5.6 POSITIVE INTEGRAL EXPONENTS

In this section

- The Product and Quotient Rules
- Raising an Exponential Expression to a Power
- Power of a Product
- Power of a Quotient
- Summary of Rules

The product rule for positive integral exponents was presented in Section 5.2, and the quotient rule was presented in Section 5.5. In this section we review those rules and then further investigate the properties of exponents.

The Product and Quotient Rules

The rules that we have already discussed are summarized below.

The following rules hold for nonnegative integers m and n and $a \neq 0$.

$$a^m \cdot a^n = a^{m+n} \qquad \text{Product rule}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{if } m \geq n \qquad \text{Quotient rule}$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad \text{if } n > m$$

$$a^0 = 1 \qquad \text{Zero exponent}$$

CAUTION The product and quotient rules apply only if the bases of the expressions are identical. For example, $3^2 \cdot 3^4 = 3^6$, but the product rule cannot be applied to $5^2 \cdot 3^4$. Note also that the bases are not multiplied: $3^2 \cdot 3^4 \neq 9^6$.

Note that in the quotient rule the exponents are always subtracted, as in

$$\frac{x^7}{x^3} = x^4 \qquad \text{and} \qquad \frac{y^5}{y^8} = \frac{1}{y^3}.$$

If the larger exponent is in the denominator, then the result is placed in the denominator.

EXAMPLE 1 **Using the product and quotient rules**

Use the rules of exponents to simplify each expression. Assume that all variables represent nonzero real numbers.

a) $2^3 \cdot 2^2$

b) $(3x)^0(5x^2)(4x)$

c) $\dfrac{8x^2}{-2x^5}$

d) $\dfrac{(3a^2b)b^9}{(6a^5)a^3b^2}$

Solution

a) Because the bases are both 2, we can use the product rule:

$$2^3 \cdot 2^2 = 2^5 \qquad \text{Product rule}$$
$$= 32 \qquad \text{Simplify.}$$

b) $(3x)^0(5x^2)(4x) = 1 \cdot 5x^2 \cdot 4x$ — Definition of zero exponent

$= 20x^3$ — Product rule

c) $\dfrac{8x^2}{-2x^5} = -\dfrac{4}{x^3}$ — Quotient rule

study tip

Keep track of your time for one entire week. Account for how you spend every half hour. Add up your totals for sleep, study, work, and recreation. You should be sleeping 50–60 hours per week and studying 1–2 hours for every hour you spend in the classroom.

d) First use the product rule to simplify the numerator and denominator:

$$\frac{(3a^2b)b^9}{(6a^5)a^3b^2} = \frac{3a^2b^{10}}{6a^8b^2} \quad \text{Product rule}$$

$$= \frac{b^8}{2a^6} \quad \text{Quotient rule}$$

Raising an Exponential Expression to a Power

When we raise an exponential expression to a power, we can use the product rule to find the result, as shown in the following example:

$$(w^4)^3 = w^4 \cdot w^4 \cdot w^4 \quad \text{Three factors of } w^4 \text{ because of the exponent 3}$$

$$= w^{12} \quad \text{Product rule}$$

By the product rule we add the three 4's to get 12, but 12 is also the product of 4 and 3. This example illustrates the **power rule** for exponents.

Power Rule

If m and n are nonnegative integers and $a \neq 0$, then

$$(a^m)^n = a^{mn}.$$

In the next example we use the new rule along with the other rules.

EXAMPLE 2

Using the power rule

Use the rules of exponents to simplify each expression. Assume that all variables represent nonzero real numbers.

a) $3x^2(x^3)^5$ **b)** $\frac{(2^3)^4 \cdot 2^7}{2^5 \cdot 2^9}$ **c)** $\frac{3(x^5)^4}{15x^{22}}$

Solution

a) $3x^2(x^3)^5 = 3x^2x^{15}$ Power rule

$= 3x^{17}$ Product rule

b) $\frac{(2^3)^4 \cdot 2^7}{2^5 \cdot 2^9} = \frac{2^{12} \cdot 2^7}{2^{14}}$ Power rule and product rule

$= \frac{2^{19}}{2^{14}}$ Product rule

$= 2^5$ Quotient rule

$= 32$ Evaluate 2^5.

c) $\frac{3(x^5)^4}{15x^{22}} = \frac{3x^{20}}{15x^{22}} = \frac{1}{5x^2}$

Power of a Product

Consider an example of raising a monomial to a power. We will use known rules to rewrite the expression.

$$(2x)^3 = 2x \cdot 2x \cdot 2x \quad \text{Definition of exponent 3}$$

$$= 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \quad \text{Commutative and associative properties}$$

$$= 2^3x^3 \quad \text{Definition of exponents}$$

Note that the power 3 is applied to each factor of the product. This example illustrates the **power of a product rule.**

Power of a Product Rule

If a and b are real numbers and n is a positive integer, then

$$(ab)^n = a^n b^n.$$

EXAMPLE 3

Using the power of a product rule

Simplify. Assume that the variables are nonzero.

a) $(xy^3)^5$ **b)** $(-3m)^3$ **c)** $(2x^3y^2z^7)^3$

Solution

a) $(xy^3)^5 = x^5(y^3)^5$ Power of a product rule

$= x^5y^{15}$ Power rule

b) $(-3m)^3 = (-3)^3m^3$ Power of a product rule

$= -27m^3$ $(-3)(-3)(-3) = -27$

c) $(2x^3y^2z^7)^3 = 2^3(x^3)^3(y^2)^3(z^7)^3 = 8x^9y^6z^{21}$

Power of a Quotient

helpful hint

Note that these rules of exponents are not absolutely necessary. We could simplify every expression here by using only the definition of exponent. However, these rules make it a lot simpler.

Raising a quotient to a power is similar to raising a product to a power:

$$\left(\frac{x}{5}\right)^3 = \frac{x}{5} \cdot \frac{x}{5} \cdot \frac{x}{5}$$ Definition of exponent 3

$$= \frac{x \cdot x \cdot x}{5 \cdot 5 \cdot 5}$$ Definition of multiplication of fractions

$$= \frac{x^3}{5^3}$$ Definition of exponents

The power is applied to both the numerator and denominator. This example illustrates the **power of a quotient rule.**

Power of a Quotient Rule

If a and b are real numbers, $b \neq 0$, and n is a positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

EXAMPLE 4

Using the power of a quotient rule

Simplify. Assume that the variables are nonzero.

a) $\left(\frac{2}{5x^3}\right)^2$ **b)** $\left(\frac{3x^4}{2y^3}\right)^3$ **c)** $\left(\frac{-12a^5b}{4a^2b^7}\right)^3$

Solution

a) $\left(\frac{2}{5x^3}\right)^2 = \frac{2^2}{(5x^3)^2}$ Power of a quotient rule

$= \frac{4}{25x^6}$ $(5x^3)^2 = 5^2(x^3)^2 = 25x^6$

b) $\left(\dfrac{3x^4}{2y^3}\right)^3 = \dfrac{3^3x^{12}}{2^3y^9}$ Power of a quotient and power of a product rule

$= \dfrac{27x^{12}}{8y^9}$ Simplify.

c) Use the quotient rule to simplify the expression inside the parentheses before using the power of a quotient rule.

$$\left(\frac{-12a^5b}{4a^2b^7}\right)^3 = \left(\frac{-3a^3}{b^6}\right)^3 \quad \text{Use the quotient rule first.}$$

$$= \frac{-27a^9}{b^{18}} \quad \text{Power of a quotient rule}$$

Summary of Rules

helpful hint

Note that the rules of exponents show how exponents behave with respect to multiplication and division only. We studied the more complicated problem of using exponents with addition and subtraction in Section 5.4 when we learned rules for $(a + b)^2$ and $(a - b)^2$.

The rules for exponents are summarized in the following box.

Rules for Nonnegative Integral Exponents

The following rules hold for nonzero real numbers a and b and nonnegative integers m and n.

1. $a^0 = 1$ Definition of zero exponent
2. $a^m \cdot a^n = a^{m+n}$ Product rule
3. $\dfrac{a^m}{a^n} = a^{m-n}$ for $m \geq n$,

   $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$ for $n > m$ Quotient rule
4. $(a^m)^n = a^{mn}$ Power rule
5. $(ab)^n = a^n \cdot b^n$ Power of a product rule
6. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ Power of a quotient rule

WARM-UPS

True or false? Assume that all variables represent nonzero real numbers. A statement involving variables is to be marked true only if it is an identity. Explain your answer.

1. $-3^0 = 1$
2. $2^5 \cdot 2^8 = 4^{13}$
3. $2^3 \cdot 3^3 = 6^5$
4. $(2x)^4 = 2x^4$
5. $(q^3)^5 = q^8$
6. $(-3x^2)^3 = 27x^6$
7. $(ab^3)^4 = a^4b^{12}$
8. $\dfrac{a^{12}}{a^4} = a^3$
9. $\dfrac{6w^4}{3w^9} = 2w^5$
10. $\left(\dfrac{2y^3}{9}\right)^2 = \dfrac{4y^6}{81}$

5.6 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the product rule for exponents?
2. What is the quotient rule for exponents?
3. Why must the bases be the same in these rules?
4. What is the power rule for exponents?
5. What is the power of a product rule?
6. What is the power of a quotient rule?

For all exercises in this section, assume that the variables represent nonzero real numbers.

Simplify the exponential expressions. See Example 1.

7. $2^2 \cdot 2^5$ **8.** $x^6 \cdot x^7$

9. $(-3u^8)(-2u^2)$ **10.** $(3r^4)(-6r^2)$

11. $a^3b^4 \cdot ab^6(ab)^0$ **12.** $x^2y \cdot x^3y^6(x + y)^0$

13. $\dfrac{-2a^3}{4a^7}$ **14.** $\dfrac{-3t^9}{6t^{18}}$

15. $\dfrac{2a^5b \cdot 3a^7b^3}{15a^6b^8}$ **16.** $\dfrac{3xy^8 \cdot 5xy^9}{20x^3y^{14}}$

17. $2^3 \cdot 5^2$ **18.** $2^2 \cdot 10^3$

Simplify. See Example 2.

19. $(x^2)^3$ **20.** $(y^2)^4$

21. $2x^2 \cdot (x^2)^5$ **22.** $(y^2)^6 \cdot 3y^5$

23. $\dfrac{(t^2)^5}{(t^3)^4}$ **24.** $\dfrac{(r^4)^2}{(r^5)^3}$

25. $\dfrac{3x(x^5)^2}{6x^3(x^2)^4}$ **26.** $\dfrac{5y^3(y^5)^2}{10y^5(y^2)^6}$

Simplify. See Example 3.

27. $(xy^2)^3$

28. $(wy^2)^6$

29. $(-2t^5)^3$

30. $(-3r^3)^3$

31. $(-2x^2y^5)^3$

32. $(-3y^2z^3)^3$

33. $\dfrac{(a^4b^2c^5)^3}{a^3b^4c}$

34. $\dfrac{(2ab^2c^3)^5}{(2a^3bc)^4}$

Simplify. See Example 4.

35. $\left(\dfrac{x^4}{4}\right)^3$ **36.** $\left(\dfrac{y^2}{2}\right)^3$

37. $\left(\dfrac{-2a^2}{b^3}\right)^4$ **38.** $\left(\dfrac{-9r^3}{t^5}\right)^2$

39. $\left(\dfrac{2x^2y}{-4y^2}\right)^3$ **40.** $\left(\dfrac{3y^8}{2zy^2}\right)^4$

41. $\left(\dfrac{-6x^2y^4z^9}{3x^6y^4z^3}\right)^2$ **42.** $\left(\dfrac{-10rs^9t^4}{2rs^2t^7}\right)^3$

Simplify each expression. Your answer should be an integer or a fraction. Do not use a calculator.

43. $3^2 + 6^2$ **44.** $(5 - 3)^2$

45. $(3 + 6)^2$ **46.** $5^2 - 3^2$

47. $2^3 - 3^3$ **48.** $3^3 + 4^3$

49. $(2 - 3)^3$ **50.** $(3 + 4)^3$

51. $\left(\dfrac{2}{5}\right)^3$ **52.** $\left(\dfrac{3}{4}\right)^3$

53. $5^2 \cdot 2^3$ **54.** $10^3 \cdot 3^3$

55. $2^3 \cdot 2^4$ **56.** $10^2 \cdot 10^4$

57. $\left(\dfrac{2^3}{2^5}\right)^2$ **58.** $\left(\dfrac{3}{3^3}\right)^2$

Simplify each expression.

59. $3x^4 \cdot 5x^7$ **60.** $-2y^3(3y)$

61. $(-5x^4)^3$ **62.** $(4z^3)^3$

63. $-3y^5z^{12} \cdot 9yz^7$ **64.** $2a^4b^5 \cdot 2a^9b^2$

65. $\dfrac{-9u^4v^9}{-3u^5v^8}$ **66.** $\dfrac{-20a^5b^{13}}{5a^4b^{13}}$

67. $(-xt^2)(-2x^2t)^4$ **68.** $(-ab)^3(-3ba^2)^4$

69. $\left(\dfrac{2x^2}{x^4}\right)^3$ **70.** $\left(\dfrac{3y^8}{y^5}\right)^2$

71. $\left(\dfrac{-8a^3b^4}{4c^5}\right)^5$ **72.** $\left(\dfrac{-10a^5c}{5a^5b^4}\right)^5$

73. $\left(\dfrac{-8x^4y^7}{-16x^5y^6}\right)^5$ **74.** $\left(\dfrac{-5x^2yz^3}{-5x^2yz}\right)^5$

Solve each problem.

75. ***Long-term investing.*** Sheila invested P dollars at annual rate r for 10 years. At the end of 10 years her investment was worth $P(1 + r)^{10}$ dollars. She then reinvested this money for another 5 years at annual rate r. At the end of the second time period her investment was worth $P(1 + r)^{10}(1 + r)^5$ dollars. Which law of exponents can be used to simplify the last expression? Simplify it.

76. ***CD rollover.*** Ronnie invested P dollars in a 2-year CD with an annual rate of return of r. After the CD rolled over two times, its value was $P((1 + r)^2)^3$. Which law of exponents can be used to simplify the expression? Simplify it.

GETTING MORE INVOLVED

77. ***Writing.*** When we square a product, we square each factor in the product. For example, $(3b)^2 = 9b^2$. Explain why we cannot square a sum by simply squaring each term of the sum.

78. ***Writing.*** Explain why we define 2^0 to be 1. Explain why $-2^0 \neq 1$.

5.7 NEGATIVE EXPONENTS AND SCIENTIFIC NOTATION

In this section

- Negative Integral Exponents
- Rules for Integral Exponents
- Converting from Scientific Notation
- Converting to Scientific Notation
- Computations with Scientific Notation

We defined exponential expressions with positive integral exponents in Chapter 1 and learned the rules for positive integral exponents in Section 5.6. In this section you will first study negative exponents and then see how positive and negative integral exponents are used in scientific notation.

Negative Integral Exponents

If x is nonzero, the reciprocal of x is written as $\frac{1}{x}$. For example, the reciprocal of 2^3 is written as $\frac{1}{2^3}$. To write the reciprocal of an exponential expression in a simpler way, we use a negative exponent. So $2^{-3} = \frac{1}{2^3}$. In general we have the following definition.

Negative Integral Exponents

If a is a nonzero real number and n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}. \quad \text{(If } n \text{ is positive, } -n \text{ is negative.)}$$

Since a^{-n} and a^n are reciprocals, their product is 1. Using a negative exponent for the reciprocal allows us to get this result with the product rule for exponents:

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1$$

EXAMPLE 1 **Simplifying expressions with negative exponents**

Simplify.

a) 2^{-5} **b)** $(-2)^{-5}$ **c)** $\frac{2^{-3}}{3^{-2}}$

Solution

a) $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

b) $(-2)^{-5} = \frac{1}{(-2)^5}$ Definition of negative exponent

$= \frac{1}{-32} = -\frac{1}{32}$

c) $\frac{2^{-3}}{3^{-2}} = 2^{-3} \div 3^{-2}$

$= \frac{1}{2^3} \div \frac{1}{3^2}$

$= \frac{1}{8} \div \frac{1}{9} = \frac{1}{8} \cdot \frac{9}{1} = \frac{9}{8}$ ■

calculator close-up

You can evaluate expressions with negative exponents on a calculator as shown here.

```
2^-5▸Frac
              1/32
(-2)^-5▸Frac
             -1/32
2^-3/3^-2▸Frac
               9/8
```