GRAPHING CALCULATOR EXERCISES

Solve each equation by locating the x-intercepts on the graph of a corresponding function. Round approximate answers to two decimal places.

66. $(5x - 7)^2 - (5x - 7) - 6 = 0$

67. $x^4 - 116x^2 + 1600 = 0$

68. $(x^2 + 3x)^2 - 7(x^2 + 3x) + 9 = 0$

69. $x^2 - 3x^{1/2} - 12 = 0$

10.5 QUADRATIC AND RATIONAL INEQUALITIES

In this section

- Solving Quadratic Inequalities with a Sign Graph
- Solving Rational Inequalities with a Sign Graph
- Quadratic Inequalities That Cannot Be Factored
- Applications

In this section we solve inequalities involving quadratic polynomials. We use a new technique based on the rules for multiplying real numbers.

Solving Quadratic Inequalities with a Sign Graph

An inequality involving a quadratic polynomial is called a **quadratic** inequality.

Quadratic Inequality

A quadratic inequality is an inequality of the form

$$ax^2 + bx + c > 0,$$

where a, b, and c are real numbers with $a \neq 0$. The inequality symbols $<$, $\leq$, and $\geq$ may also be used.

If we can factor a quadratic inequality, then the inequality can be solved with a **sign graph,** which shows where each factor is positive, negative, or zero.

EXAMPLE 1

Solving a quadratic inequality

Use a sign graph to solve the inequality $x^2 + 3x - 10 > 0$.

Solution

Because the left-hand side can be factored, we can write the inequality as

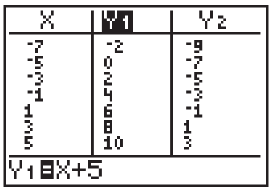
$$(x + 5)(x - 2) > 0.$$

This inequality says that the product of $x + 5$ and $x - 2$ is positive. If both factors are negative or both are positive, the product is positive. To analyze the signs of each factor, we make a sign graph as follows. First consider the possible values of the factor $x + 5$:

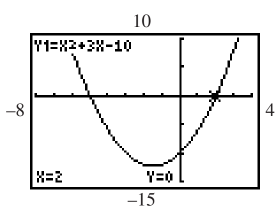
Value	Where	On the number line
$x + 5 = 0$	if $x = -5$	Put a 0 above -5.
$x + 5 > 0$	if $x > -5$	Put $+$ signs to the right of -5.
$x + 5 < 0$	if $x < -5$	Put $-$ signs to the left of -5.

calculator close-up

Use Y= to set $y_1 = x + 5$ and $y_2 = x - 2$. Now make a table and scroll through the table. The table numerically supports the sign graph in Fig. 10.10.



Note that the graph of $y = x^2 + 3x - 10$ is above the x-axis when $x < -5$ or when $x > 2$.



The sign graph shown in Fig. 10.9 for the factor $x + 5$ is made from the information in the preceding table.

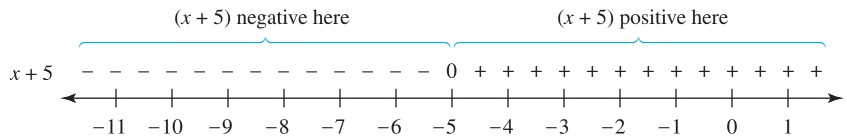


FIGURE 10.9

Now consider the possible values of the factor $x - 2$:

Value	Where	On the number line
$x - 2 = 0$	if $x = 2$	Put a 0 above 2.
$x - 2 > 0$	if $x > 2$	Put + signs to the right of 2.
$x - 2 < 0$	if $x < 2$	Put − signs to the left of 2.

We put the information for the factor $x - 2$ on the sign graph for the factor $x + 5$ as shown in Fig. 10.10. We can see from Fig. 10.10 that the product is positive if $x < -5$ and the product is positive if $x > 2$. The solution set for the quadratic inequality is shown in Fig. 10.11. Note that -5 and 2 are not included in the graph because for those values of x the product is zero. The solution set is $(-\infty, -5) \cup (2, \infty)$.

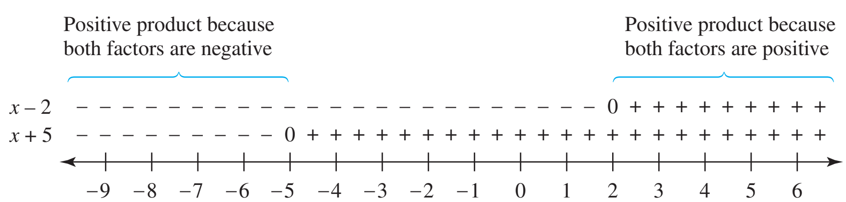


FIGURE 10.10

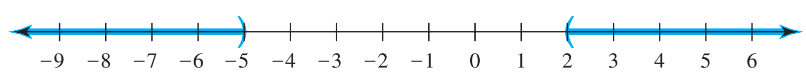


FIGURE 10.11

In the next example we will make the procedure from Example 1 a bit more efficient.

EXAMPLE 2 **Solving a quadratic inequality**

Solve $2x^2 + 5x \le 3$ and graph the solution set.

Solution

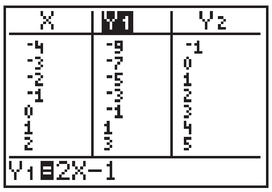
Rewrite the inequality with 0 on one side:

$$2x^2 + 5x - 3 \le 0$$

$$(2x - 1)(x + 3) \le 0 \quad \text{Factor.}$$

calculator close-up

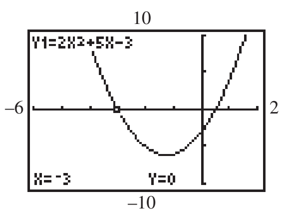
Use Y= to set $y_1 = 2x - 1$ and $y_2 = x + 3$. The table of values for y_1 and y_2 supports the sign graph in Fig. 10.12.



X	Y1	Y2
-4	-9	-1
-3	-7	0
-2	-5	1
-1	-3	2
0	-1	3
1	1	4
2	3	5

Y1=2X-1

Note that the graph of $y = 2x^2 + 5x - 3$ is below the x-axis when x is between -3 and $\frac{1}{2}$.



Examine the signs of each factor:

$$2x - 1 = 0 \text{ if } x = \frac{1}{2}$$

$$2x - 1 > 0 \text{ if } x > \frac{1}{2}$$

$$2x - 1 < 0 \text{ if } x < \frac{1}{2}$$

$$x + 3 = 0 \text{ if } x = -3$$

$$x + 3 > 0 \text{ if } x > -3$$

$$x + 3 < 0 \text{ if } x < -3$$

Make a sign graph as shown in Fig. 10.12. The product of the factors is negative between -3 and $\frac{1}{2}$, when one factor is negative and the other is positive. The product is 0 at -3 and at $\frac{1}{2}$. So the solution set is the interval $\left[-3, \frac{1}{2}\right]$. The graph of the solution set is shown in Fig. 10.13.

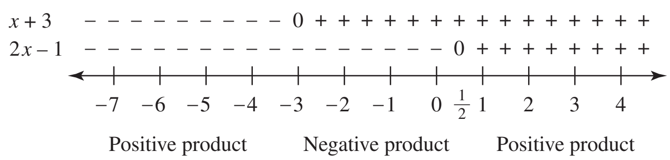


FIGURE 10.12

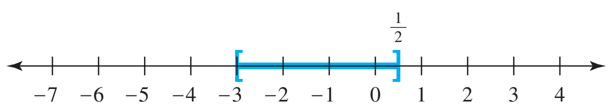


FIGURE 10.13

We summarize the strategy used for solving a quadratic inequality as follows.

Strategy for Solving a Quadratic Inequality with a Sign Graph

1. Write the inequality with 0 on the right.
2. Factor the quadratic polynomial on the left.
3. Make a sign graph showing where each factor is positive, negative, or zero.
4. Use the rules for multiplying signed numbers to determine which regions satisfy the original inequality.

Solving Rational Inequalities with a Sign Graph

The inequalities

$$\frac{x + 2}{x - 3} \le 2, \quad \frac{2x - 3}{x + 5} \le 0 \quad \text{and} \quad \frac{2}{x + 4} \ge \frac{1}{x + 1}$$

are called **rational inequalities.** When we solve *equations* that involve rational expressions, we usually multiply each side by the LCD. However, if we multiply each side of any inequality by a negative number, we must reverse the inequality, and when we multiply by a positive number, we do not reverse the inequality. For this reason we generally *do not multiply inequalities by expressions involving variables.* The values of the expressions might be positive or negative. The next two examples show how to use a sign graph to solve rational inequalities that have variables in the denominator.

EXAMPLE 3 Solving a rational inequality

Solve $\frac{x + 2}{x - 3} \le 2$ and graph the solution set.

Solution

We *do not* multiply each side by $x - 3$. Instead, subtract 2 from each side to get 0 on the right:

$$\frac{x + 2}{x - 3} - 2 \le 0$$

$$\frac{x + 2}{x - 3} - \frac{2(x - 3)}{x - 3} \le 0 \quad \text{Get a common denominator.}$$

$$\frac{x + 2}{x - 3} - \frac{2x - 6}{x - 3} \le 0 \quad \text{Simplify.}$$

$$\frac{x + 2 - 2x + 6}{x - 3} \le 0 \quad \text{Subtract the rational expressions.}$$

$$\frac{-x + 8}{x - 3} \le 0 \quad \text{The quotient of } -x + 8 \text{ and } x - 3 \text{ is less than or equal to 0.}$$

Examine the signs of the numerator and denominator:

$$x - 3 = 0 \text{ if } x = 3 \qquad -x + 8 = 0 \text{ if } x = 8$$
$$x - 3 > 0 \text{ if } x > 3 \qquad -x + 8 > 0 \text{ if } x < 8$$
$$x - 3 < 0 \text{ if } x < 3 \qquad -x + 8 < 0 \text{ if } x > 8$$

Make a sign graph as shown in Fig. 10.14. Using the rule for dividing signed numbers and the sign graph, we can identify where the quotient is negative or zero. The solution set is $(-\infty, 3) \cup [8, \infty)$. Note that 3 is not in the solution set because the quotient is undefined if $x = 3$. The graph of the solution set is shown in Fig. 10.15.

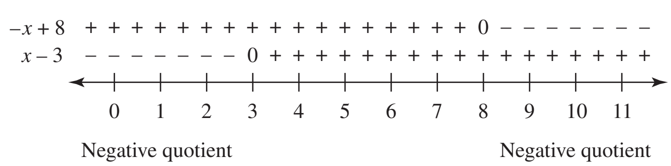


FIGURE 10.14

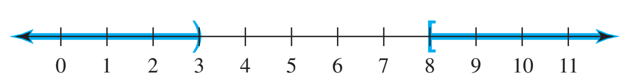


FIGURE 10.15

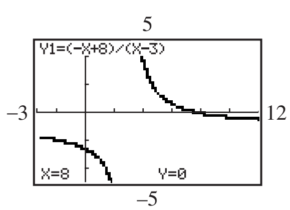
CAUTION Remember to reverse the inequality sign when multiplying or dividing by a negative number. For example, $x - 3 > 0$ is equivalent to $x > 3$. But $-x + 8 > 0$ is equivalent to $-x > -8$, or $x < 8$.

helpful hint

By getting 0 on one side of the inequality, we can use the rules for dividing signed numbers. The only way to obtain a negative result is to divide numbers with opposite signs.

calculator close-up

Graph $y = \frac{-x + 8}{x - 3}$ to support the conclusion that $y \le 0$ when $x < 3$ or $x \ge 8$.



EXAMPLE 4

Solving a rational inequality

Solve $\frac{2}{x+4} \ge \frac{1}{x+1}$ and graph the solution set.

Solution

We do not multiply by the LCD as we do in solving equations. Instead, subtract $\frac{1}{x+1}$ from each side:

$$\frac{2}{x+4} - \frac{1}{x+1} \ge 0$$

$$\frac{2(x+1)}{(x+4)(x+1)} - \frac{1(x+4)}{(x+1)(x+4)} \ge 0 \quad \text{Get a common denominator.}$$

$$\frac{2x+2-x-4}{(x+1)(x+4)} \ge 0 \quad \text{Simplify.}$$

$$\frac{x-2}{(x+1)(x+4)} \ge 0$$

Make a sign graph as shown in Fig. 10.16.

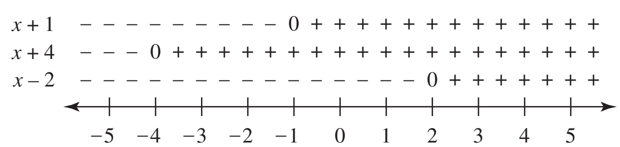


FIGURE 10.16

The computation of

$$\frac{x-2}{(x+1)(x+4)}$$

involves multiplication and division. The result of this computation is positive if all of the three binomials are positive or if only one is positive and the other two are negative. The sign graph shows that this rational expression will have a positive value when x is between -4 and -1 and again when x is larger than 2. The solution set is $(-4, -1) \cup [2, \infty)$. Note that -1 and -4 are not in the solution set because they make the denominator zero. The graph of the solution set is shown in Fig. 10.17.

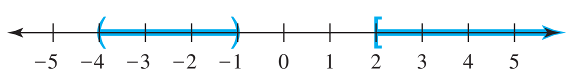


FIGURE 10.17

Solving rational inequalities with a sign graph is summarized below.

Strategy for Solving a Rational Inequality with a Sign Graph

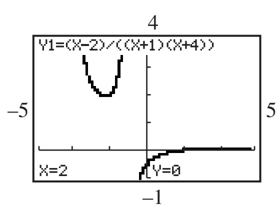
1. Rewrite the inequality with 0 on the right-hand side.
2. Use only addition and subtraction to get an equivalent inequality.
3. Factor the numerator and denominator if possible.
4. Make a sign graph showing where each factor is positive, negative, or zero.
5. Use the rules for multiplying and dividing signed numbers to determine the regions that satisfy the original inequality.

study tip

If you must miss class, let your instructor know. Be sure to get notes from a reliable classmate. Take good notes yourself in case a classmate comes to you for notes.

calculator close-up

Graph $y = \frac{x-2}{(x+1)(x+4)}$ to support the conclusion that $y \ge 0$ when x is between -4 and -1 or when $x \ge 2$.



Another method for solving quadratic and rational inequalities will be shown in Example 5. This method, called the **test point method,** can be used instead of the sign graph to solve the inequalities of Examples 1, 2, 3, and 4.

Quadratic Inequalities That Cannot Be Factored

The following example shows how to solve a quadratic inequality that involves a prime polynomial.

EXAMPLE 5 **Solving a quadratic inequality using the quadratic formula**

Solve $x^2 - 4x - 6 > 0$ and graph the solution set.

Solution

The quadratic polynomial is prime, but we can solve $x^2 - 4x - 6 = 0$ by the quadratic formula:

$$x = \frac{4 \pm \sqrt{16 - 4(1)(-6)}}{2(1)} = \frac{4 \pm \sqrt{40}}{2} = \frac{4 \pm 2\sqrt{10}}{2} = 2 \pm \sqrt{10}$$

As in the previous examples, the solutions to the equation divide the number line into the intervals $(-\infty, 2 - \sqrt{10})$, $(2 - \sqrt{10}, 2 + \sqrt{10})$, and $(2 + \sqrt{10}, \infty)$ on which the quadratic polynomial has either a positive or negative value. To determine which, we select an arbitrary **test point** in each interval. Because $2 + \sqrt{10} \approx 5.2$ and $2 - \sqrt{10} \approx -1.2$, we choose a test point that is less than -1.2, one between -1.2 and 5.2, and one that is greater than 5.2. We have selected -2, 0, and 7 for test points, as shown in Fig. 10.18. Now evaluate $x^2 - 4x - 6$ at each test point.

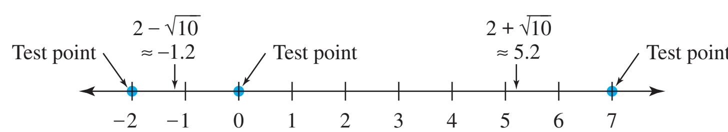


FIGURE 10.18

calculator close-up

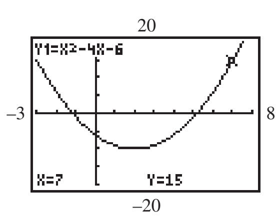
Notice that the graph of

$y = x^2 - 4x - 6$

lies above the x-axis when

$x < 2 - \sqrt{10}$

or $x > 2 + \sqrt{10}$.



Test point	Value of $x^2 - 4x - 6$ at the test point	Sign of $x^2 - 4x - 6$ in interval of test point
-2	6	Positive
0	-6	Negative
7	15	Positive

Because $x^2 - 4x - 6$ is positive at the test points -2 and 7, it is positive at every point in the intervals containing those test points. So the solution set to the inequality $x^2 - 4x - 6 > 0$ is

$$(-\infty, 2 - \sqrt{10}) \cup (2 + \sqrt{10}, \infty),$$

and its graph is shown in Fig. 10.19.

FIGURE 10.19

The test point method used in Example 5 can be used also on inequalities that do factor. We summarize the strategy for solving inequalities using test points in the following box.

Strategy for Solving Quadratic Inequalities Using Test Points

1. Rewrite the inequality with 0 on the right.
2. Solve the quadratic equation that results from replacing the inequality symbol with the equals symbol.
3. Locate the solutions to the quadratic equation on a number line.
4. Select a test point in each interval determined by the solutions to the quadratic equation.
5. Test each point in the original quadratic inequality to determine which intervals satisfy the inequality.

Applications

The following example shows how a quadratic inequality can be used to solve a problem.

EXAMPLE 6 **Making a profit**

Charlene's daily profit P (in dollars) for selling x magazine subscriptions is determined by the formula

$$P = -x^2 + 80x - 1500.$$

For what values of x is her profit positive?

Solution

We can find the values of x for which $P > 0$ by solving a quadratic inequality:

$$-x^2 + 80x - 1500 > 0$$

$$x^2 - 80x + 1500 < 0 \quad \text{Multiply each side by } -1.$$

$$(x - 30)(x - 50) < 0 \quad \text{Factor.}$$

Make a sign graph as shown in Fig. 10.20. The product of the two factors is negative for x between 30 and 50. Because the last inequality is equivalent to the first, the profit is positive when the number of magazine subscriptions sold is greater than 30 and less than 50.

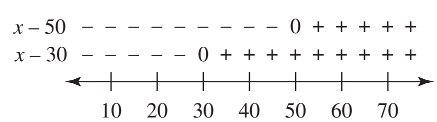


FIGURE 10.20

WARM-UPS

True or false? Explain.

1. The solution set to $x^2 > 4$ is $(2, \infty)$.
2. The inequality $\frac{x}{x-3} > 2$ is equivalent to $x > 2x - 6$.
3. The inequality $(x - 1)(x + 2) < 0$ is equivalent to $x - 1 < 0$ or $x + 2 < 0$.

WARM-UPS

(continued)

4. We cannot solve quadratic inequalities that do not factor.

5. One technique for solving quadratic inequalities is based on the rules for multiplying signed numbers.

6. Multiplying each side of an inequality by a variable should be avoided.

7. In solving quadratic or rational inequalities, we always get 0 on one side.

8. The inequality $\frac{x}{2} > 3$ is equivalent to $x > 6$.

9. The inequality $\frac{x-3}{x+2} < 1$ is equivalent to $\frac{x-3}{x+2} - 1 < 0$.

10. The solution set to $\frac{x+2}{x-4} \geq 0$ is $(-\infty, -2] \cup [4, \infty)$.

10.5 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a quadratic inequality?

2. What is a sign graph?

3. What is a rational inequality?

4. Why don't we usually multiply each side of an inequality by an expression involving a variable?

Solve each inequality. State the solution set using interval notation and graph the solution set. See Examples 1 and 2.

5. $x^2 + x - 6 < 0$

6. $x^2 - 3x - 4 \geq 0$

7. $y^2 - 4 > 0$

8. $z^2 - 16 < 0$

9. $2u^2 + 5u \geq 12$

10. $2v^2 + 7v < 4$

11. $4x^2 - 8x \geq 0$

12. $x^2 + x > 0$

13. $5x - 10x^2 < 0$

14. $3x - x^2 > 0$

15. $x^2 + 6x + 9 \geq 0$

16. $x^2 + 25 < 10x$

Solve each rational inequality. State and graph the solution set. See Examples 3 and 4.

17. $\dfrac{x}{x-3} > 0$

18. $\dfrac{a}{a+2} > 0$

19. $\frac{x + 2}{x} \le 0$

20. $\frac{w - 6}{w} \le 0$

21. $\frac{t - 3}{t + 6} > 0$

22. $\frac{x - 2}{2x + 5} < 0$

23. $\frac{x}{x + 2} > -1$

24. $\frac{x + 3}{x} \le -2$

25. $\frac{2}{x - 5} > \frac{1}{x + 4}$

26. $\frac{3}{x + 2} > \frac{2}{x - 1}$

27. $\frac{m}{m - 5} + \frac{3}{m - 1} > 0$

28. $\frac{p}{p - 16} + \frac{2}{p - 6} \le 0$

29. $\frac{x}{x - 3} \le \frac{-8}{x - 6}$

30. $\frac{x}{x + 20} > \frac{2}{x + 8}$

Solve each inequality. State and graph the solution set. See Example 5.

31. $x^2 - 2x - 4 > 0$

32. $x^2 - 2x - 5 \le 0$

33. $2x^2 - 6x + 3 \ge 0$

34. $2x^2 - 8x + 3 < 0$

35. $y^2 - 3y - 9 \le 0$

36. $z^2 - 5z - 7 < 0$

In Exercises 37–60, solve each inequality. State the solution set using interval notation.

37. $x^2 \le 9$

38. $x^2 \ge 36$

39. $16 - x^2 > 0$

40. $9 - x^2 < 0$

41. $x^2 - 4x \ge 0$

42. $4x^2 - 9 > 0$

43. $3(2w^2 - 5) < w$

44. $6(y^2 - 2) + y < 0$

45. $z^2 \ge 4(z + 3)$

46. $t^2 < 3(2t - 3)$

47. $(q + 4)^2 > 10q + 31$

48. $(2p + 4)(p - 1) < (p + 2)^2$

49. $\frac{1}{2}x^2 \ge 4 - x$

50. $\frac{1}{2}x^2 \le x + 12$