62. $-3|6 - x| \geq -3$

63. $2|3 - 2x| - 6 \geq 18$

64. $2|5 - 2x| - 15 \geq 5$

Solve each absolute value inequality and graph the solution set. See Examples 8 and 9.

65. $|x - 2| > 0$

66. $|6 - x| \geq 0$

67. $|x - 5| \geq 0$

68. $|3x - 7| \geq -3$

69. $-2|3x - 7| > 6$

70. $-3|7x - 42| > 18$

71. $|2x + 3| + 6 > 0$

72. $|5 - x| + 5 > 5$

Solve each inequality. Write the solution set using interval notation.

73. $1 < |x + 2|$

74. $5 \geq |x - 4|$

75. $5 > |x| + 1$

76. $4 \leq |x| - 6$

77. $3 - 5|x| > -2$

78. $1 - 2|x| < -7$

79. $|5.67x - 3.124| < 1.68$

80. $|4.67 - 3.2x| \geq 1.43$

81. $|2x - 1| < 3$ and $2x - 3 > 2$

82. $|5 - 3x| \geq 3$ and $5 - 2x > 3$

83. $|x - 2| < 3$ and $|x - 7| < 3$

84. $|x - 5| < 4$ and $|x - 6| > 2$

Solve each problem by using an absolute value equation or inequality. See Example 10.

85. Famous battles. In the Hundred Years' War, Henry V defeated a French army in the battle of Agincourt and Joan of Arc defeated an English army in the battle of Orleans (*The Doubleday Almanac*). Suppose you know only that these two famous battles were 14 years apart and that the battle of Agincourt occurred in 1415. Use an absolute value equation to find the possibilities for the year in which the battle of Orleans occurred.

86. World records. In July 1985 Steve Cram of Great Britain set a world record of 3 minutes 29.67 seconds for the 1500-meter race and a world record of 3 minutes 46.31 seconds for the 1-mile race (*The Doubleday Almanac*). Suppose you know only that these two events occurred 11 days apart and that the 1500-meter record was set on July 16. Use an absolute value equation to find the possible dates for the 1-mile record run.

87. Weight difference. Research at a major university has shown that identical twins generally differ by less than 6 pounds in body weight. If Kim weighs 127 pounds, then in what range is the weight of her identical twin sister Kathy?

88. Intelligence quotient. Jude's IQ score is more than 15 points away from Sherry's. If Sherry scored 110, then in what range is Jude's score?

89. Unidentified flying objects. The formula

$$S = -16t^2 + v_0t + s_0$$

gives height in feet above the earth at time t seconds for an object projected into the air with an initial velocity of v_0 feet per second (ft/sec) from an initial height of s_0 feet. Two balls are tossed into the air simultaneously, one from the ground at 50 ft/sec and one from a height of 10 feet at 40 ft/sec. See the accompanying graph.

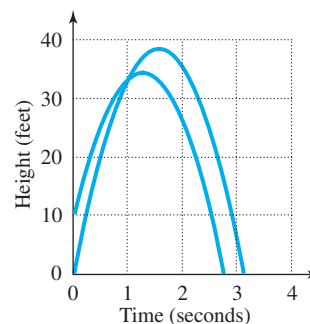


FIGURE FOR EXERCISE 89

- a) Use the graph to estimate the time at which the balls are at the same height.
- b) Find the time from part (a) algebraically.
- c) For what values of t will their heights above the ground differ by less than 5 feet (while they are both in the air)?

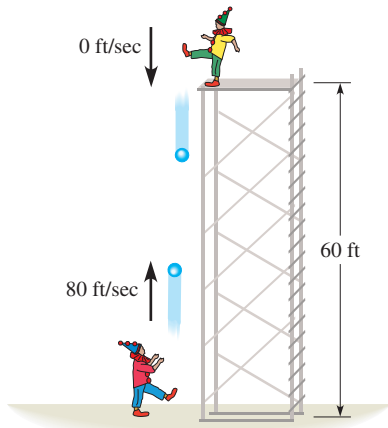


FIGURE FOR EXERCISE 90

90. **Playing catch.** A circus clown at the top of a 60-foot platform is playing catch with another clown on the ground. The clown on the platform drops a ball at the same time as the one on the ground tosses a ball upward at 80 ft/sec. For what length of time is the distance between the balls less than or equal to 10 feet? (*Hint:* Use the formula given in Exercise 89. The initial velocity of a ball that is dropped is 0 ft/sec.)

GETTING MORE INVOLVED



91. **Discussion.** For which real numbers m and n is each equation satisfied?

- a) $|m - n| = |n - m|$
- b) $|mn| = |m| \cdot |n|$
- c) $\frac{|m|}{|n|} = \frac{|m|}{|n|}$



92. **Exploration.** a) Evaluate $|m + n|$ and $|m| + |n|$ for
- i) $m = 3$ and $n = 5$
 - ii) $m = -3$ and $n = 5$
 - iii) $m = 3$ and $n = -5$
 - iv) $m = -3$ and $n = -5$
- b) What can you conclude about the relationship between $|m + n|$ and $|m| + |n|$?

COLLABORATIVE ACTIVITIES

Everyday Algebra

Every day, people use algebra without even knowing it. Any time you solve for an unknown quantity, you are using algebra. Here is an example of a simple problem that you could solve without even thinking of algebra.

- While shopping Joe notices a store brand that is available at a lower price than the name brand. If the name brand product costs \$3.79 and Joe has a coupon for 50 cents, what would the store brand price need to be for Joe to save money with the coupon?

This is a problem you could solve mentally by subtracting the two quantities and finding that if the store brand costs \$3.29 or more, Joe would save money with the coupon. If the store brand is less than \$3.29, then Joe would save money by buying it instead. The beauty of algebra is not apparent in cases like this because you already know what operation to perform to find the unknown quantity. Algebra becomes useful when it is not clear what to do.

Grouping: 2 to 4 students per group

Topic: Use of algebra in common occurrences

We will consider another situation in which we want to find the best price of an item. For this situation we will use the following formulas.

The formula for markup of an item is

$$P = C + rC,$$

where P is the price of the item, C is the wholesale cost of the item, and r is the percent of markup.

The formula for discounting an item is

$$S = P - dP,$$

where S is the discounted price and d is the percent discount.

- Lane belongs to a wholesale buying club. She can order items through the club with a markup of 8% above the wholesale cost. She also can buy the same items in a store where she can get a 10% discount off the shelf price. Lane wants to know what the store markup on any particular item must be for it to be cheaper to order through the club.