87. ***Tomato soup.*** The amount of metal S (in square inches) that it takes to make a can for tomato soup is a function of the radius r and height h:

$$S = 2\pi r^2 + 2\pi rh$$

a) Rewrite this formula by factoring out the greatest common factor on the right-hand side.

b) If $h = 5$ in., then S is a function of r. Write a formula for that function.

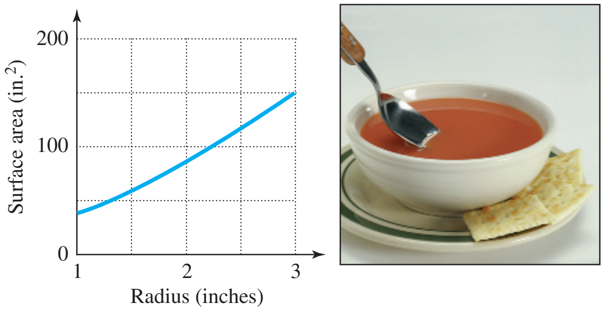

FIGURE FOR EXERCISE 87

c) The accompanying graph shows S for r between 1 in. and 3 in. (with $h = 5$ in.). Which of these r-values gives the maximum surface area?

88. ***Amount of an investment.*** The amount of an investment of P dollars for t years at simple interest rate r is given by $A = P + Prt$.

a) Rewrite this formula by factoring out the greatest common factor on the right-hand side.

b) Find A if \$8300 is invested for 3 years at a simple interest rate of 15%.

GETTING MORE INVOLVED

89. ***Discussion.*** Is the greatest common factor of $-6x^2 + 3x$ positive or negative? Explain.

90. ***Writing.*** Explain in your own words why you use the smallest power of each common prime factor when finding the GCF of two or more integers.

6.2 FACTORING THE SPECIAL PRODUCTS AND FACTORING BY GROUPING

In this section

- Factoring a Difference of Two Squares
- Factoring a Perfect Square Trinomial
- Factoring Completely
- Factoring by Grouping

In Section 5.4 you learned how to find the special products: the square of a sum, the square of a difference, and the product of a sum and a difference. In this section you will learn how to reverse those operations.

Factoring a Difference of Two Squares

In Section 5.4 you learned that the product of a sum and a difference is a difference of two squares:

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

So a difference of two squares can be factored as a product of a sum and a difference, using the following rule.

Factoring a Difference of Two Squares

For any real numbers a and b,

$$a^2 - b^2 = (a + b)(a - b).$$

Note that the square of an integer is a perfect square. For example, 64 is a perfect square because $64 = 8^2$. The square of a monomial in which the coefficient is an integer is also called a **perfect square** or simply a **square.** For example, $9m^2$ is a perfect square because $9m^2 = (3m)^2$.

EXAMPLE 1 **Factoring a difference of two squares**

Factor each polynomial.

a) $y^2 - 81$ **b)** $9m^2 - 16$ **c)** $4x^2 - 9y^2$

Solution

a) Because $81 = 9^2$, the binomial $y^2 - 81$ is a difference of two squares:

$$y^2 - 81 = y^2 - 9^2 \quad \text{Rewrite as a difference of two squares.}$$
$$= (y + 9)(y - 9) \quad \text{Factor.}$$

Check by multiplying.

b) Because $9m^2 = (3m)^2$ and $16 = 4^2$, the binomial $9m^2 - 16$ is a difference of two squares:

$$9m^2 - 16 = (3m)^2 - 4^2 \quad \text{Rewrite as a difference of two squares.}$$
$$= (3m + 4)(3m - 4) \quad \text{Factor.}$$

Check by multiplying.

c) Because $4x^2 = (2x)^2$ and $9y^2 = (3y)^2$, the binomial $4x^2 - 9y^2$ is a difference of two squares:

$$4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$$

Factoring a Perfect Square Trinomial

In Section 5.4 you learned how to square a binomial using the rule

$$(a + b)^2 = a^2 + 2ab + b^2.$$

You can reverse this rule to factor a trinomial such as $x^2 + 6x + 9$. Notice that

$$x^2 + 6x + 9 = \underset{a^2}{x^2} + \underbrace{2 \cdot x \cdot 3}_{2ab} + \underset{b^2}{3^2}.$$

So if $a = x$ and $b = 3$, then $x^2 + 6x + 9$ fits the form $a^2 + 2ab + b^2$, and

$$x^2 + 6x + 9 = (x + 3)^2.$$

A trinomial that is of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is called a **perfect square trinomial.** A perfect square trinomial is the square of a binomial. Perfect square trinomials can be identified by using the following strategy.

Strategy for Identifying a Perfect Square Trinomial

A trinomial is a perfect square trinomial if:

1. the first and last terms are of the form a^2 and b^2 (perfect squares), and

2. the middle term is $2ab$ or $-2ab$.

EXAMPLE 2 **Identifying the special products**

Determine whether each binomial is a difference of two squares and whether each trinomial is a perfect square trinomial.

a) $x^2 - 14x + 49$

b) $4x^2 - 81$

c) $4a^2 + 24a + 25$

d) $9y^2 - 24y - 16$

Solution

a) The first term is x^2, and the last term is 7^2. The middle term, $-14x$, is $-2 \cdot x \cdot 7$. So this trinomial is a perfect square trinomial.

study tip

A lumber mill turns logs into plywood, adding value to the logs. College is like a lumber mill. If you are not changing, growing, and learning, you may not be increasing in value. Everything that you learn increases your value.

b) Both terms of $4x^2 - 81$ are perfect squares, $(2x)^2$ and 9^2. So $4x^2 - 81$ is a difference of two squares.

c) The first term of $4a^2 + 24a + 25$ is $(2a)^2$ and the last term is 5^2. However, $2 \cdot 2a \cdot 5$ is $20a$. Because the middle term is $24a$, this trinomial is not a perfect square trinomial.

d) The first and last terms in a perfect square trinomial are both positive. Because the last term in $9y^2 - 24y - 16$ is negative, the trinomial is not a perfect square trinomial. ■

Note that the middle term in a perfect square trinomial may have a positive or a negative coefficient, while the first and last terms must be positive. Any perfect square trinomial can be factored as the square of a binomial by using the following rule.

Factoring Perfect Square Trinomials

For any real numbers a and b,

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

EXAMPLE 3 **Factoring perfect square trinomials**

Factor.

a) $x^2 - 4x + 4$ **b)** $a^2 + 16a + 64$ **c)** $4x^2 - 12x + 9$

Solution

a) The first term is x^2, and the last term is 2^2. Because the middle term is $-2 \cdot 2 \cdot x$, or $-4x$, this polynomial is a perfect square trinomial:

$$x^2 - 4x + 4 = (x - 2)^2$$

Check by expanding $(x - 2)^2$.

b) $a^2 + 16a + 64 = (a + 8)^2$

Check by expanding $(a + 8)^2$.

c) The first term is $(2x)^2$, and the last term is 3^2. Because $-2 \cdot 2x \cdot 3 = -12x$, the polynomial is a perfect square trinomial. So

$$4x^2 - 12x + 9 = (2x - 3)^2.$$

Check by expanding $(2x - 3)^2$. ■

Factoring Completely

To factor a polynomial means to write it as a product of simpler polynomials. A polynomial that cannot be factored is called a **prime** or **irreducible polynomial.** The polynomials $3x$, $w + 1$, and $4m - 5$ are prime polynomials. A polynomial is **factored completely** when it is written as a product of prime polynomials. So $(y - 8)(y + 1)$ is a complete factorization. When factoring polynomials, we usually do not factor integers that occur as common factors. So $6x(x - 7)$ is considered to be factored completely even though 6 could be factored.

Some polynomials have a factor common to all terms. To factor such polynomials completely, it is simpler to factor out the greatest common factor (GCF) and then factor the remaining polynomial. The following example illustrates factoring completely.

EXAMPLE 4 **Factoring completely**

Factor each polynomial completely.

a) $2x^3 - 50x$ **b)** $8x^2y - 32xy + 32y$

Solution

a) The greatest common factor of $2x^3$ and $50x$ is $2x$:

$$2x^3 - 50x = 2x(x^2 - 25)$$ Check this step by multiplying.

$$= 2x(x + 5)(x - 5)$$ Difference of two squares

b) $$8x^2y - 32xy + 32y = 8y(x^2 - 4x + 4)$$ Check this step by multiplying.

$$= 8y(x - 2)^2$$ Perfect square trinomial

Remember that factoring reverses multiplication and *every step of factoring can be checked by multiplication.*

Factoring by Grouping

The product of two binomials may be a polynomial with four terms. For example,

$$(x + a)(x + 3) = (x + a)x + (x + a)3$$
$$= x^2 + ax + 3x + 3a.$$

We can factor a four-term polynomial of this type by simply reversing the steps we used to find the product. To reverse these steps, we factor out common factors from the first two terms and from the last two terms. For example,

$$w^2 - bw + 3w - 3b = w(w - b) + 3(w - b)$$
$$= (w + 3)(w - b)$$

This procedure is called **factoring by grouping.** Sometimes we must factor out a negative common factor or rearrange the terms as shown in the next example.

EXAMPLE 5 **Factoring by grouping**

Use grouping to factor each polynomial completely.

a) $xy + 2y + 3x + 6$ **b)** $2x^3 - 3x^2 - 2x + 3$ **c)** $ax + 3y - 3x - ay$

Solution

a) Notice that the first two terms have a common factor of y and the last two terms have a common factor of 3:

$$xy + 2y + 3x + 6 = (xy + 2y) + (3x + 6)$$ Use the associative property to group the terms.

$$= y(x + 2) + 3(x + 2)$$ Factor out the common factors in each group.

$$= (y + 3)(x + 2)$$ Factor out $x + 2$.

b) We can factor x^2 out of the first two terms and 1 out of the last two terms:

$$2x^3 - 3x^2 - 2x + 3 = (2x^3 - 3x^2) + (-2x + 3)$$ Group the terms.

$$= x^2(2x - 3) + 1(-2x + 3)$$

However, we cannot proceed any further because $2x - 3$ and $-2x + 3$ are not the same. To get $2x - 3$ as a common factor, we must factor out -1 from the last two terms:

$$2x^3 - 3x^2 - 2x + 3 = x^2(2x - 3) - 1(2x - 3)$$ Factor out the common factors.

$$= (x^2 - 1)(2x - 3)$$ Factor out $2x - 3$.

$$= (x - 1)(x + 1)(2x - 3)$$ Difference of two squares

study tip

When you take notes, leave space. Go back later and fill in more details, make corrections, or work another problem of the same type.

c) In $ax + 3y - 3x - ay$ there are no common factors in the first two or the last two terms. However, if we use the commutative property to rewrite the polynomial as $ax - 3x - ay + 3y$, then we can factor by grouping:

$$\begin{aligned} ax + 3y - 3x - ay &= ax - 3x - ay + 3y && \text{Rearrange the terms.} \\ &= x(a - 3) - y(a - 3) && \text{Factor out } x \text{ and } -y. \\ &= (x - y)(a - 3) && \text{Factor out } a - 3. \end{aligned}$$

WARM-UPS

True or false? Explain your answer.

1. The polynomial $x^2 + 16$ is a difference of two squares.
2. The polynomial $x^2 - 8x + 16$ is a perfect square trinomial.
3. The polynomial $9x^2 + 21x + 49$ is a perfect square trinomial.
4. $4x^2 + 4 = (2x + 2)^2$ for any real number x.
5. A difference of two squares is equal to a product of a sum and a difference.
6. The polynomial $16y + 1$ is a prime polynomial.
7. The polynomial $x^2 + 9$ can be factored as $(x + 3)(x + 3)$.
8. The polynomial $4x^2 - 4$ is factored completely as $4(x^2 - 1)$.
9. $y^2 - 2y + 1 = (y - 1)^2$ for any real number y.
10. $2x^2 - 18 = 2(x - 3)(x + 3)$ for any real number x.

6.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a perfect square?
2. How do we factor a difference of two squares?
3. How can you recognize if a trinomial is a perfect square?
4. What is a prime polynomial.
5. When is a polynomial factored completely?
6. What should you always look for first when attempting to factor a polynomial completely?

Factor each polynomial. See Example 1.

7. $a^2 - 4$
8. $h^2 - 9$
9. $x^2 - 49$
10. $y^2 - 36$
11. $y^2 - 9x^2$
12. $16x^2 - y^2$
13. $25a^2 - 49b^2$
14. $9a^2 - 64b^2$
15. $121m^2 - 1$
16. $144n^2 - 1$
17. $9w^2 - 25c^2$
18. $144w^2 - 121a^2$

Determine whether each binomial is a difference of two squares and whether each trinomial is a perfect square trinomial. See Example 2.

19. $x^2 - 20x + 100$
20. $x^2 - 10x - 25$
21. $y^2 - 40$
22. $a^2 - 49$
23. $4y^2 + 12y + 9$
24. $9a^2 - 30a - 25$
25. $x^2 - 8x + 64$
26. $x^2 + 4x + 4$
27. $9y^2 - 25c^2$

28. $9x^2 + 4$
29. $9a^2 + 6ab + b^2$
30. $4x^2 - 4xy + y^2$

Factor each perfect square trinomial. See Example 3.

31. $x^2 + 12x + 36$
32. $y^2 + 14y + 49$
33. $a^2 - 4a + 4$
34. $b^2 - 6b + 9$
35. $4w^2 + 4w + 1$
36. $9m^2 + 6m + 1$
37. $16x^2 - 8x + 1$
38. $25y^2 - 10y + 1$
39. $4t^2 + 20t + 25$
40. $9y^2 - 12y + 4$
41. $9w^2 + 42w + 49$
42. $144x^2 + 24x + 1$
43. $n^2 + 2nt + t^2$
44. $x^2 - 2xy + y^2$

Factor each polynomial completely. See Example 4.

45. $5x^2 - 125$
46. $3y^2 - 27$
47. $-2x^2 + 18$
48. $-5y^2 + 20$
49. $a^3 - ab^2$
50. $x^2y - y$
51. $3x^2 + 6x + 3$
52. $12a^2 + 36a + 27$
53. $-5y^2 + 50y - 125$
54. $-2a^2 - 16a - 32$
55. $x^3 - 2x^2y + xy^2$
56. $x^3y + 2x^2y^2 + xy^3$
57. $-3x^2 + 3y^2$
58. $-8a^2 + 8b^2$
59. $2ax^2 - 98a$
60. $32x^2y - 2y^3$
61. $3ab^2 - 18ab + 27a$
62. $-2a^2b + 8ab - 8b$
63. $-4m^3 + 24m^2n - 36mn^2$
64. $10a^3 - 20a^2b + 10ab^2$

Use grouping to factor each polynomial completely. See Example 5.

65. $bx + by + cx + cy$
66. $3x + 3z + ax + az$
67. $x^3 + x^2 - 4x - 4$
68. $x^3 + x^2 - x - 1$
69. $3a - 3b - xa + xb$
70. $ax - bx - 4a + 4b$
71. $a^3 + 3a^2 + a + 3$
72. $y^3 - 5y^2 + 8y - 40$
73. $xa + ay + 3y + 3x$
74. $x^3 + ax + 3a + 3x^2$
75. $abc - 3 + c - 3ab$
76. $xa + tb + ba + tx$
77. $x^2a - b + bx^2 - a$
78. $a^2m - b^2n + a^2n - b^2m$
79. $y^2 + y + by + b$
80. $ac + mc + aw^2 + mw^2$

Factor each polynomial completely.

81. $6a^3y + 24a^2y^2 + 24ay^3$
82. $8b^5c - 8b^4c^2 + 2b^3c^3$
83. $24a^3y - 6ay^3$
84. $27b^3c - 12bc^3$
85. $2a^3y^2 - 6a^2y$
86. $9x^3y - 18x^2y^2$
87. $ab + 2bw - 4aw - 8w^2$
88. $3am - 6n - an + 18m$

Use factoring to solve each problem.

89. ***Skydiving.*** The height (in feet) above the earth for a sky diver t seconds after jumping from an airplane at 6400 ft is approximated by the formula $h = -16t^2 + 6400$, provided that $t < 5$. Rewrite the formula with the right-hand side factored completely. Use your revised formula to find h when $t = 2$.

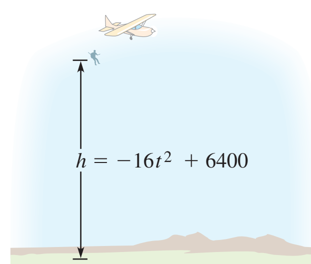

FIGURE FOR EXERCISE 89

90. ***Demand for pools.*** Tropical Pools sells an above-ground model for p dollars each. The monthly revenue from the sale of this model is a function of the price, given by

$$R = -0.08p^2 + 300p.$$

Revenue is the product of the price p and the demand (quantity sold).

a) Factor out the price on the right-hand side of the formula.

b) What is an expression for the monthly demand?

c) What is the monthly demand for this pool when the price is \$3000?

d) Use the graph on page 290 to estimate the price at which the revenue is maximized. Approximately how many pools will be sold monthly at this price?

e) What is the approximate maximum revenue?

f) Use the accompanying graph to estimate the price at which the revenue is zero.

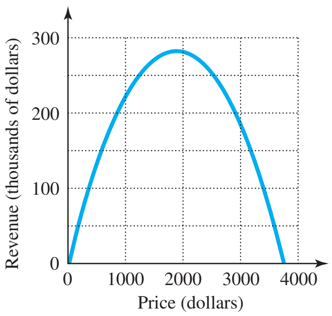

FIGURE FOR EXERCISE 90

91. ***Volume of a tank.*** The volume of a fish tank with a square base and height y is $y^3 - 6y^2 + 9y$ cubic inches. Find the length of a side of the square base.

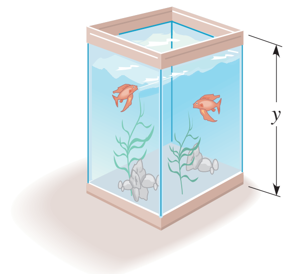

FIGURE FOR EXERCISE 91

GETTING MORE INVOLVED

92. ***Discussion.*** For what real number k, does $3x^2 - k$ factor as $3(x - 2)(x + 2)$?

93. ***Writing.*** Explain in your own words how to factor a four-term polynomial by grouping.

94. ***Writing.*** Explain how you know that $x^2 + 1$ is a prime polynomial.

6.3 FACTORING $ax^2 + bx + c$ WITH $a = 1$

In this section

- Factoring $ax^2 + bx + c$ with $a = 1$
- Prime Polynomials
- Factoring with Two Variables
- Factoring Completely

In this section we will factor the type of trinomials that result from multiplying two different binomials. We will do this only for trinomials in which the coefficient of x^2, the leading coefficient, is 1. Factoring trinomials with leading coefficient not equal to 1 will be done in Section 6.4.

Factoring $ax^2 + bx + c$ with $a = 1$

Let's look closely at an example of finding the product of two binomials using the distributive property:

$$(x + 2)(x + 3) = (x + 2)x + (x + 2)3 \quad \text{Distributive property}$$
$$= x^2 + 2x + 3x + 6 \quad \text{Distributive property}$$
$$= x^2 + 5x + 6 \quad \text{Combine like terms.}$$

To factor $x^2 + 5x + 6$, we reverse these steps as shown in our first example.

EXAMPLE 1 **Factoring a trinomial**

Factor.

a) $x^2 + 5x + 6$ **b)** $x^2 + 8x + 12$ **c)** $a^2 - 9a + 20$

Solution

a) The coefficient 5 is the sum of two numbers that have a product of 6. The only integers that have a product of 6 and a sum of 5 are 2 and 3. So write $5x$ as $2x + 3x$, then factor by grouping:

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6 \quad \text{Replace } 5x \text{ by } 2x + 3x.$$
$$= (x^2 + 2x) + (3x + 6) \quad \text{Group terms together.}$$
$$= x(x + 2) + 3(x + 2) \quad \text{Factor out common factors.}$$
$$= (x + 2)(x + 3) \quad \text{Factor out } x + 2.$$