

7.1 REDUCING RATIONAL EXPRESSIONS

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Rational expressions in algebra are similar to the rational numbers in arithmetic. In this section you will learn the basic ideas of rational expressions.

Rational Expressions

A rational number is the ratio of two integers with the denominator not equal to 0. For example,

$$\frac{3}{4}, \quad \frac{-9}{-6}, \quad 7, \quad \text{and} \quad 0$$

are rational numbers. A **rational expression** is the ratio of two polynomials with the denominator not equal to 0. Because an integer is a monomial, a rational number is a rational expression. The following expressions are rational expressions:

$$\frac{x^2 - 1}{x + 8}, \quad \frac{3a^2 + 5a - 3}{a - 9}, \quad \frac{3}{7}, \quad w$$

We say that w is a rational expression because w can be written as $\frac{w}{1}$.

A rational expression involving a variable has no value unless we assign a value to the variable. Once the variable is given a value we can evaluate the expression. We can discuss the value of a rational expression with or without the function notation introduced in Chapter 4.

EXAMPLE 1

Evaluating a rational expression

- a) Find the value of $\frac{4x - 1}{x + 2}$ for $x = -3$.
- b) If $R(x) = \frac{3x + 2}{2x - 1}$, find $R(4)$.

Solution

- a) To find the value of $\frac{4x - 1}{x + 2}$ for $x = -3$, replace x by -3 in the rational expression:

$$\frac{4(-3) - 1}{-3 + 2} = \frac{-13}{-1} = 13$$

So the value of the rational expression is 13.

- b) $R(4)$ is the function notation for the value of the expression when $x = 4$. To find $R(4)$ replace x by 4 in $R(x) = \frac{3x + 2}{2x - 1}$:

$$R(4) = \frac{3(4) + 2}{2(4) - 1}$$

$$R(4) = \frac{14}{7} = 2$$

So the value of the rational expression is 2 when $x = 4$, or $R(4) = 2$ (read “ R of 4 is 2”).

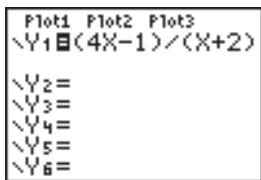
Because the denominator cannot be zero, any number can be used in place of the variable *except* numbers that cause the denominator to be zero.

calculator

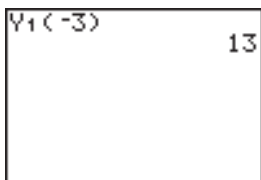


close-up

To evaluate the rational expression in Example 1(a) with a calculator, first use $Y=$ to define the rational expression. Be sure to enclose both numerator and denominator in parentheses.



Then find $y_1(-3)$.



EXAMPLE 2**Ruling out values for x**

Which numbers cannot be used in place of x in each rational expression?

$$\text{a) } \frac{x^2 - 1}{x + 8} \qquad \text{b) } \frac{x + 2}{2x + 1} \qquad \text{c) } \frac{x + 5}{x^2 - 4}$$

Solution

a) The denominator is 0 if $x + 8 = 0$, or $x = -8$. So -8 cannot be used in place of x .

b) The denominator is zero if $2x + 1 = 0$, or $x = -\frac{1}{2}$. So we cannot use $-\frac{1}{2}$ in place of x .

c) The denominator is zero if $x^2 - 4 = 0$. Solve this equation:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x - 2)(x + 2) &= 0 && \text{Factor.} \\ x - 2 = 0 &\quad \text{or} \quad x + 2 = 0 && \text{Zero factor property} \\ x = 2 &\quad \text{or} \quad x = -2 \end{aligned}$$

So 2 and -2 cannot be used in place of x . ■

The **domain** of an expression is the set of all real numbers that can be used in place of the variable. So the domain of $\frac{x^2 - 1}{x + 8}$ is the set of all real numbers except -8 . The domain of $\frac{x + 2}{2x + 1}$ is the set of all real numbers except $-\frac{1}{2}$. When dealing with rational expressions we generally assume that the variables represent numbers in the domain of the expression.

Reducing to Lowest Terms

Rational expressions are a generalization of rational numbers. The operations that we perform on rational numbers can be performed on rational expressions in exactly the same manner.

Each rational number can be written in infinitely many equivalent forms. For example,

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{12}{20} = \frac{15}{25} = \dots$$

Each equivalent form of $\frac{3}{5}$ is obtained from $\frac{3}{5}$ by multiplying both numerator and denominator by the same nonzero number. This is equivalent to multiplying the fraction by 1, which does not change its value. For example,

$$\frac{3}{5} = \frac{3}{5} \cdot 1 = \frac{3}{5} \cdot \frac{2}{2} = \frac{6}{10} \quad \text{and} \quad \frac{3}{5} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{9}{15}$$

If we start with $\frac{6}{10}$ and convert it into $\frac{3}{5}$, we say that we are **reducing $\frac{6}{10}$ to lowest terms**. We reduce by dividing the numerator and denominator by the common factor 2:

$$\frac{6}{10} = \frac{\cancel{2} \cdot 3}{\cancel{2} \cdot 5} = \frac{3}{5}$$

A rational number is expressed in lowest terms when the numerator and the denominator have no common factors other than 1.

helpful hint

Most students learn to convert $\frac{3}{5}$ into $\frac{6}{10}$ by dividing 5 into 10 to get 2 and then multiply 2 by 3 to get 6. In algebra it is better to do this conversion by multiplying the numerator and denominator of $\frac{3}{5}$ by 2 as shown here.

CAUTION We can reduce fractions only by dividing the numerator and the denominator by a common factor. Although it is true that

$$\frac{6}{10} = \frac{2 + 4}{2 + 8},$$

we cannot eliminate the 2's, because they are not factors. Removing them from the sums in the numerator and denominator would not result in $\frac{3}{5}$. Likewise, you cannot eliminate the 2's in an expression such as $\frac{y+2}{x+2}$.

Reducing Fractions

If $a \neq 0$ and $c \neq 0$, then

$$\frac{ab}{ac} = \frac{b}{c}.$$

To reduce rational expressions to lowest terms, we use exactly the same procedure as with fractions:

Reducing Rational Expressions

1. Factor the numerator and denominator completely.
2. Divide the numerator and denominator by the greatest common factor.

Dividing the numerator and denominator by the GCF is often referred to as **dividing out the GCF**.

EXAMPLE 3

Reducing

Reduce to lowest terms.

a) $\frac{30}{42}$

b) $\frac{x^2 - 9}{6x + 18}$

Solution

$$\begin{aligned} \text{a) } \frac{30}{42} &= \frac{\cancel{2} \cdot \cancel{3} \cdot 5}{\cancel{2} \cdot \cancel{3} \cdot 7} && \text{Factor.} \\ &= \frac{5}{7} && \text{Divide out the GCF: } 2 \cdot 3 \text{ or } 6. \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{x^2 - 9}{6x + 18} &= \frac{(x - 3)\cancel{(x + 3)}}{6\cancel{(x + 3)}} && \text{Factor.} \\ &= \frac{x - 3}{6} && \text{Divide out the GCF: } x + 3. \end{aligned}$$

study tip

Find a clean, comfortable, well-lit place to study. But don't get too comfortable. Sitting at a desk is preferable to lying in bed. Where you study can influence your concentration and your study habits.

If two rational expressions are equivalent, then they have the same numerical value for any replacement of the variables. Of course, the replacement must not give us an undefined expression (0 in the denominator). So in Example 2(b) the equation

$$\frac{x^2 - 9}{6x + 18} = \frac{x - 3}{6}$$

is satisfied by all real numbers except $x = -3$. The equation is an identity.

Reducing with the Quotient Rule

To reduce rational expressions involving exponential expressions, we use the quotient rule for exponents from Chapter 5. We restate it here for reference.

Quotient Rule

Suppose $a \neq 0$, and m and n are positive integers.

$$\text{If } m \geq n, \text{ then } \frac{a^m}{a^n} = a^{m-n}.$$

$$\text{If } m < n, \text{ then } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

EXAMPLE 4 Using the quotient rule in reducing

Reduce to lowest terms.

a) $\frac{3a^{15}}{6a^7}$

b) $\frac{6x^4y^2}{4xy^5}$

Solution

$$\begin{aligned} \text{a) } \frac{3a^{15}}{6a^7} &= \frac{\cancel{3}a^{15}}{\cancel{3} \cdot 2a^7} && \text{Factor.} \\ &= \frac{a^{15-7}}{2} && \text{Quotient rule} \\ &= \frac{a^8}{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{6x^4y^2}{4xy^5} &= \frac{\cancel{2} \cdot 3x^4y^2}{\cancel{2} \cdot 2xy^5} && \text{Factor.} \\ &= \frac{3x^{4-1}}{2y^{5-2}} && \text{Quotient rule} \\ &= \frac{3x^3}{2y^3} \end{aligned}$$

The essential part of reducing is getting a complete factorization for the numerator and denominator. To get a complete factorization, you must use the techniques for factoring from Chapter 6. If there are large integers in the numerator and denominator, you can use the technique shown in Section 6.1 to get a prime factorization of each integer.

EXAMPLE 5 Reducing expressions involving large integers

Reduce $\frac{420}{616}$ to lowest terms.

Solution

Use the method of Section 6.1 to get a prime factorization of 420 and 616:

$$\begin{array}{r} 7 \\ 5 \overline{)35} \\ 3 \overline{)105} \\ 2 \overline{)210} \\ \text{Start here} \rightarrow 2 \overline{)420} \end{array} \qquad \begin{array}{r} 11 \\ 7 \overline{)77} \\ 2 \overline{)154} \\ 2 \overline{)308} \\ 2 \overline{)616} \end{array}$$

The complete factorization for 420 is $2^2 \cdot 3 \cdot 5 \cdot 7$, and the complete factorization for 616 is $2^3 \cdot 7 \cdot 11$. To reduce the fraction, we divide out the common factors:

$$\begin{aligned}\frac{420}{616} &= \frac{2^2 \cdot 3 \cdot 5 \cdot 7}{2^3 \cdot 7 \cdot 11} \\ &= \frac{3 \cdot 5}{2 \cdot 11} \\ &= \frac{15}{22}\end{aligned}$$

Dividing $a - b$ by $b - a$

In Section 5.2 you learned that $a - b = -(b - a) = -1(b - a)$. So if $a - b$ is divided by $b - a$, the quotient is -1 :

$$\begin{aligned}\frac{a - b}{b - a} &= \frac{-1(b - a)}{b - a} \\ &= -1\end{aligned}$$

We will use this fact in the next example.

EXAMPLE 6 Expressions with $a - b$ and $b - a$

Reduce to lowest terms.

a) $\frac{5x - 5y}{4y - 4x}$

b) $\frac{m^2 - n^2}{n - m}$

Solution

$$\begin{aligned}\text{a) } \frac{5x - 5y}{4y - 4x} &= \frac{5(x - y)}{4(y - x)} \quad \text{Factor.} \\ &= \frac{5}{4} \cdot (-1) \frac{x - y}{y - x} = -1 \\ &= -\frac{5}{4}\end{aligned}$$

$$\begin{aligned}\text{b) } \frac{m^2 - n^2}{n - m} &= \frac{(m - n)(m + n)}{n - m} \quad \text{Factor.} \\ &= -1(m + n) \quad \frac{m - n}{n - m} = -1 \\ &= -m - n\end{aligned}$$

CAUTION We can reduce $\frac{a - b}{b - a}$ to -1 , but we cannot reduce $\frac{a - b}{a + b}$. There is no factor that is common to the numerator and denominator of $\frac{a - b}{a + b}$.

Factoring Out the Opposite of a Common Factor

If we can factor out a common factor, we can also factor out the opposite of that common factor. For example, from $-3x - 6y$ we can factor out the common factor 3 or the common factor -3 :

$$-3x - 6y = 3(-x - 2y) \quad \text{or} \quad -3x - 6y = -3(x + 2y)$$

To reduce an expression, it is sometimes necessary to factor out the opposite of a common factor.

EXAMPLE 7 Factoring out the opposite of a common factor

Reduce $\frac{-3w - 3w^2}{w^2 - 1}$ to lowest terms.

Solution

We can factor $3w$ or $-3w$ from the numerator. If we factor out $-3w$, we get a common factor in the numerator and denominator:

$$\begin{aligned} \frac{-3w - 3w^2}{w^2 - 1} &= \frac{-3w(1 + w)}{(w - 1)(w + 1)} && \text{Factor.} \\ &= \frac{-3w}{w - 1} && \text{Since } 1 + w = w + 1, \text{ we divide out } w + 1. \\ &= \frac{3w}{1 - w} && \text{Multiply numerator and denominator by } -1. \end{aligned}$$

helpful hint

Note that $w + 1$ and $1 + w$ are equivalent, but $w - 1$ and $1 - w$ are not equivalent.

The last step in this reduction is not absolutely necessary, but we usually perform it to make the answer look a little simpler. ■

The main points to remember for reducing rational expressions are summarized in the following reducing strategy.

Strategy for Reducing Rational Expressions

1. Reducing is done by dividing out all common factors.
2. Factor the numerator and denominator completely to see the common factors.
3. Use the quotient rule to reduce a ratio of two monomials.
4. You may have to factor out a common factor with a negative sign to get identical factors in the numerator and denominator.
5. The quotient of $a - b$ and $b - a$ is -1 .

Writing Rational Expressions

Rational expressions occur naturally in applications involving rates.

EXAMPLE 8 Writing rational expressions

Answer each question with a rational expression.

- a) If a trucker drives 500 miles in $x + 1$ hours, then what is his average speed?
- b) If a wholesaler buys 100 pounds of shrimp for x dollars, then what is the price per pound?
- c) If a painter completes an entire house in $2x$ hours, then at what rate is she painting?

Solution

- a) Because $R = \frac{D}{T}$, he is averaging $\frac{500}{x+1}$ mph.
- b) At x dollars for 100 pounds, the wholesaler is paying $\frac{x}{100}$ dollars per pound or $\frac{x}{100}$ dollars/pound.
- c) By completing 1 house in $2x$ hours, her rate is $\frac{1}{2x}$ house/hour. ■

WARM - U P S**True or false? Explain your answer.**

- A complete factorization of 3003 is $2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$.
- A complete factorization of 120 is $2^3 \cdot 3 \cdot 5$.
- Any number can be used in place of x in the expression $\frac{x-2}{5}$.
- We cannot replace x by -1 or 3 in the expression $\frac{x+1}{x-3}$.
- The rational expression $\frac{x+2}{2}$ reduces to x .
- $\frac{2x}{2} = x$ for any real number x .
- $\frac{x^{13}}{x^{20}} = \frac{1}{x^7}$ for any nonzero value of x .
- $\frac{a^2+b^2}{a+b}$ reduced to lowest terms is $a+b$.
- If $a \neq b$, then $\frac{a-b}{b-a} = 1$.
- The expression $\frac{-3x-6}{x+2}$ reduces to -3 .

7.1 EXERCISES

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

- What is a rational number?
- What is a rational expression?
- How do you reduce a rational number to lowest terms?
- How do you reduce a rational expression to lowest terms?
- How is the quotient rule used in reducing rational expressions?
- What is the relationship between $a - b$ and $b - a$?

Evaluate each rational expression. See Example 1.

- Evaluate $\frac{3x-3}{x+5}$ for $x = -2$.
- Evaluate $\frac{3x+1}{4x-4}$ for $x = 5$.
- If $R(x) = \frac{2x+9}{x}$, find $R(3)$.
- If $R(x) = \frac{-20x-2}{x-8}$, find $R(-1)$.
- If $R(x) = \frac{x-5}{x+3}$, find $R(2)$, $R(-4)$, $R(-3.02)$, and $R(-2.96)$.
- If $R(x) = \frac{x^2-2x-3}{x-2}$, find $R(3)$, $R(5)$, $R(2.05)$, and $R(1.999)$.

Which numbers cannot be used in place of the variable in each rational expression? See Example 2.

13. $\frac{x}{x+1}$

14. $\frac{3x}{x-7}$

15. $\frac{7a}{3a-5}$

16. $\frac{84}{3-2a}$

17. $\frac{2x+3}{x^2-16}$

18. $\frac{2y+1}{y^2-y-6}$

19. $\frac{p-1}{2}$

20. $\frac{m+31}{5}$

Reduce each rational expression to lowest terms. Assume that the variables represent only numbers for which the denominators are nonzero. See Example 3.

21. $\frac{6}{27}$

22. $\frac{14}{21}$

23. $\frac{42}{90}$

24. $\frac{42}{54}$

25. $\frac{36a}{90}$

26. $\frac{56y}{40}$

27. $\frac{78}{30w}$

28. $\frac{68}{44y}$

29. $\frac{6x+2}{6}$

30. $\frac{2w+2}{2w}$

31. $\frac{2x+4y}{6y+3x}$

32. $\frac{3m+9w}{3m-6w}$

33. $\frac{w^2-49}{w+7}$

34. $\frac{a^2-b^2}{a-b}$

35. $\frac{a^2-1}{a^2+2a+1}$

36. $\frac{x^2-y^2}{x^2+2xy+y^2}$

37. $\frac{2x^2+4x+2}{4x^2-4}$

38. $\frac{2x^2+10x+12}{3x^2-27}$

39. $\frac{3x^2+18x+27}{21x+63}$

40. $\frac{x^3-3x^2-4x}{x^2-4x}$

Reduce each expression to lowest terms. Assume that all denominators are nonzero. See Example 4.

41. $\frac{x^{10}}{x^7}$

42. $\frac{y^8}{y^5}$

43. $\frac{z^3}{z^8}$

44. $\frac{w^9}{w^{12}}$

45. $\frac{4x^7}{-2x^5}$

46. $\frac{-6y^3}{3y^9}$

47. $\frac{-12m^9n^{18}}{8m^6n^{16}}$

48. $\frac{-9u^9v^{19}}{6u^9v^{14}}$

49. $\frac{6b^{10}c^4}{-8b^{10}c^7}$

51. $\frac{30a^3bc}{18a^7b^{17}}$

50. $\frac{9x^{20}y}{-6x^{25}y^3}$

52. $\frac{15m^{10}n^3}{24m^{12}np}$

Reduce each expression to lowest terms. See Example 5.

53. $\frac{210}{264}$

54. $\frac{616}{660}$

55. $\frac{231}{168}$

56. $\frac{936}{624}$

57. $\frac{630x^5}{300x^9}$

58. $\frac{96y^2}{108y^5}$

59. $\frac{924a^{23}}{448a^{19}}$

60. $\frac{270b^{75}}{165b^{12}}$

Reduce each expression to lowest terms. See Example 6.

61. $\frac{3a-2b}{2b-3a}$

62. $\frac{5m-6n}{6n-5m}$

63. $\frac{h^2-t^2}{t-h}$

64. $\frac{r^2-s^2}{s-r}$

65. $\frac{2g-6h}{9h^2-g^2}$

66. $\frac{5a-10b}{4b^2-a^2}$

67. $\frac{x^2-x-6}{9-x^2}$

68. $\frac{1-a^2}{a^2+a-2}$

Reduce each expression to lowest terms. See Example 7.

69. $\frac{-x-6}{x+6}$

70. $\frac{-5x-20}{3x+12}$

71. $\frac{-2y-6y^2}{3+9y}$

72. $\frac{y^2-16}{-8-2y}$

73. $\frac{-3x-6}{3x-6}$

74. $\frac{8-4x}{-8x-16}$

75. $\frac{-12a-6}{2a^2+7a+3}$

76. $\frac{-2b^2-6b-4}{b^2-1}$

Reduce each expression to lowest terms.

77. $\frac{2x^{12}}{4x^8}$

78. $\frac{4x^2}{2x^9}$

79. $\frac{2x+4}{4x}$

80. $\frac{2x+4x^2}{4x}$

81. $\frac{a-4}{4-a}$

82. $\frac{2b-4}{2b+4}$

83. $\frac{2c-4}{4-c^2}$

84. $\frac{-2t-4}{4-t^2}$

85. $\frac{x^2+4x+4}{x^2-4}$

86. $\frac{3x-6}{x^2-4x+4}$

87. $\frac{-2x-4}{x^2+5x+6}$

88. $\frac{-2x-8}{x^2+2x-8}$

89. $\frac{2q^8 + q^7}{2q^6 + q^5}$

90. $\frac{8s^{12}}{12s^6 - 16s^5}$

91. $\frac{u^2 - 6u - 16}{u^2 - 16u + 64}$

92. $\frac{v^2 + 3v - 18}{v^2 + 12v + 36}$

93. $\frac{a^3 - 8}{2a - 4}$

94. $\frac{4w^2 - 12w + 36}{2w^3 + 54}$

95. $\frac{y^3 - 2y^2 - 4y + 8}{y^2 - 4y + 4}$

96. $\frac{mx + 3x + my + 3y}{m^2 - 3m - 18}$

Answer each question with a rational expression. Be sure to include the units. See Example 8.

97. If Sergio drove 300 miles at $x + 10$ miles per hour, then how many hours did he drive?

98. If Carrie walked 40 miles in x hours, then how fast did she walk?

99. If $x + 4$ pounds of peaches cost \$4.50, then what is the cost per pound?

100. If 9 pounds of pears cost x dollars, then what is the price per pound?

101. If Ayesha can clean the entire swimming pool in x hours, then how much of the pool does she clean per hour?

102. If Ramon can mow the entire lawn in $x - 3$ hours, then how much of the lawn does he mow per hour?

Solve each problem.

103. **Annual reports.** The Crest Meat Company found that the cost per report for printing x annual reports at

Peppy Printing is given by the formula

$$C(x) = \frac{150 + 0.60x}{x},$$

where $C(x)$ is in dollars.

- Use the accompanying graph to estimate the cost per report for printing 1000 reports.
- Use the formula to find the cost per report for printing 1000, 5000, and 10,000 reports.
- What happens to the cost per report as the number of reports gets very large?

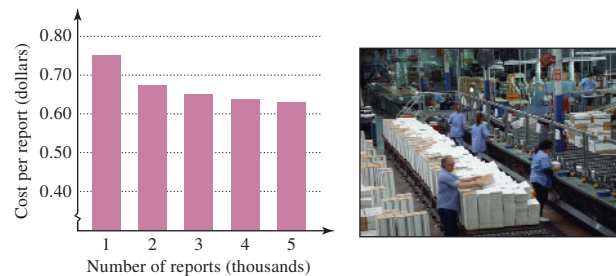


FIGURE FOR EXERCISE 103

104. **Toxic pollutants.** The annual cost in dollars for removing $p\%$ of the toxic chemicals from a town's water supply is given by the formula

$$C(p) = \frac{500,000}{100 - p}.$$

- Use the accompanying graph to estimate the cost for removing 90% and 95% of the toxic chemicals.
- Use the formula to determine the cost for removing 99.5% of the toxic chemicals.
- What happens to the cost as the percentage of pollutants removed approaches 100%?

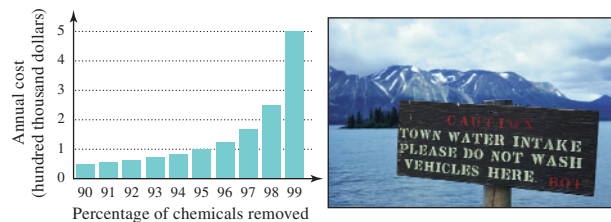


FIGURE FOR EXERCISE 104