6.1 FACTORING OUT COMMON FACTORS

In this section

- Prime Factorization of Integers
- Greatest Common Factor
- Finding the Greatest Common Factor for Monomials
- Factoring Out the Greatest Common Factor
- Factoring Out the Opposite of the GCF

In Chapter 5 you learned how to multiply a monomial and a polynomial. In this section you will learn how to reverse that multiplication by finding the greatest common factor for the terms of a polynomial and then factoring the polynomial.

Prime Factorization of Integers

To **factor** an expression means to write the expression as a product. For example, if we start with 12 and write $12 = 4 \cdot 3$, we have factored 12. Both 4 and 3 are **factors** or **divisors** of 12. There are other factorizations of 12:

$$12 = 2 \cdot 6 \qquad 12 = 1 \cdot 12 \qquad 12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

The one that is most useful to us is $12 = 2^2 \cdot 3$, because it expresses 12 as a product of *prime numbers.*

Prime Number

A positive integer larger than 1 that has no integral factors other than itself and 1 is called a **prime number.**

The numbers 2, 3, 5, 7, 11, 13, 17, 19, and 23 are the first nine prime numbers. A positive integer larger than 1 that is not a prime is a **composite number.** The numbers 4, 6, 8, 9, 10, and 12 are the first six composite numbers. Every composite number is a product of prime numbers. The **prime factorization** for 12 is $2^2 \cdot 3$.

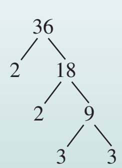
EXAMPLE 1

Prime factorization

Find the prime factorization for 36.

helpful hint

The prime factorization of 36 can be found also with a *factoring tree:*



So $36 = 2 \cdot 2 \cdot 3 \cdot 3$.

Solution

We start by writing 36 as a product of two integers:

$$\begin{aligned} 36 &= 2 \cdot 18 && \text{Write 36 as } 2 \cdot 18. \\ &= 2 \cdot 2 \cdot 9 && \text{Replace 18 by } 2 \cdot 9. \\ &= 2 \cdot 2 \cdot 3 \cdot 3 && \text{Replace 9 by } 3 \cdot 3. \\ &= 2^2 \cdot 3^2 && \text{Use exponential notation.} \end{aligned}$$

The prime factorization of 36 is $2^2 \cdot 3^2$. ■

For larger numbers it is helpful to use the method shown in the next example.

EXAMPLE 2

Factoring a large number

Find the prime factorization for 420.

Solution

Start by dividing 420 by the smallest prime number that will divide into it evenly (without remainder). The smallest prime divisor of 420 is 2.

$$2\overline{)420}^{\;210}$$

helpful hint

If a number is even, then it is divisible by 2. If the sum of the digits of a number is divisible by 3, then the number is divisible by 3. A number that ends in 0 or 5 is divisible by 5.

Now find the smallest prime that will divide evenly into the quotient, 210. The smallest prime divisor of 210 is 2. Continue this procedure, as follows, until the quotient is a prime number:

$$\begin{array}{rl} 7 & \quad 35 \div 5 = 7 \\ 5\overline{)35} & \quad 105 \div 3 = 35 \\ 3\overline{)105} & \quad 210 \div 2 = 105 \\ 2\overline{)210} & \\ \text{Start here} \rightarrow \; 2\overline{)420} & \end{array}$$

The prime factorization of 420 is $2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$, or $2^2 \cdot 3 \cdot 5 \cdot 7$. Note that it is really not necessary to divide by the smallest prime divisor at each step. We obtain the same factorization if we divide by any prime divisor at each step. ■

Greatest Common Factor

The largest integer that is a factor of two or more integers is called the **greatest common factor (GCF)** of the integers. For example, 1, 2, 3, and 6 are common factors of 18 and 24. Because 6 is the largest, 6 is the GCF of 18 and 24. We can use prime factorizations to find the GCF. For example, to find the GCF of 8 and 12, we first factor 8 and 12:

$$8 = 2 \cdot 2 \cdot 2 = 2^3 \qquad 12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

We see that the factor 2 appears twice in both 8 and 12. So 2^2, or 4, is the GCF of 8 and 12. Notice that 2 is a factor in both 2^3 and $2^2 \cdot 3$ and that 2^2 is the smallest power of 2 in these factorizations. In general, we can use the following strategy to find the GCF.

Strategy for Finding the GCF for Positive Integers

1. Find the prime factorization of each integer.
2. Determine which primes appear in all of the factorizations and the smallest exponent that appears on each of the common prime factors.
3. The GCF is the product of the common prime factors using the exponents from part (2).

If two integers have no common prime factors, then their greatest common factor is 1, because 1 is a factor of every integer. For example, 6 and 35 have no common prime factors ($6 = 2 \cdot 3$ and $35 = 5 \cdot 7$). So the GCF for 6 and 35 is 1.

EXAMPLE 3 **Greatest common factor**

Find the GCF for each group of numbers.

a) 150, 225 **b)** 216, 360, 504 **c)** 55, 168

Solution

a) First find the prime factorization for each number:

$$\begin{array}{r} 5 \\ 5\overline{)25} \\ 3\overline{)75} \\ 2\overline{)150} \end{array} \qquad \begin{array}{r} 5 \\ 5\overline{)25} \\ 3\overline{)75} \\ 3\overline{)225} \end{array}$$

$$150 = 2 \cdot 3 \cdot 5^2 \qquad 225 = 3^2 \cdot 5^2$$

Because 2 is not a factor of 225, it is not a common factor of 150 and 225. Only 3 and 5 appear in both factorizations. Looking at both $2 \cdot 3 \cdot 5^2$ and $3^2 \cdot 5^2$, we see that the smallest power of 5 is 2 and the smallest power of 3 is 1. So the GCF of 150 and 225 is $3 \cdot 5^2$, or 75.

b) First find the prime factorization for each number:

$$216 = 2^3 \cdot 3^3 \qquad 360 = 2^3 \cdot 3^2 \cdot 5 \qquad 504 = 2^3 \cdot 3^2 \cdot 7$$

The only common prime factors are 2 and 3. The smallest power of 2 in the factorizations is 3, and the smallest power of 3 is 2. So the GCF is $2^3 \cdot 3^2$, or 72.

c) First find the prime factorization for each number:

$$55 = 5 \cdot 11 \qquad 168 = 2^3 \cdot 3 \cdot 7$$

Because there are no common factors other than 1, the GCF is 1. ■

helpful hint

The fact that every composite number has a unique prime factorization is known as the fundamental theorem of arithmetic.

Finding the Greatest Common Factor for Monomials

To find the GCF for a group of monomials, we use the same procedure as that used for integers.

Strategy for Finding the GCF for Monomials

1. Find the GCF for the coefficients of the monomials.
2. Form the product of the GCF of the coefficients and each variable that is common to all of the monomials, where the exponent on each variable is the smallest power of that variable in any of the monomials.

EXAMPLE 4 **Greatest common factor of monomials**

Find the greatest common factor for each group of monomials.

a) $15x^2, 9x^3$ **b)** $12x^2y^2, 30x^2yz, 42x^3y$

Solution

a) The GCF for 15 and 9 is 3, and the smallest power of x is 2. So the GCF for the monomials is $3x^2$. If we write these monomials as

$$15x^2 = 5 \cdot 3 \cdot x \cdot x \qquad \text{and} \qquad 9x^3 = 3 \cdot 3 \cdot x \cdot x \cdot x,$$

we can see that $3x^2$ is the GCF.

b) The GCF for 12, 30, and 42 is 6. For the common variables x and y, 2 is the smallest power of x and 1 is the smallest power of y. So the GCF for the monomials is $6x^2y$. ■

Factoring Out the Greatest Common Factor

In Chapter 5 we used the distributive property to multiply monomials and polynomials. For example,

$$6(5x - 3) = 30x - 18.$$

If we start with $30x - 18$ and write

$$30x - 18 = 6(5x - 3),$$

we have factored $30x - 18$. Because multiplication is the last operation to be performed in $6(5x - 3)$, the expression $6(5x - 3)$ is a product. Because 6 is the GCF of 30 and 18, we have **factored out** the GCF.

EXAMPLE 5 **Factoring out the greatest common factor**

Factor the following polynomials by factoring out the GCF.

a) $25a^2 + 40a$ **b)** $6x^4 - 12x^3 + 3x^2$

c) $x^2y^5 + x^6y^3$ **d)** $(a + b)w + (a + b)6$

Solution

a) The GCF of the coefficients 25 and 40 is 5. Because the smallest power of the common factor a is 1, we can factor $5a$ out of each term:

$$25a^2 + 40a = 5a \cdot 5a + 5a \cdot 8$$
$$= 5a(5a + 8)$$

b) The GCF of 6, 12, and 3 is 3. We can factor x^2 out of each term, since the smallest power of x in the three terms is 2. So factor $3x^2$ out of each term as follows:

$$6x^4 - 12x^3 + 3x^2 = 3x^2 \cdot 2x^2 - 3x^2 \cdot 4x + 3x^2 \cdot 1$$
$$= 3x^2(2x^2 - 4x + 1)$$

Check by multiplying: $3x^2(2x^2 - 4x + 1) = 6x^4 - 12x^3 + 3x^2$.

c) The GCF of the numerical coefficients is 1. Both x and y are common to each term. Using the lowest powers of x and y, we get

$$x^2y^5 + x^6y^3 = x^2y^3 \cdot y^2 + x^2y^3 \cdot x^4$$
$$= x^2y^3(y^2 + x^4).$$

Check by multiplying.

d) Even though this expression looks different from the rest, we can factor it in the same way. The binomial $a + b$ is a common factor, and we can factor it out just as we factor out a monomial:

$$(a + b)w + (a + b)6 = (a + b)(w + 6)$$

■

study tip

The keys to college success are motivation and time management. Anyone who tells you that they are making great grades without studying is probably not telling the truth. Success in college takes effort.

CAUTION If the GCF is one of the terms of the polynomial, then you must remember to leave a 1 in place of that term when the GCF is factored out. For example,

$$ab + b = a \cdot b + 1 \cdot b = b(a + 1).$$

You should always check your answer by multiplying the factors.

Factoring Out the Opposite of the GCF

Because the greatest common factor for $-4x + 2xy$ is $2x$, we write

$$-4x + 2xy = 2x(-2 + y).$$

We could factor out $-2x$, the opposite of the greatest common factor:

$$-4x + 2xy = -2x(2 - y).$$

It will be necessary to factor out the opposite of the greatest common factor when you learn factoring by grouping in Section 6.2. Remember that you can check all factoring by multiplying the factors to see whether you get the original polynomial.

EXAMPLE 6 **Factoring out the opposite of the GCF**

Factor each polynomial twice. First factor out the greatest common factor, and then factor out the opposite of the GCF.

a) $3x - 3y$ **b)** $a - b$ **c)** $-x^3 + 2x^2 - 8x$

Solution

a) $3x - 3y = 3(x - y)$ Factor out 3.

$= -3(-x + y)$ Factor out -3.

Note that the signs of the terms in parentheses change when -3 is factored out. Check the answers by multiplying.

b) $a - b = 1(a - b)$ Factor out 1, the GCF of a and b.

$= -1(-a + b)$ Factor out -1.

We can also write $a - b = -1(b - a)$.

c) $-x^3 + 2x^2 - 8x = x(-x^2 + 2x - 8)$ Factor out x.

$= -x(x^2 - 2x + 8)$ Factor out $-x$. ■

CAUTION Be sure to change the sign of each term in parentheses when you factor out the opposite of the greatest common factor.

In the next example we factor to find the length of a rectangle.

EXAMPLE 7

An application of factoring

The width of a rectangle is w meters and its area is $w^2 + 30w$ square meters. Find an expression for the length of the rectangle.

Solution

The area of a rectangle is the product of its length and width. Since

$$A = w^2 + 30w = w(w + 30)$$

and w is the width, the length is $w + 30$ meters. ■

WARM-UPS

True or false? Explain your answer.

1. There are only nine prime numbers.
2. The prime factorization of 32 is $2^3 \cdot 3$.
3. The integer 51 is a prime number.
4. The GCF of the integers 12 and 16 is 4.
5. The GCF of the integers 10 and 21 is 1.
6. The GCF of the polynomial $x^5y^3 - x^4y^7$ is x^4y^3.
7. For the polynomial $2x^2y - 6xy^2$ we can factor out either $2xy$ or $-2xy$.
8. The greatest common factor of the polynomial $8a^3b - 12a^2b$ is $4ab$.
9. $x - 7 = 7 - x$ for any real number x.
10. $-3x^2 + 6x = -3x(x - 2)$ for any real number x.

6.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What does it mean to factor an expression?
2. What is a prime number?
3. How do you find the prime factorization of a number?
4. What is the greatest common factor for two numbers?

5. What is the greatest common factor for two monomials?

6. How can you check if you have factored an expression correctly?

Find the prime factorization of each integer. See Examples 1 and 2.

7. 18
8. 20
9. 52
10. 76
11. 98
12. 100
13. 460
14. 345
15. 924
16. 585

Find the greatest common factor (GCF) for each group of integers. See Example 3.

17. 8, 20
18. 18, 42
19. 36, 60
20. 42, 70
21. 40, 48, 88
22. 15, 35, 45
23. 76, 84, 100
24. 66, 72, 120
25. 39, 68, 77
26. 81, 200, 539

Find the greatest common factor (GCF) for each group of monomials. See Example 4.

27. $6x, 8x^3$
28. $12x^2, 4x^3$
29. $12x^3, 4x^2, 6x$
30. $3y^5, 9y^4, 15y^3$
31. $3x^2y, 2xy^2$
32. $7a^2x^3, 5a^3x$
33. $24a^2bc, 60ab^2$
34. $30x^2yz^3, 75x^3yz^6$
35. $12u^3v^2, 25s^2t^4$
36. $45m^2n^5, 56a^4b^8$
37. $18a^3b, 30a^2b^2, 54ab^3$
38. $16x^2z, 40xz^2, 72z^3$

Complete the factoring of each monomial.

39. $27x = 9(\quad)$
40. $51y = 3y(\quad)$
41. $24t^2 = 8t(\quad)$
42. $18u^2 = 3u(\quad)$
43. $36y^5 = 4y^2(\quad)$
44. $42z^4 = 3z^2(\quad)$
45. $u^4v^3 = uv(\quad)$
46. $x^5y^3 = x^2y(\quad)$
47. $-14m^4n^3 = 2m^4(\quad)$
48. $-8y^3z^4 = 4z^3(\quad)$
49. $-33x^4y^3z^2 = -3x^3yz(\quad)$
50. $-96a^3b^4c^5 = -12ab^3c^3(\quad)$

Factor out the GCF in each expression. See Example 5.

51. $x^3 - 6x$
52. $10y^4 - 30y^2$
53. $5ax + 5ay$
54. $6wz + 15wa$
55. $h^5 - h^3$
56. $y^6 + y^5$
57. $-2k^7m^4 + 4k^3m^6$
58. $-6h^5t^2 + 3h^3t^6$
59. $2x^3 - 6x^2 + 8x$
60. $6x^3 + 18x^2 - 24x$
61. $12x^4t + 30x^3t - 24x^2t^2$
62. $15x^2y^2 - 9xy^2 + 6x^2y$
63. $(x - 3)a + (x - 3)b$
64. $(y + 4)3 + (y + 4)z$
65. $a(y + 1)^2 + b(y + 1)^2$
66. $w(w + 2)^2 + 8(w + 2)^2$
67. $36a^3b^5 - 27a^2b^4 + 18a^2b^9$
68. $56x^3y^5 - 40x^2y^6 + 8x^2y^3$

First factor out the GCF, and then factor out the opposite of the GCF. See Example 6.

69. $8x - 8y$
70. $2a - 6b$
71. $-4x + 8x^2$
72. $-5x^2 + 10x$
73. $x - 5$
74. $a - 6$
75. $4 - 7a$
76. $7 - 5b$
77. $-24a^3 + 16a^2$
78. $-30b^4 + 75b^3$
79. $-12x^2 - 18x$
80. $-20b^2 - 8b$
81. $-2x^3 - 6x^2 + 14x$
82. $-8x^4 + 6x^3 - 2x^2$
83. $4a^3b - 6a^2b^2 - 4ab^3$
84. $12u^5v^6 + 18u^2v^3 - 15u^4v^5$

Solve each problem by factoring. See Example 7.

85. ***Uniform motion.*** Helen traveled a distance of $20x + 40$ miles at 20 miles per hour on the Yellowhead Highway. Find a binomial that represents the time that she traveled.

86. ***Area of a painting.*** A rectangular painting with a width of x centimeters has an area of $x^2 + 50x$ square centimeters. Find a binomial that represents the length.

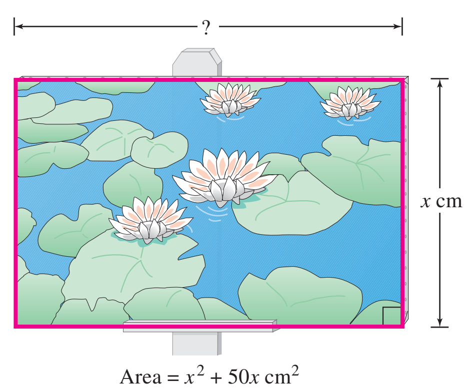


FIGURE FOR EXERCISE 86

87. ***Tomato soup.*** The amount of metal S (in square inches) that it takes to make a can for tomato soup is a function of the radius r and height h:

$$S = 2\pi r^2 + 2\pi rh$$

a) Rewrite this formula by factoring out the greatest common factor on the right-hand side.

b) If $h = 5$ in., then S is a function of r. Write a formula for that function.

c) The accompanying graph shows S for r between 1 in. and 3 in. (with $h = 5$ in.). Which of these r-values gives the maximum surface area?

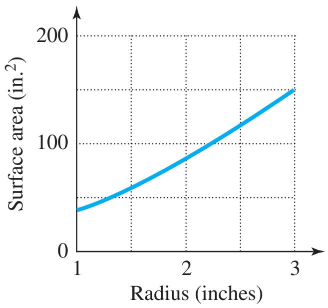


FIGURE FOR EXERCISE 87

88. ***Amount of an investment.*** The amount of an investment of P dollars for t years at simple interest rate r is given by $A = P + Prt$.

a) Rewrite this formula by factoring out the greatest common factor on the right-hand side.

b) Find A if \$8300 is invested for 3 years at a simple interest rate of 15%.

GETTING MORE INVOLVED

89. ***Discussion.*** Is the greatest common factor of $-6x^2 + 3x$ positive or negative? Explain.

90. ***Writing.*** Explain in your own words why you use the smallest power of each common prime factor when finding the GCF of two or more integers.

6.2 FACTORING THE SPECIAL PRODUCTS AND FACTORING BY GROUPING

In this section

- Factoring a Difference of Two Squares
- Factoring a Perfect Square Trinomial
- Factoring Completely
- Factoring by Grouping

In Section 5.4 you learned how to find the special products: the square of a sum, the square of a difference, and the product of a sum and a difference. In this section you will learn how to reverse those operations.

Factoring a Difference of Two Squares

In Section 5.4 you learned that the product of a sum and a difference is a difference of two squares:

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

So a difference of two squares can be factored as a product of a sum and a difference, using the following rule.

Factoring a Difference of Two Squares

For any real numbers a and b,

$$a^2 - b^2 = (a + b)(a - b).$$

Note that the square of an integer is a perfect square. For example, 64 is a perfect square because $64 = 8^2$. The square of a monomial in which the coefficient is an integer is also called a **perfect square** or simply a **square.** For example, $9m^2$ is a perfect square because $9m^2 = (3m)^2$.

EXAMPLE 1 **Factoring a difference of two squares**

Factor each polynomial.

a) $y^2 - 81$ **b)** $9m^2 - 16$ **c)** $4x^2 - 9y^2$