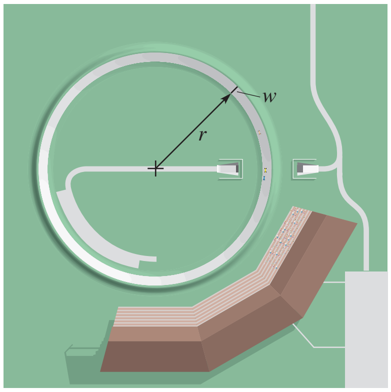


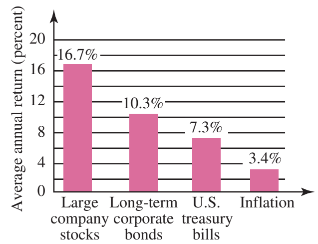

**FIGURE FOR EXERCISE 86**

Use a special product rule to simplify this formula. What is the cost of paving the track if the inside radius is 1000 feet and the width of the track is 40 feet?

**87.** ***Compounded annually.*** $P$ dollars is invested at annual interest rate $r$ for 2 years. If the interest is compounded annually, then the polynomial $P(1 + r)^2$ represents the value of the investment after 2 years. Rewrite this expression without parentheses. Evaluate the polynomial if $P = \$200$ and $r = 10\%$.

**88.** ***Compounded semiannually.*** $P$ dollars is invested at annual interest rate $r$ for 1 year. If the interest is compounded semiannually, then the polynomial $P\left(1 + \frac{r}{2}\right)^2$ represents the value of the investment after 1 year. Rewrite this expression without parentheses. Evaluate the polynomial if $P = \$200$ and $r = 10\%$.

**89.** ***Investing in treasury bills.*** An investment advisor uses the polynomial $P(1 + r)^{10}$ to predict the value in 10 years of a client's investment of $P$ dollars with an average annual return $r$. The accompanying graph shows historic average annual returns for the last 20 years for various asset classes (T. Rowe Price, www.troweprice.com). Use the historical average return to predict the value in 10 years of an investment of \$10,000 in U.S. treasury bills?



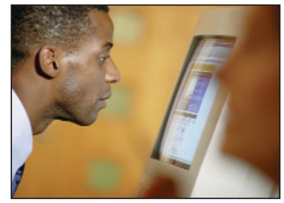



**FIGURE FOR EXERCISES 89 AND 90**

**90.** ***Comparing investments.*** How much more would the investment in Exercise 89 be worth in 10 years if the client invests in large company stocks rather than U.S. treasury bills?

## GETTING MORE INVOLVED

**91.** ***Writing.*** What is the difference between the equations $(x + 5)^2 = x^2 + 10x + 25$ and $(x + 5)^2 = x^2 + 25$?

**92.** ***Writing.*** Is it possible to square a sum or a difference without using the rules presented in this section? Why should you learn the rules given in this section?

# 5.5 DIVISION OF POLYNOMIALS

**In this section**

- Dividing Monomials Using the Quotient Rule
- Dividing a Polynomial by a Monomial
- Dividing a Polynomial by a Binomial

You multiplied polynomials in Section 5.2. In this section you will learn to divide polynomials.

## Dividing Monomials Using the Quotient Rule

In Chapter 1 we used the definition of division to divide signed numbers. Because the definition of division applies to any division, we restate it here.

**Division of Real Numbers**

If $a$, $b$, and $c$ are any numbers with $b \neq 0$, then

$$a \div b = c \qquad \text{provided that} \qquad c \cdot b = a.$$

**study tip**

Establish a regular routine of eating, sleeping, and exercise. The ability to concentrate depends on adequate sleep, decent nutrition, and the physical well-being that comes with exercise.

If $a \div b = c$, we call $a$ the **dividend,** $b$ the **divisor,** and $c$ (or $a \div b$) the **quotient.**

You can find the quotient of two monomials by writing the quotient as a fraction and then reducing the fraction. For example,

$$x^5 \div x^2 = \frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = x^3.$$

You can be sure that $x^3$ is correct by checking that $x^3 \cdot x^2 = x^5$. You can also divide $x^2$ by $x^5$, but the result is not a monomial:

$$x^2 \div x^5 = \frac{x^2}{x^5} = \frac{1 \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^3}$$

Note that the exponent 3 can be obtained in either case by subtracting 5 and 2. These examples illustrate the quotient rule for exponents.

**Quotient Rule**

Suppose $a \neq 0$, and $m$ and $n$ are positive integers.

$$\text{If } m \geq n, \text{ then } \frac{a^m}{a^n} = a^{m-n}.$$

$$\text{If } n > m, \text{ then } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

Note that if you use the quotient rule to subtract the exponents in $x^4 \div x^4$, you get the expression $x^{4-4}$, or $x^0$, which has not been defined yet. Because we must have $x^4 \div x^4 = 1$ if $x \neq 0$, we define the zero power of a nonzero real number to be 1. We do not define the expression $0^0$.

**Zero Exponent**

For any nonzero real number $a$,

$$a^0 = 1.$$

**EXAMPLE 1** **Using the definition of zero exponent**

Simplify each expression. Assume that all variables are nonzero real numbers.

**a)** $5^0$ **b)** $(3xy)^0$ **c)** $a^0 + b^0$

**Solution**

**a)** $5^0 = 1$ **b)** $(3xy)^0 = 1$ **c)** $a^0 + b^0 = 1 + 1 = 2$ ■

With the definition of zero exponent the quotient rule is valid for all positive integers as stated.

**EXAMPLE 2** **Using the quotient rule in dividing monomials**

Find each quotient.

**a)** $\frac{y^9}{y^5}$ **b)** $\frac{12b^2}{3b^7}$ **c)** $-6x^3 \div (2x^9)$ **d)** $\frac{x^8y^2}{x^2y^2}$

**helpful hint**

Recall that the order of operations gives multiplication and division an equal ranking and says to do them in order from left to right. So without parentheses,

$$-6x^3 \div 2x^9$$

actually means

$$\frac{-6x^3}{2} \cdot x^9.$$

## Solution

**a)** $\frac{y^9}{y^5} = y^{9-5} = y^4$

Use the definition of division to check that $y^4 \cdot y^5 = y^9$.

**b)** $\frac{12b^2}{3b^7} = \frac{12}{3} \cdot \frac{b^2}{b^7} = 4 \cdot \frac{1}{b^{7-2}} = \frac{4}{b^5}$

Use the definition of division to check that

$$\frac{4}{b^5} \cdot 3b^7 = \frac{12b^7}{b^5} = 12b^2.$$

**c)** $-6x^3 \div (2x^9) = \frac{-6x^3}{2x^9} = \frac{-3}{x^6}$

Use the definition of division to check that

$$\frac{-3}{x^6} \cdot 2x^9 = \frac{-6x^9}{x^6} = -6x^3.$$

**d)** $\frac{x^8y^2}{x^2y^2} = \frac{x^8}{x^2} \cdot \frac{y^2}{y^2} = x^6 \cdot y^0 = x^6$

Use the definition of division to check that $x^6 \cdot x^2y^2 = x^8y^2$. ■

We showed more steps in Example 2 than are necessary. For division problems like these you should try to write down only the quotient.

## Dividing a Polynomial by a Monomial

We divided some simple polynomials by monomials in Chapter 1. For example,

$$\frac{6x + 8}{2} = \frac{1}{2}(6x + 8) = \frac{6x}{2} + \frac{8}{2} = 3x + 4.$$

We use the distributive property to take one-half of $6x$ and one-half of 8 to get $3x + 4$. So both $6x$ and 8 are divided by 2. To divide any polynomial by a monomial, we divide each term of the polynomial by the monomial.

**EXAMPLE 3**

**Dividing a polynomial by a monomial**

Find the quotient for $(-8x^6 + 12x^4 - 4x^2) \div (4x^2)$.

**study tip**

Play offensive math, not defensive math. A student who says, "Give me a question and I'll see if I can answer it," is playing defensive math. The student is taking a passive approach to learning. A student who takes an active approach and knows the usual questions and answers for each topic is playing offensive math.

## Solution

$$\frac{-8x^6 + 12x^4 - 4x^2}{4x^2} = \frac{-8x^6}{4x^2} + \frac{12x^4}{4x^2} - \frac{4x^2}{4x^2}$$
$$= -2x^4 + 3x^2 - 1$$

The quotient is $-2x^4 + 3x^2 - 1$. We can check by multiplying.

$$4x^2(-2x^4 + 3x^2 - 1) = -8x^6 + 12x^4 - 4x^2.$$ ■

Because division by zero is undefined, we will always assume that the divisor is nonzero in any quotient involving variables. For example, the division in Example 3 is valid only if $4x^2 \neq 0$, or $x \neq 0$.

## Dividing a Polynomial by a Binomial

Division of whole numbers is often done with a procedure called **long division.** For example, 253 is divided by 7 as follows:

$$\begin{array}{r l}
36 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow 7\overline{)253} & \leftarrow \text{Dividend} \\
\underline{21} & \\
43 & \\
\underline{42} & \\
1 & \leftarrow \text{Remainder}
\end{array}$$

Note that $36 \cdot 7 + 1 = 253$. It is always true that

$$(\text{quotient})(\text{divisor}) + (\text{remainder}) = \text{dividend}.$$

To divide a polynomial by a binomial, we perform the division like long division of whole numbers. For example, to divide $x^2 - 3x - 10$ by $x + 2$, we get the first term of the quotient by dividing the first term of $x + 2$ into the first term of $x^2 - 3x - 10$. So divide $x^2$ by $x$ to get $x$, then multiply and subtract as follows:

$$\begin{array}{lll}
1 \quad \text{Divide:} & \phantom{x + 2)}x & x^2 \div x = x \\
2 \quad \text{Multiply:} & x + 2\overline{)x^2 - 3x - 10} & \\
 & \phantom{x + 2)}\underline{x^2 + 2x} & x \cdot (x + 2) = x^2 + 2x \\
3 \quad \text{Subtract:} & \phantom{x + 2)x^2 + }-5x & -3x - 2x = -5x
\end{array}$$

Now bring down $-10$ and continue the process. We get the second term of the quotient (below) by dividing the first term of $x + 2$ into the first term of $-5x - 10$. So divide $-5x$ by $x$ to get $-5$:

$$\begin{array}{lrl}
1 \quad \text{Divide:} & x - 5 & -5x \div x = -5 \\
2 \quad \text{Multiply:} & x + 2\overline{)x^2 - 3x - 10} & \\
 & \underline{x^2 + 2x} \quad \downarrow & \text{Bring down } -10. \\
 & -5x - 10 & \\
 & \underline{-5x - 10} & -5(x + 2) = -5x - 10 \\
3 \quad \text{Subtract:} & 0 & -5x - (-5x) = 0, -10 - (-10) = 0
\end{array}$$

So the quotient is $x - 5$, and the remainder is 0.

In the next example we must rearrange the dividend before dividing.

**EXAMPLE 4** **Dividing a polynomial by a binomial**

Divide $2x^3 - 4 - 7x^2$ by $2x - 3$, and identify the quotient and the remainder.

### Solution

Rearrange the dividend as $2x^3 - 7x^2 - 4$. Because the $x$-term in the dividend is missing, we write $0 \cdot x$ for it:

$$\begin{array}{rl}
x^2 - 2x - 3 & 2x^3 \div (2x) = x^2 \\
2x - 3\overline{)2x^3 - 7x^2 + 0 \cdot x - 4} & \\
\underline{2x^3 - 3x^2} & \\
-4x^2 + 0 \cdot x & -7x^2 - (-3x^2) = -4x^2 \\
\underline{-4x^2 + 6x} & \\
-6x - 4 & 0 \cdot x - 6x = -6x \\
\underline{-6x + 9} & \\
-13 & -4 - (9) = -13
\end{array}$$

**helpful hint**

Students usually have the most difficulty with the subtraction part of long division. So pay particular attention to that step and double check your work.

The quotient is $x^2 - 2x - 3$, and the remainder is $-13$. Note that the degree of the remainder is 0 and the degree of the divisor is 1. To check, we must verify that

$$(2x - 3)(x^2 - 2x - 3) - 13 = 2x^3 - 7x^2 - 4.$$

**CAUTION** To avoid errors, always write the terms of the divisor and the dividend in descending order of the exponents and insert a zero for any term that is missing.

If we divide both sides of the equation

$$\text{dividend} = (\text{quotient})(\text{divisor}) + (\text{remainder})$$

by the divisor, we get the equation

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

This fact is used in expressing improper fractions as mixed numbers. For example, if 19 is divided by 5, the quotient is 3 and the remainder is 4. So

$$\frac{19}{5} = 3 + \frac{4}{5} = 3\frac{4}{5}.$$

We can also use this form to rewrite algebraic fractions.

**EXAMPLE 5** **Rewriting algebraic fractions**

Express $\frac{-3x}{x-2}$ in the form

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

### Solution

Use long division to get the quotient and remainder:

$$\begin{array}{r} -3 \\ x - 2\overline{)-3x + 0} \\ \underline{-3x + 6} \\ -6 \end{array}$$

Because the quotient is $-3$ and the remainder is $-6$, we can write

$$\frac{-3x}{x-2} = -3 + \frac{-6}{x-2}.$$

To check, we must verify that $-3(x - 2) - 6 = -3x$.

**CAUTION** When dividing polynomials by long division, we do not stop until the remainder is 0 or the degree of the remainder is smaller than the degree of the divisor. For example, we stop dividing in Example 5 because the degree of the remainder $-6$ is 0 and the degree of the divisor $x - 2$ is 1.

## WARM-UPS

**True or false? Explain your answer.**

**1.** $y^{10} \div y^2 = y^5$ for any nonzero value of $y$.

**2.** $\frac{7x + 2}{7} = x + 2$ for any value of $x$.

## WARM-UPS

**(*continued*)**

**3.** $\frac{7x^2}{7} = x^2$ for any value of $x$.

**4.** If $3x^2 + 6$ is divided by 3, the quotient is $x^2 + 6$.

**5.** If $4y^2 - 6y$ is divided by $2y$, the quotient is $2y - 3$.

**6.** The quotient times the remainder plus the dividend equals the divisor.

**7.** $(x + 2)(x + 1) + 3 = x^2 + 3x + 5$ for any value of $x$.

**8.** If $x^2 + 3x + 5$ is divided by $x + 2$, then the quotient is $x + 1$.

**9.** If $x^2 + 3x + 5$ is divided by $x + 2$, the remainder is 3.

**10.** If the remainder is zero, then (divisor)(quotient) $=$ dividend.

## 5.5 EXERCISES

***Reading and Writing*** *After reading this section, write out the answers to these questions. Use complete sentences.*

**1.** What rule is important for dividing monomials?

**2.** What is the meaning of a zero exponent?

**3.** How many terms should you get when dividing a polynomial by a monomial?

**4.** How should the terms of the polynomials be written when dividing with long division?

**5.** How do you know when to stop the process in long division of polynomials?

**6.** How do you handle missing terms in the dividend polynomial when doing long division?

*Simplify each expression. See Example 1.*

**7.** $9^0$ **8.** $m^0$ **9.** $(-2x^3)^0$

**10.** $(5a^3b)^0$ **11.** $2 \cdot 5^0 - 3^0$ **12.** $-4^0 - 8^0$

**13.** $(2x - y)^0$ **14.** $(a^2 + b^2)^0$

*Find each quotient. Try to write only the answer. See Example 2.*

**15.** $\frac{x^8}{x^2}$ **16.** $\frac{y^9}{y^3}$

**17.** $\frac{6a^7}{2a^{12}}$ **18.** $\frac{30b^2}{3b^6}$

**19.** $-12x^5 \div (3x^9)$ **20.** $-6y^5 \div (-3y^{10})$

**21.** $-6y^2 \div (6y)$ **22.** $-3a^2b \div (3ab)$

**23.** $\frac{-6x^3y^2}{2x^2y^2}$ **24.** $\frac{-4h^2k^4}{-2hk^3}$

**25.** $\frac{-9x^2y^2}{3x^5y^2}$ **26.** $\frac{-12z^4y^2}{-2z^{10}y^2}$

*Find the quotients. See Example 3.*

**27.** $\frac{3x - 6}{3}$

**28.** $\frac{5y - 10}{-5}$

**29.** $\frac{x^5 + 3x^4 - x^3}{x^2}$

**30.** $\frac{6y^6 - 9y^4 + 12y^2}{3y^2}$

**31.** $\frac{-8x^2y^2 + 4x^2y - 2xy^2}{-2xy}$

**32.** $\frac{-9ab^2 - 6a^3b^3}{-3ab^2}$

**33.** $(x^2y^3 - 3x^3y^2) \div (x^2y)$

**34.** $(4h^5k - 6h^2k^2) \div (-2h^2k)$

*Find the quotient and remainder for each division. Check by using the fact that dividend = (divisor)(quotient) + remainder. See Example 4.*

**35.** $(x^2 + 5x + 13) \div (x + 3)$

**36.** $(x^2 + 3x + 6) \div (x + 3)$

**37.** $(2x) \div (x + 5)$

**38.** $(5x) \div (x - 1)$

**39.** $(a^3 + 4a - 3) \div (a - 2)$

**40.** $(w^3 + 2w^2 - 3) \div (w - 2)$

**41.** $(x^2 - 3x) \div (x + 1)$

**42.** $(3x^2) \div (x + 1)$

**43.** $(h^3 - 27) \div (h - 3)$

**44.** $(w^3 + 1) \div (w + 1)$