7.7 APPLICATIONS OF RATIOS AND PROPORTIONS

In this section

- Ratios
- Proportions

In this section we will use the ideas of rational expressions in ratio and proportion problems. We will solve proportions in the same way we solved equations in Section 7.6.

Ratios

In Chapter 1 we defined a rational number as the *ratio of two integers.* We will now give a more general definition of ratio. If a and b are any real numbers (not just integers), with $b \neq 0$, then the expression $\frac{a}{b}$ is called the **ratio of *a* and *b*** or the **ratio of *a* to *b*.** The ratio of a to b is also written as $a:b$. A ratio is a comparison of two numbers. Some examples of ratios are

$$\frac{3}{4},\quad \frac{4.2}{2.1},\quad \frac{\frac{1}{4}}{\frac{1}{2}},\quad \frac{3.6}{5},\quad \text{and}\quad \frac{100}{1}.$$

Ratios are treated just like fractions. We can reduce ratios, and we can build them up. We generally express ratios as ratios of integers. When possible, we will convert a ratio into an equivalent ratio of integers in lowest terms.

EXAMPLE 1 **Finding equivalent ratios**

Find an equivalent ratio of integers in lowest terms for each ratio.

a) $\frac{4.2}{2.1}$ **b)** $\frac{\frac{1}{4}}{\frac{1}{2}}$ **c)** $\frac{3.6}{5}$

Solution

a) Because both the numerator and the denominator have one decimal place, we will multiply the numerator and denominator by 10 to eliminate the decimals:

$$\frac{4.2}{2.1} = \frac{4.2(10)}{2.1(10)} = \frac{42}{21} = \frac{21 \cdot 2}{21 \cdot 1} = \frac{2}{1} \qquad \text{Do not omit the 1 in a ratio.}$$

So the ratio of 4.2 to 2.1 is equivalent to the ratio 2 to 1.

b) This ratio is a complex fraction. We can simplify this expression using the LCD method as shown in Section 7.5. Multiply the numerator and denominator of this ratio by 4:

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{1}{2}$$

c) We can get a ratio of integers if we multiply the numerator and denominator by 10.

$$\frac{3.6}{5} = \frac{3.6(10)}{5(10)} = \frac{36}{50}$$

$$= \frac{18}{25} \qquad \text{Reduce to lowest terms.}$$

In the next example a ratio is used to compare quantities.

EXAMPLE 2 **Nitrogen to potash**

In a 50-pound bag of lawn fertilizer there are 8 pounds of nitrogen and 12 pounds of potash. What is the ratio of nitrogen to potash?

Solution

The nitrogen and potash occur in this fertilizer in the ratio of 8 pounds to 12 pounds:

$$\frac{8}{12} = \frac{2 \cdot \cancel{4}}{3 \cdot \cancel{4}} = \frac{2}{3}$$

So the ratio of nitrogen to potash is 2 to 3. ■

EXAMPLE 3 **Males to females**

In a class of 50 students, there were exactly 20 male students. What was the ratio of males to females in this class?

Solution

Because there were 20 males in the class of 50, there were 30 females. The ratio of males to females was 20 to 30, or 2 to 3. ■

Ratios give us a means of comparing the size of two quantities. For this reason *the numbers compared in a ratio should be expressed in the same units*. For example, if one dog is 24 inches high and another is 1 foot high, then the ratio of their heights is 2 to 1, not 24 to 1.

EXAMPLE 4 **Quantities with different units**

What is the ratio of length to width for a poster with a length of 30 inches and a width of 2 feet?

Solution

Because the width is 2 feet, or 24 inches, the ratio of length to width is 30 to 24. Reduce as follows:

$$\frac{30}{24} = \frac{5 \cdot 6}{4 \cdot 6} = \frac{5}{4}$$

So the ratio of length to width is 5 to 4. ■

Proportions

A **proportion** is any statement expressing the equality of two ratios. The statement

$$\frac{a}{b} = \frac{c}{d} \qquad \text{or} \qquad a:b = c:d$$

is a proportion. In any proportion the numbers in the positions of a and d above are called the **extremes.** The numbers in the positions of b and c above are called the **means.** In the proportion

$$\frac{30}{24} = \frac{5}{4},$$

the means are 24 and 5, and the extremes are 30 and 4. Note that $30 \cdot 4 = 5 \cdot 24$.

study tip

To get the "big picture," survey the chapter that you are studying. Read the headings to get the general idea of the chapter content. Read the chapter summary to see what is important in the chapter. Repeat this survey procedure several times while you are working in a chapter.

If we multiply each side of the proportion

$$\frac{a}{b} = \frac{c}{d}$$

by the LCD, bd, we get

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$$

or

$$a \cdot d = b \cdot c.$$

We can express this result by saying that *the product of the extremes is equal to the product of the means.* We call this fact the **extremes-means property** or **cross-multiplying.**

Extremes-Means Property (Cross-Multiplying)

Suppose a, b, c, and d are real numbers with $b \neq 0$ and $d \neq 0$. If

$$\frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

We use the extremes-means property to solve proportions.

EXAMPLE 5

Using the extremes-means property

Solve the proportion $\frac{3}{x} = \frac{5}{x + 5}$ for x.

helpful hint

The extremes-means property or cross-multiplying is nothing new. You can accomplish the same thing by multiplying each side of the equation by the LCD.

Solution

Instead of multiplying each side by the LCD, we use the extremes-means property:

$$\frac{3}{x} = \frac{5}{x + 5} \quad \text{Original proportion}$$

$$3(x + 5) = 5x \quad \text{Extremes-means property}$$

$$3x + 15 = 5x \quad \text{Distributive property}$$

$$15 = 2x$$

$$\frac{15}{2} = x$$

Check:

$$\frac{3}{\frac{15}{2}} = 3 \cdot \frac{2}{15} = \frac{2}{5}$$

$$\frac{5}{\frac{15}{2} + 5} = \frac{5}{\frac{25}{2}} = 5 \cdot \frac{2}{25} = \frac{2}{5}$$

So $\frac{15}{2}$ is the solution to the equation or the solution to the proportion. ■

EXAMPLE 6 **The capture-recapture proportion**

To estimate the number of catfish in her pond, a catfish farmer caught, tagged, and released 30 of them. Later, only 3 tagged catfish were found in a sample of 500. Estimate the number of catfish in the pond.

Solution

Let x be the number of catfish in the pond. The ratio $\frac{30}{x}$ is the ratio of tagged catfish to the total population. The ratio $\frac{3}{500}$ is the ratio of tagged catfish in the sample to the sample size. If the tagged catfish are well-mixed and the sample is truly random, then these ratios should be equal:

$$\frac{30}{x} = \frac{3}{500}$$

$$3x = 15{,}000 \quad \text{Extremes-means property}$$

$$x = 5000$$

So there are approximately 5000 catfish in the pond. ■

Note that any proportion can be solved by multiplying each side by the LCD as we did when we solved other equations involving rational expressions. The extremes-means property gives us a shortcut for solving proportions.

EXAMPLE 7 **Solving a proportion**

In a conservative portfolio the ratio of the amount invested in bonds to the amount invested in stocks should be 3 to 1. A conservative investor invested \$2850 more in bonds than she did in stocks. How much did she invest in each category?

Solution

Because the ratio of the amount invested in bonds to the amount invested in stocks is 3 to 1, we have

$$\frac{\text{Amount invested in bonds}}{\text{Amount invested in stocks}} = \frac{3}{1}.$$

If x represents the amount invested in stocks and $x + 2850$ represents the amount invested in bonds, then we can write and solve the following proportion:

$$\frac{x + 2850}{x} = \frac{3}{1}$$

$$3x = x + 2850 \quad \text{Extremes-means property}$$

$$2x = 2850$$

$$x = 1425$$

$$x + 2850 = 4275$$

So she invested \$4275 in bonds and \$1425 in stocks. Note that these amounts are in the ratio of 3 to 1. ■

The next example shows how conversions from one unit of measurement to another can be done by using proportions.

EXAMPLE 8 **Converting measurements**

There are 3 feet in 1 yard. How many feet are there in 12 yards?

Solution

Let x represent the number of feet in 12 yards. There are two proportions that we can write to solve the problem:

$$\frac{3 \text{ feet}}{x \text{ feet}} = \frac{1 \text{ yard}}{12 \text{ yards}} \qquad \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{x \text{ feet}}{12 \text{ yards}}$$

The ratios in the second proportion violate the rule of comparing only measurements that are expressed in the same units. Note that each side of the second proportion is actually the ratio 1 to 1, since 3 feet $=$ 1 yard and x feet $=$ 12 yards. For doing conversions we can use ratios like this to compare measurements in different units. Applying the extremes-means property to either proportion gives

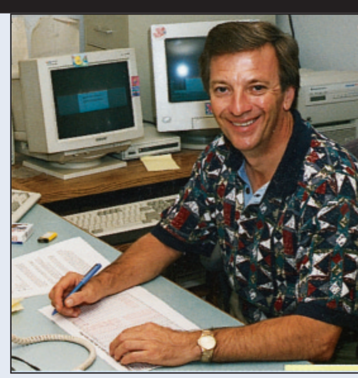
$$3 \cdot 12 = x \cdot 1,$$

or

$$x = 36.$$

So there are 36 feet in 12 yards. ■

MATH AT WORK



Did you ever wonder how your local store calculates how much of your favorite cosmetic to stock on the shelf? Mike Pittman, National Account Manager for a major cosmetic company, is responsible for providing more than 2000 stores across the United States with personal care products such as skin lotions, fragrances, and cosmetics.

Data on what has been sold is transmitted from the point of sale across a number of satellite dishes and computers to Mr. Pittman. The data usually includes size, color, and other pertinent facts. The information is then combined with demographics for certain geographic areas and movement data, as well as advertising and promotional information to answer questions such as: What color is selling best? Is it time to stock sunscreen? Is this a trend-setting area of the country? On the basis of his analysis of these and many other questions, Mr. Pittman recommends changes in packaging, promotional programs, and the quantities of products to be shipped.

Mr. Pittman's job requires a unique blend of sales, marketing, and quantitative skills. Of course, knowledge of computers and an understanding of people help. So the next time you see a whole aisle of personal care and cosmetic products, think of all the information that has been analyzed to put it there.

In Exercise 63 of this section you will see how Mr. Pittman uses a proportion to determine the quantity of mascara needed in a warehouse.

SALES ANALYST

WARM-UPS

True or false? Explain your answer.

1. The ratio of 40 men to 30 women can be expressed as the ratio 4 to 3.
2. The ratio of 3 feet to 2 yards can be expressed as the ratio 3 to 2.
3. If the ratio of men to women in the Chamber of Commerce is 3 to 2 and there are 20 men, then there must be 30 women.
4. The ratio of 1.5 to 2 is equivalent to the ratio of 3 to 4.
5. A statement that two ratios are equal is called a proportion.
6. The product of the extremes is equal to the product of the means.
7. If $\frac{2}{x} = \frac{3}{5}$, then $5x = 6$.
8. The ratio of the height of a 12-inch cactus to the height of a 3-foot cactus is 4 to 1.
9. If 30 out of 100 lawyers preferred aspirin and the rest did not, then the ratio of lawyers that preferred aspirin to those who did not is 30 to 100.
10. If $\frac{x + 5}{x} = \frac{2}{3}$, then $3x + 15 = 2x$.

7.7 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a ratio?
2. What are the different ways of expressing a ratio?
3. What are equivalent ratios?
4. What is a proportion?
5. What are the means and what are the extremes?
6. What is the extremes-means property?

For each ratio, find an equivalent ratio of integers in lowest terms. See Example 1.

7. $\frac{2.5}{3.5}$ **8.** $\frac{4.8}{1.2}$ **9.** $\frac{0.32}{0.6}$

10. $\frac{0.05}{0.8}$ **11.** $\frac{35}{10}$ **12.** $\frac{88}{33}$

13. $\frac{4.5}{7}$ **14.** $\frac{3}{2.5}$ **15.** $\frac{\frac{1}{2}}{\frac{1}{5}}$

16. $\frac{\frac{2}{3}}{\frac{3}{4}}$ **17.** $\frac{5}{\frac{1}{3}}$ **18.** $\frac{4}{\frac{1}{4}}$

Find a ratio for each of the following, and write it as a ratio of integers in lowest terms. See Examples 2–4.

19. ***Men and women.*** Find the ratio of men to women in a bowling league containing 12 men and 8 women.

20. ***Coffee drinkers.*** Among 100 coffee drinkers, 36 said that they preferred their coffee black and the rest did not prefer their coffee black. Find the ratio of those who prefer black coffee to those who prefer nonblack coffee.

FIGURE FOR EXERCISE 20