

8.6

GRAPHING LINEAR INEQUALITIES
IN TWO VARIABLESIn this
section

- Definition
- Graph of a Linear Inequality
- Using a Test Point to Graph an Inequality
- Applications

You studied linear inequalities in one variable in Chapter 3. In this section we extend the ideas of linear equations in two variables to study linear inequalities in two variables.

Definition

Linear inequalities in two variables have the same form as linear equations in two variables. An inequality symbol is used in place of the equal sign.

Linear Inequality in Two Variables

If A , B , and C are real numbers with A and B not both zero, then

$$Ax + By < C$$

is called a **linear inequality in two variables**. In place of $<$, we can also use \leq , $>$, or \geq .

The inequalities

$$3x - 4y \leq 8, \quad y > 2x - 3, \quad \text{and} \quad x - y + 9 < 0$$

are linear inequalities. Not all of these are in the form of the definition, but they could all be rewritten in that form.

An ordered pair is a solution to an inequality in two variables if the ordered pair satisfies the inequality.

EXAMPLE 1**Satisfying a linear inequality**

Determine whether each point satisfies the inequality $2x - 3y \geq 6$.

- a) (4, 1) b) (3, 0) c) (3, -2)

Solution

- a) To determine whether (4, 1) is a solution to the inequality, we replace x by 4 and y by 1 in the inequality $2x - 3y \geq 6$:

$$2(4) - 3(1) \geq 6$$

$$8 - 3 \geq 6$$

$$5 \geq 6 \quad \text{Incorrect}$$

So (4, 1) does not satisfy the inequality $2x - 3y \geq 6$.

- b) Replace x by 3 and y by 0:

$$2(3) - 3(0) \geq 6$$

$$6 \geq 6 \quad \text{Correct}$$

So the point (3, 0) satisfies the inequality $2x - 3y \geq 6$.

- c) Replace x by 3 and y by -2:

$$2(3) - 3(-2) \geq 6$$

$$6 + 6 \geq 6$$

$$12 \geq 6 \quad \text{Correct}$$

So the point (3, -2) satisfies the inequality $2x - 3y \geq 6$. ■

study tip

Write about what you read in the text. Sum things up in your own words. Write out important facts on note cards. When you have a few spare minutes in between classes review your note cards. Try to get the information on the cards into your memory.

Graph of a Linear Inequality

The graph of a linear inequality in two variables consists of all points in the rectangular coordinate system that satisfy the inequality. For example, the graph of the inequality

$$y > x + 2$$

consists of all points where the y -coordinate is larger than the x -coordinate plus 2. Consider the point $(3, 5)$ on the line

$$y = x + 2.$$

The y -coordinate of $(3, 5)$ is equal to the x -coordinate plus 2. If we choose a point with a larger y -coordinate, such as $(3, 6)$, it satisfies the inequality and it is above the line $y = x + 2$. In fact, any point above the line $y = x + 2$ satisfies $y > x + 2$. Likewise, all points below the line $y = x + 2$ satisfy the inequality $y < x + 2$. See Fig. 8.7.

helpful hint

Why do we keep drawing graphs? When we solve $2x + 1 = 7$, we don't bother to draw a graph showing 3, because the solution set is so simple. However, the solution set to a linear inequality is a very large set of ordered pairs. Graphing gives us a way to visualize the solution set.

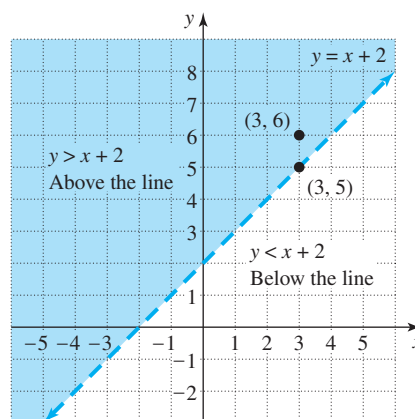


FIGURE 8.7

To graph the inequality, we shade all points above the line $y = x + 2$. To indicate that the line is not included in the graph of $y > x + 2$, we use a dashed line.

The procedure for graphing linear inequalities is summarized as follows.

Strategy for Graphing a Linear Inequality in Two Variables

- Solve the inequality for y , then graph $y = mx + b$.
 - $y > mx + b$ is the region above the line.
 - $y = mx + b$ is the line itself.
 - $y < mx + b$ is the region below the line.
- If the inequality involves only x , then graph the vertical line $x = k$.
 - $x > k$ is the region to the right of the line.
 - $x = k$ is the line itself.
 - $x < k$ is the region to the left of the line.

EXAMPLE 2 Graphing a linear inequality

Graph each inequality.

a) $y < \frac{1}{3}x + 1$

b) $y \geq -2x + 3$

c) $2x - 3y < 6$

Solution

a) The set of points satisfying this inequality is the region below the line

$$y = \frac{1}{3}x + 1.$$

To show this region, we first graph the boundary line. The slope of the line is $\frac{1}{3}$, and the y-intercept is (0, 1). We draw the line dashed because it is not part of the graph of $y < \frac{1}{3}x + 1$. In Fig. 8.8 the graph is the shaded region.

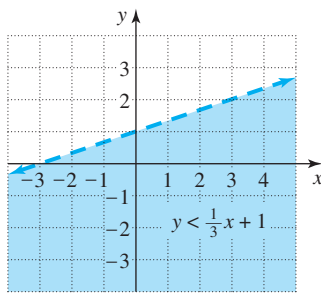


FIGURE 8.8

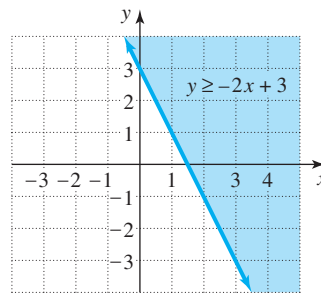


FIGURE 8.9

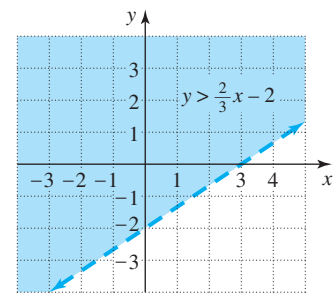


FIGURE 8.10

b) Because the inequality symbol is \geq , every point on or above the line satisfies this inequality. We use the fact that the slope of this line is -2 and the y-intercept is (0, 3) to draw the graph of the line. To show that the line $y = -2x + 3$ is included in the graph, we make it a solid line and shade the region above. See Fig. 8.9.

c) First solve for y:

$$\begin{aligned} 2x - 3y &< 6 \\ -3y &< -2x + 6 \\ y &> \frac{2}{3}x - 2 \quad \text{Divide by } -3 \text{ and reverse the inequality.} \end{aligned}$$

To graph this inequality, we first graph the line with slope $\frac{2}{3}$ and y-intercept (0, -2). We use a dashed line for the boundary because it is not included, and we shade the region above the line. Remember, “less than” means below the line and “greater than” means above the line only when the inequality is solved for y. See Fig. 8.10 for the graph. ■

EXAMPLE 3 Horizontal and vertical boundary lines

Graph each inequality.

a) $y \leq 4$

b) $x > 3$

Solution

a) The line $y = 4$ is the horizontal line with y-intercept (0, 4). We draw a solid horizontal line and shade below it as in Fig. 8.11.

- b) The line $x = 3$ is a vertical line through $(3, 0)$. Any point to the right of this line has an x -coordinate larger than 3. The graph is shown in Fig. 8.12.

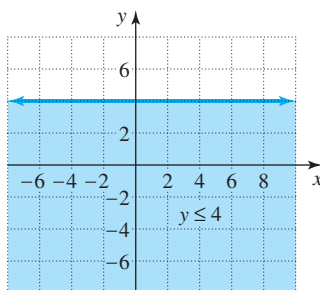


FIGURE 8.11

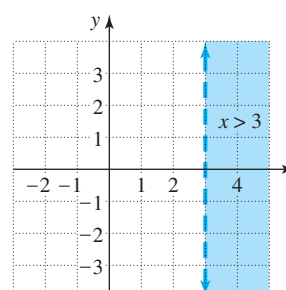


FIGURE 8.12

Using a Test Point to Graph an Inequality

The graph of a linear equation such as $2x - 3y = 6$ separates the coordinate plane into two regions. One region satisfies the inequality $2x - 3y > 6$, and the other region satisfies the inequality $2x - 3y < 6$. We can tell which region satisfies which inequality by testing a point in one region. With this method it is not necessary to solve the inequality for y .

EXAMPLE 4

Using a test point

Graph the inequality $2x - 3y > 6$.

Solution

First graph the equation $2x - 3y = 6$ using the x -intercept $(3, 0)$ and the y -intercept $(0, -2)$ as shown in Fig. 8.13. Select a point on one side of the line, say $(0, 1)$, to test in the inequality. Because

$$2(0) - 3(1) > 6$$

is false, the region on the other side of the line satisfies the inequality. The graph of $2x - 3y > 6$ is shown in Fig. 8.14.

helpful hint

Some people always like to choose $(0, 0)$ as the test point for lines that do not go through $(0, 0)$. The arithmetic for testing $(0, 0)$ is generally easier than for any other point.

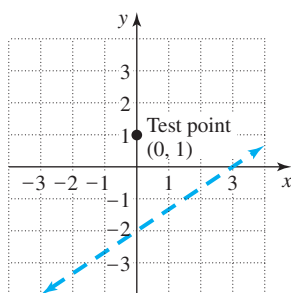


FIGURE 8.13

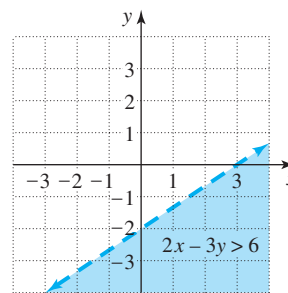


FIGURE 8.14

Applications

The values of variables used in applications are often restricted to nonnegative numbers. So solutions to inequalities in these applications are graphed in the first quadrant only.

EXAMPLE 5 Manufacturing tables

The Ozark Furniture Company can obtain at most 8000 board feet of oak lumber for making two types of tables. It takes 50 board feet to make a round table and 80 board feet to make a rectangular table. Write an inequality that limits the possible number of tables of each type that can be made. Draw a graph showing all possibilities for the number of tables that can be made.

Solution

If x is the number of round tables and y is the number of rectangular tables, then x and y satisfy the inequality

$$50x + 80y \leq 8000.$$

Now find the intercepts for the line $50x + 80y = 8000$:

$$50 \cdot 0 + 80y = 8000$$

$$80y = 8000$$

$$y = 100$$

$$50x + 80 \cdot 0 = 8000$$

$$50x = 8000$$

$$x = 160$$

Draw the line through $(0, 100)$ and $(160, 0)$. Because $(0, 0)$ satisfies the inequality, the number of tables must be below the line. Since the number of tables cannot be negative, the number of tables made must be below the line and in the first quadrant as shown in Fig. 8.15. Assuming that Ozark will not make a fraction of a table, only points in Fig. 8.15 with whole-number coordinates are practical.

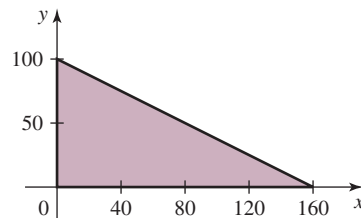


FIGURE 8.15

WARM - U P S**True or false? Explain your answer.**

1. The point $(-1, 4)$ satisfies the inequality $y > 3x + 1$. **True**
2. The point $(2, -3)$ satisfies the inequality $3x - 2y \geq 12$. **True**
3. The graph of the inequality $y > x + 9$ is the region above the line $y = x + 9$. **True**
4. The graph of the inequality $x < y + 2$ is the region below the line $x = y + 2$. **False**
5. The graph of $x = 3$ is a single point on the x -axis. **False**
6. The graph of $y \leq 5$ is the region below the horizontal line $y = 5$. **False**
7. The graph of $x < 3$ is the region to the left of the vertical line $x = 3$. **True**

WARM-UPS

(continued)

8. In graphing the inequality $y \geq x$ we use a dashed boundary line. **False**
 9. The point $(0, 0)$ is on the graph of the inequality $y \geq x$. **True**
 10. The point $(0, 0)$ lies above the line $y = 2x + 1$. **False**

8.6 EXERCISES

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

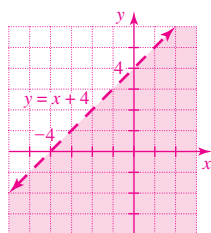
- What is a linear inequality in two variables?
A linear inequality has the same form as a linear equation except that an inequality symbol is used.
- How can you tell if an ordered pair satisfies a linear inequality in two variables?
An ordered pair satisfies a linear inequality if the inequality is correct when the variables are replaced by the coordinates of the ordered pair.
- How do you determine whether to draw the boundary line of the graph of a linear inequality dashed or solid?
If the inequality symbol includes equality, then the boundary line is solid; otherwise it is dashed.
- How do you decide which side of the boundary line to shade?
We shade the side that satisfies the inequality.
- What is the test point method?
In the test point method we test a point to see which side of the boundary line satisfies the inequality.
- What is the advantage of the test point method?
With the test point method you can use the inequality in any form.

Determine which of the points following each inequality satisfy that inequality. See Example 1.

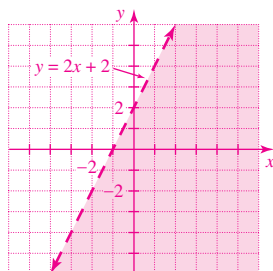
- $x - y > 5$ $(2, 3)$, $(-3, -9)$, $(8, 3)$ **$(-3, -9)$**
- $2x + y < 3$ $(-2, 6)$, $(0, 3)$, $(3, 0)$ **$(-2, 6)$**
- $y \geq -2x + 5$ $(3, 0)$, $(1, 3)$, $(-2, 5)$ **$(3, 0)$, $(1, 3)$**
- $y \leq -x + 6$ $(2, 0)$, $(-3, 9)$, $(-4, 12)$ **$(2, 0)$, $(-3, 9)$**
- $x > -3y + 4$ $(2, 3)$, $(7, -1)$, $(0, 5)$ **$(2, 3)$, $(0, 5)$**
- $x < -y - 3$ $(1, 2)$, $(-3, -4)$, $(0, -3)$ **$(-3, -4)$**

Graph each inequality. See Examples 2 and 3.

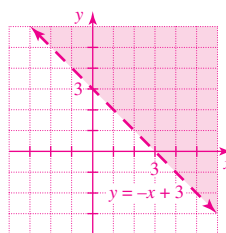
13. $y < x + 4$



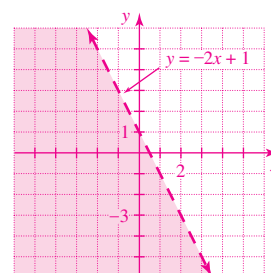
14. $y < 2x + 2$



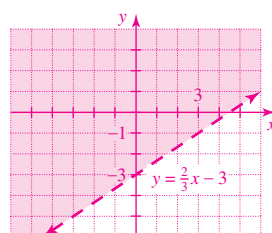
15. $y > -x + 3$



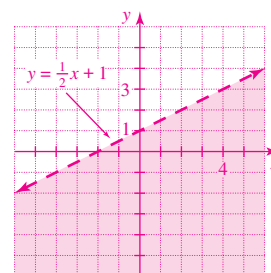
16. $y < -2x + 1$



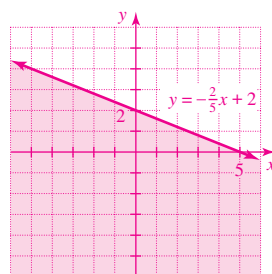
17. $y > \frac{2}{3}x - 3$



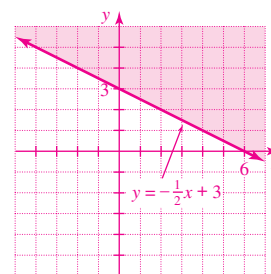
18. $y < \frac{1}{2}x + 1$



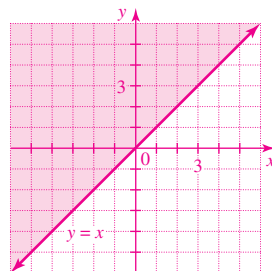
19. $y \leq -\frac{2}{5}x + 2$



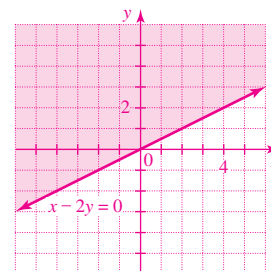
20. $y \geq -\frac{1}{2}x + 3$



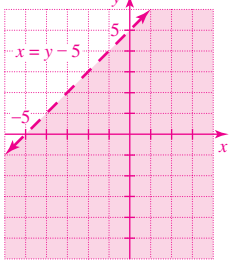
21. $y - x \geq 0$



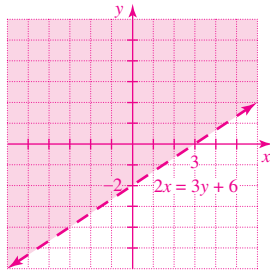
22. $x - 2y \leq 0$



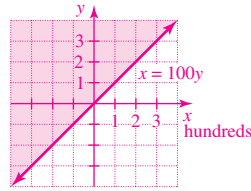
23. $x > y - 5$



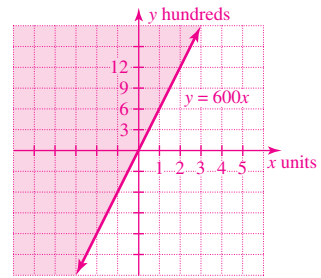
24. $2x < 3y + 6$



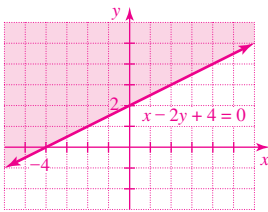
33. $x \leq 100y$



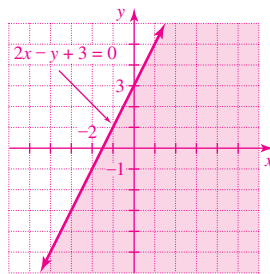
34. $y \geq 600x$



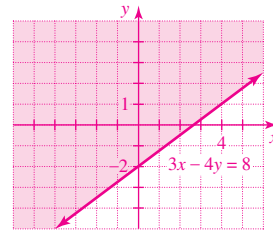
25. $x - 2y + 4 \leq 0$



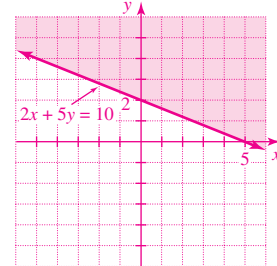
26. $2x - y + 3 \geq 0$



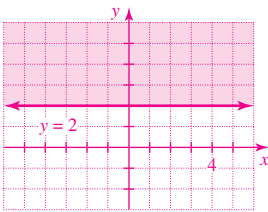
35. $3x - 4y \leq 8$



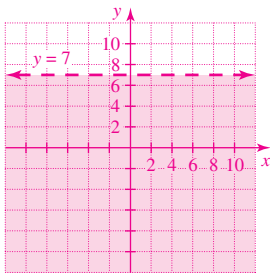
36. $2x + 5y \geq 10$



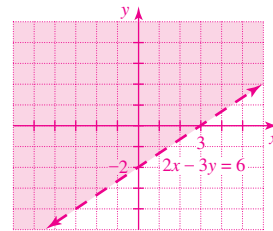
27. $y \geq 2$



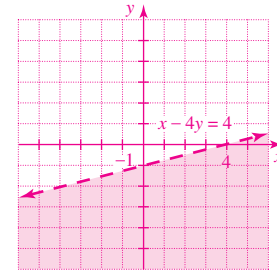
28. $y < 7$



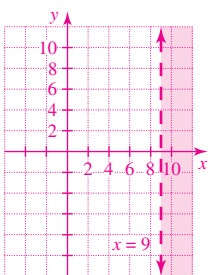
37. $2x - 3y < 6$



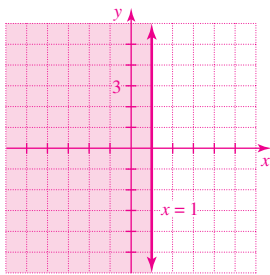
38. $x - 4y > 4$



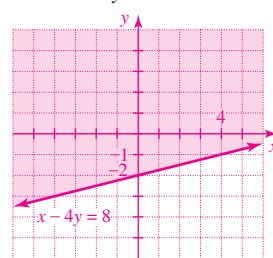
29. $x > 9$



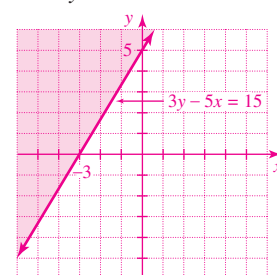
30. $x \leq 1$



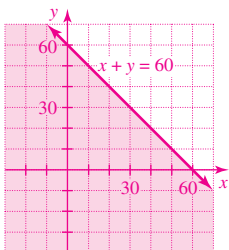
39. $x - 4y \leq 8$



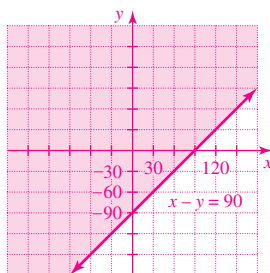
40. $3y - 5x \geq 15$



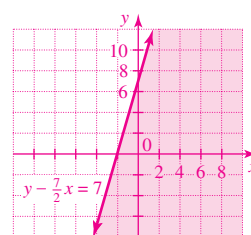
31. $x + y \leq 60$



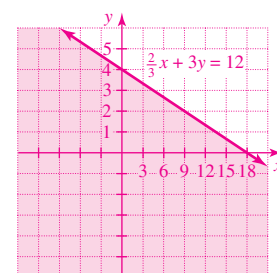
32. $x - y \leq 90$



41. $y - \frac{7}{2}x \leq 7$

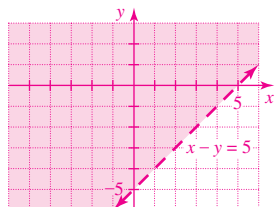


42. $\frac{2}{3}x + 3y \leq 12$

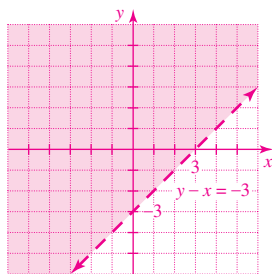


Graph each inequality. Use the test point method of Example 4.

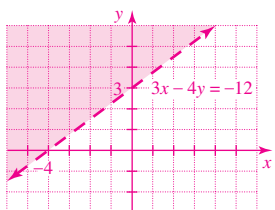
43. $x - y < 5$



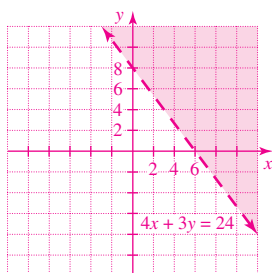
44. $y - x > -3$



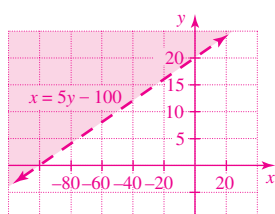
45. $3x - 4y < -12$



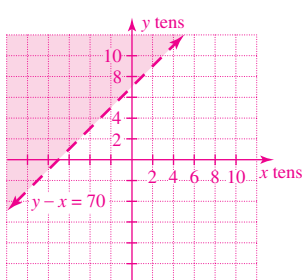
46. $4x + 3y > 24$



47. $x < 5y - 100$

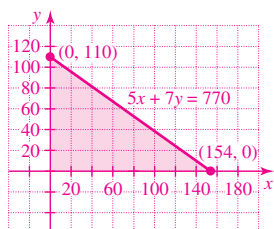


48. $-x > 70 - y$



Solve each problem. See Example 5.

49. **Storing the tables.** Ozark Furniture Company must store its oak tables before shipping. A round table is packaged in a carton with a volume of 25 cubic feet (ft^3), and a rectangular table is packaged in a carton with a volume of 35 ft^3 . The warehouse has at most 3850 ft^3 of space available for these tables. Write an inequality that limits the possible number of tables of each type that can be stored, and graph the inequality in the first quadrant.
 $5x + 7y \leq 770$



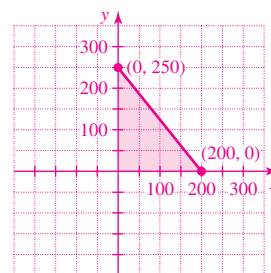
50. **Maple rockers.** Ozark Furniture Company can obtain at most 3000 board feet of maple lumber for making its classic and modern maple rocking chairs. A classic maple rocker requires 15 board feet of maple, and a



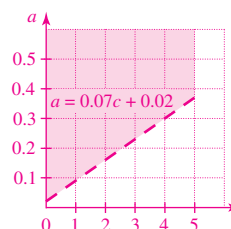
FIGURE FOR EXERCISE 50

modern rocker requires 12 board feet of maple. Write an inequality that limits the possible number of maple rockers of each type that can be made, and graph the inequality in the first quadrant.

$5x + 4y \leq 1000$



51. **Enzyme concentration.** A food chemist tests enzymes for their ability to break down pectin in fruit juices (Dennis Callas, *Snapshots of Applications in Mathematics*). Excess pectin makes juice cloudy. In one test, the chemist measures the concentration of the enzyme, c , in milligrams per milliliter and the fraction of light absorbed by the liquid, a . If $a > 0.07c + 0.02$, then the enzyme is working as it should. Graph the inequality for $0 < c < 5$.



GETTING MORE INVOLVED



52. Discussion. When asked to graph the inequality $x + 2y < 12$, a student found that $(0, 5)$ and $(8, 0)$ both satisfied $x + 2y < 12$. The student then drew a dashed line through these two points and shaded the region below the line. What is wrong with this method? Do



53. Writing. Compare and contrast the two methods presented in this section for graphing linear inequalities. What are the advantages and disadvantages of each method? How do you choose which method to use?

all of the points graphed by this student satisfy the inequality?

In this section

- The Solution to a System of Inequalities
- Graphing a System of Inequalities

8.7 GRAPHING SYSTEMS OF LINEAR INEQUALITIES

In Section 8.6 you learned how to solve a linear inequality. In this section you will solve systems of linear inequalities.

The Solution to a System of Inequalities

A **system of inequalities** consists of two or more inequalities. A point is a solution to a system of inequalities if it satisfies all of the inequalities in the system.

EXAMPLE 1

Satisfying a system of inequalities

Determine whether each point is a solution to the system of inequalities:

$$\begin{aligned} 2x + 3y &< 6 \\ y &> 2x - 1 \end{aligned}$$

- a) $(-3, 2)$ b) $(4, -3)$ c) $(5, 1)$

Solution

- a) The point $(-3, 2)$ is a solution to the system if it satisfies both inequalities. Let $x = -3$ and $y = 2$ in each inequality:

$$\begin{aligned} 2x + 3y &< 6 & y &> 2x - 1 \\ 2(-3) + 3(2) &< 6 & 2 &> 2(-3) - 1 \\ 0 &< 6 & 2 &> -7 \end{aligned}$$

Because both inequalities are satisfied, the point $(-3, 2)$ is a solution to the system.

- b) Let $x = 4$ and $y = -3$ in each inequality:

$$\begin{aligned} 2x + 3y &< 6 & y &> 2x - 1 \\ 2(4) + 3(-3) &< 6 & -3 &> 2(4) - 1 \\ -1 &< 6 & -3 &> 7 \end{aligned}$$

Because only one inequality is satisfied, the point $(4, -3)$ is not a solution to the system.

- c) Let $x = 5$ and $y = 1$ in each inequality:

$$\begin{aligned} 2x + 3y &< 6 & y &> 2x - 1 \\ 2(5) + 3(1) &< 6 & 1 &> 2(5) - 1 \\ 13 &< 6 & 1 &> 9 \end{aligned}$$

Because neither inequality is satisfied, the point $(5, 1)$ is not a solution to the system. ■

study tip

Read the text and recite to yourself what you have read. Ask questions and answer them out loud. Listen to your answers to see if they are complete and correct. Would other students understand your answers?