of a novel and every day thereafter increase their daily reading by two pages. If his students follow this suggestion, then how many pages will they read during October?

58. ***Heavy penalties.*** If an air-conditioning system is not completed by the agreed upon date, the contractor pays a penalty of \$500 for the first day that it is overdue, \$600 for the second day, \$700 for the third day, and so on. If the system is completed 10 days late, then what is the total amount of the penalties that the contractor must pay?

GETTING MORE INVOLVED

59. ***Discussion.*** Which of the following sequences is not an arithmetic sequence? Explain your answer.

a) $\frac{1}{2}, 1, \frac{3}{2}, \ldots$ **b)** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

c) $5, 0, -5, \ldots$ **d)** $2, 3, 4, \ldots$

60. ***Discussion.*** What is the smallest value of n for which $\sum_{i=1}^{n} \frac{1}{2} > 50$?

14.4 GEOMETRIC SEQUENCES AND SERIES

In this section

- Geometric Sequences
- Finite Geometric Series
- Infinite Geometric Series
- Annuities

In Section 14.3 you studied the arithmetic sequences and series. In this section you will study sequences in which each term is a *multiple* of the term preceding it. You will also learn how to find the sum of the corresponding series.

Geometric Sequences

In an arithmetic sequence such as 2, 4, 6, 8, 10, . . . there is a common difference between consecutive terms. In a geometric sequence there is a common ratio between consecutive terms. The following table contains several geometric sequences and the common ratios between consecutive terms.

Geometric Sequence	Common Ratio
$3, 6, 12, 24, 48, \ldots$	2
$27, 9, 3, 1, \frac{1}{3}, \ldots$	$\frac{1}{3}$
$1, -10, 100, -1000, \ldots$	-10

Note that every term after the first term of each geometric sequence can be obtained by multiplying the previous term by the common ratio.

Geometric Sequence

A sequence in which each term after the first is obtained by multiplying the preceding term by a constant is called a **geometric sequence.**

The constant is denoted by the letter r and is called the **common ratio.** If a_1 is the first term, then the second term is $a_1 r$. The third term is $a_1 r^2$, the fourth term is $a_1 r^3$, and so on. We can write a formula for the nth term of a geometric sequence by following this pattern.

Formula for the nth Term of a Geometric Sequence

The nth term, a_n, of a geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

The first term and the common ratio determine all of the terms of a geometric sequence.

EXAMPLE 1 **Finding the nth term**

Write a formula for the nth term of the geometric sequence

$$6, 2, \frac{2}{3}, \frac{2}{9}, \ldots.$$

Solution

We can obtain the common ratio by dividing any term after the first by the term preceding it. So

$$r = 2 \div 6 = \frac{1}{3}.$$

Because each term after the first is $\frac{1}{3}$ of the term preceding it, the nth term is given by

$$a_n = 6\left(\frac{1}{3}\right)^{n-1}.$$

Check a few terms: $a_1 = 6\left(\frac{1}{3}\right)^{1-1} = 6$, $a_2 = 6\left(\frac{1}{3}\right)^{2-1} = 2$, and $a_3 = 6\left(\frac{1}{3}\right)^{3-1} = \frac{2}{3}$. ■

EXAMPLE 2 **Finding the nth term**

Find a formula for the nth term of the geometric sequence

$$2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots.$$

Solution

We obtain the ratio by dividing a term by the term preceding it:

$$r = -1 \div 2 = -\frac{1}{2}$$

Each term after the first is obtained by multiplying the preceding term by $-\frac{1}{2}$. The formula for the nth term is

$$a_n = 2\left(-\frac{1}{2}\right)^{n-1}.$$

Check a few terms: $a_1 = 2\left(-\frac{1}{2}\right)^{1-1} = 2$, $a_2 = 2\left(-\frac{1}{2}\right)^{2-1} = -1$, and $a_3 = 2\left(-\frac{1}{2}\right)^{3-1} = \frac{1}{2}$. ■

In the next example we use the formula for the nth term to write some terms of a geometric sequence.

EXAMPLE 3 **Writing the terms**

Write the first five terms of the geometric sequence whose nth term is

$$a_n = 3(-2)^{n-1}.$$

Solution

Let n take the values 1 through 5 in the formula for the nth term:

$$\begin{aligned} a_1 &= 3(-2)^{1-1} = 3 \\ a_2 &= 3(-2)^{2-1} = -6 \\ a_3 &= 3(-2)^{3-1} = 12 \\ a_4 &= 3(-2)^{4-1} = -24 \\ a_5 &= 3(-2)^{5-1} = 48 \end{aligned}$$

Notice that $a_n = 3(-2)^{n-1}$ gives the general term for a geometric sequence with first term 3 and common ratio -2. Because every term after the first can be obtained by multiplying the previous term by -2, the terms 3, -6, 12, -24, and 48 are correct. ■

The formula for the nth term involves four variables: a_n, a_1, r, and n. If we know the value of any three of them, we can find the value of the fourth.

EXAMPLE 4 **Finding a missing term**

Find the first term of a geometric sequence whose fourth term is 8 and whose common ratio is $\frac{1}{2}$.

Solution

Let $a_4 = 8$, $r = \frac{1}{2}$, and $n = 4$ in the formula $a_n = a_1 r^{n-1}$:

$$\begin{aligned} 8 &= a_1\left(\frac{1}{2}\right)^{4-1} \\ 8 &= a_1 \cdot \frac{1}{8} \\ 64 &= a_1 \end{aligned}$$

So the first term is 64. ■

study tip

Many schools have study skills centers that offer courses, workshops, and individual help on how to study. A search for "study skills" on the World Wide Web will turn up more information than you could possibly read. If you are not having the success in school that you would like, do something about it. What you do now will affect you the rest of your life.

Finite Geometric Series

Consider the following series:

$$1 + 2 + 4 + 8 + 16 + \cdots + 512$$

The terms of this series are the terms of a finite geometric sequence. The indicated sum of a geometric sequence is called a **geometric series.**

We can find the actual sum of this finite geometric series by using a technique similar to the one used for the sum of an arithmetic series. Let

$$S = 1 + 2 + 4 + 8 + \cdots + 256 + 512.$$

Because the common ratio is 2, multiply each side by -2:

$$-2S = -2 - 4 - 8 - \cdots - 512 - 1024$$

Adding the last two equations eliminates all but two of the terms on the right:

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + \cdots + 256 + 512 \\ -2S &= \quad -2 - 4 - 8 - \cdots \qquad - 512 - 1024 \\ \hline -S &= 1 \qquad\qquad\qquad\qquad - 1024 \quad \text{Add.} \\ -S &= -1023 \\ S &= 1023 \end{aligned}$$

If $S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$ is any geometric series, we can find the sum in the same manner. Multiplying each side of this equation by $-r$ yields

$$-rS_n = -a_1r - a_1r^2 - a_1r^3 - \cdots - a_1r^n.$$

If we add S_n and $-rS_n$, all but two of the terms on the right are eliminated:

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots \qquad + a_1r^{n-1} \\ -rS_n &= \quad - a_1r - a_1r^2 - a_1r^3 - \cdots \qquad - a_1r^n \\ \hline S_n - rS_n &= a_1 \qquad\qquad\qquad\qquad - a_1r^n \quad \text{Add.} \\ (1 - r)S_n &= a_1(1 - r^n) \quad \text{Factor out common factors.} \end{aligned}$$

Now divide each side of this equation by $1 - r$ to get the formula for S_n.

Sum of *n* Terms of a Geometric Series

If S_n represents the sum of the first n terms of a geometric series with first term a_1 and common ratio r ($r \neq 1$), then

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

EXAMPLE 5 **The sum of a finite geometric series**

Find the sum of the series

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{729}.$$

Solution

The first term is $\frac{1}{3}$, and the common ratio is $\frac{1}{3}$. So the nth term can be written as

$$a_n = \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}.$$

We can use this formula to find the number of terms in the series:

$$\begin{aligned} \frac{1}{729} &= \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} \\ \frac{1}{729} &= \left(\frac{1}{3}\right)^n \end{aligned}$$

Because $3^6 = 729$, we have $n = 6$. (Of course, you could use logarithms to solve for n.) Now use the formula for the sum of six terms of this geometric series:

$$S_6 = \frac{\frac{1}{3}\left[1 - \left(\frac{1}{3}\right)^6\right]}{1 - \frac{1}{3}} = \frac{\frac{1}{3}\left[1 - \frac{1}{729}\right]}{\frac{2}{3}}$$

$$= \frac{1}{3} \cdot \frac{728}{729} \cdot \frac{3}{2}$$

$$= \frac{364}{729}$$

EXAMPLE 6 **The sum of a finite geometric series**

Find the sum of the series

$$\sum_{i=1}^{12} 3(-2)^{i-1}.$$

Solution

This series is geometric with first term 3, ratio -2, and $n = 12$. We use the formula for the sum of the first 12 terms of a geometric series:

$$S_{12} = \frac{3[1 - (-2)^{12}]}{1 - (-2)} = \frac{3[-4095]}{3} = -4095$$

Infinite Geometric Series

calculator close-up

Experiment with your calculator to see what happens to r^n as n gets larger and larger.

```
.99^100
          .3660323413
.99^1000
       4.317124741E-5
.99^10000
       2.24877485E-44
```

Consider how a very large value of n affects the formula for the sum of a finite geometric series,

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

If $|r| < 1$, then the value of r^n gets closer and closer to 0 as n gets larger and larger. For example, if $r = \frac{2}{3}$ and $n = 10$, 20, and 100, then

$$\left(\frac{2}{3}\right)^{10} \approx 0.0173415, \quad \left(\frac{2}{3}\right)^{20} \approx 0.0003007, \quad \text{and} \quad \left(\frac{2}{3}\right)^{100} \approx 2.460 \times 10^{-18}.$$

Because r^n is approximately 0 for large values of n, $1 - r^n$ is approximately 1. If we replace $1 - r^n$ by 1 in the expression for S_n, we get

$$S_n \approx \frac{a_1}{1 - r}.$$

So as n gets larger and larger, the sum of the first n terms of the infinite geometric series

$$a_1 + a_1r + a_1r^2 + \cdots$$

gets closer and closer to $\frac{a_1}{1 - r}$, provided that $|r| < 1$. Therefore we say that $\frac{a_1}{1 - r}$ is the sum of *all* of the terms of the infinite geometric series.

Sum of an Infinite Geometric Series

If $a_1 + a_1r + a_1r^2 + \cdots$ is an infinite geometric series, with $|r| < 1$, then the sum S of all of the terms of this series is given by

$$S = \frac{a_1}{1 - r}.$$

EXAMPLE 7 **Sum of an infinite geometric series**

Find the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

helpful hint

You can imagine this series in a football game. The Bears have the ball on the Lions' 1-yard line. The Lions continually get penalties that move the ball one-half of the distance to the goal. Theoretically, the ball will never reach the goal, but the total distance it moves will get closer and closer to 1 yard.

Solution

This series is an infinite geometric series with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. Because $r < 1$, we have

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

For an infinite series the index of summation i takes the values 1, 2, 3, and so on, without end. To indicate that the values for i keep increasing without bound, we say that *i takes the values from* 1 *through* ∞ (infinity). Note that the symbol "∞" does not represent a number. Using the ∞ symbol, we can write the indicated sum of an infinite geometric series (with $|r| < 1$) by using summation notation as follows:

$$a_1 + a_1r + a_1r^2 + \cdots = \sum_{i=1}^{\infty} a_1 r^{i-1}$$

EXAMPLE 8 **Sum of an infinite geometric series**

Find the value of the sum

$$\sum_{i=1}^{\infty} 8\left(\frac{3}{4}\right)^{i-1}.$$

Solution

This series is an infinite geometric series with first term 8 and ratio $\frac{3}{4}$. So

$$S = \frac{8}{1 - \frac{3}{4}} = 8 \cdot \frac{4}{1} = 32.$$

EXAMPLE 9 **Follow the bouncing ball**

Suppose a ball always rebounds $\frac{2}{3}$ of the height from which it falls and the ball is dropped from a height of 6 feet. Find the total distance that the ball travels.

Solution

The ball falls 6 feet (ft) and rebounds 4 ft, then falls 4 ft and rebounds $\frac{8}{3}$ ft. The following series gives the total distance that the ball falls:

$$F = 6 + 4 + \frac{8}{3} + \frac{16}{9} + \cdots$$

The distance that the ball rebounds is given by the following series:

$$R = 4 + \frac{8}{3} + \frac{16}{9} + \cdots$$

Each of these series is an infinite geometric series with ratio $\frac{2}{3}$. Use the formula for an infinite geometric series to find each sum:

$$F = \frac{6}{1 - \frac{2}{3}} = 6 \cdot \frac{3}{1} = 18 \text{ ft}, \qquad R = \frac{4}{1 - \frac{2}{3}} = 4 \cdot \frac{3}{1} = 12 \text{ ft}$$

The total distance traveled by the ball is the sum of F and R, 30 ft. ■

Annuities

One of the most important applications of geometric series is in calculating the value of an annuity. An **annuity** is a sequence of periodic payments. The payments might be loan payments or investments.

EXAMPLE 10 **Value of an annuity**

A deposit of \$1000 is made at the beginning of each year for 30 years and earns 6% interest compounded annually. What is the value of this annuity at the end of the thirtieth year?

Solution

The last deposit earns interest for only one year. So at the end of the thirtieth year it amounts to \$1000(1.06). The next to last deposit earns interest for 2 years and amounts to $\$1000(1.06)^2$. The first deposit earns interest for 30 years and amounts to $\$1000(1.06)^{30}$. So the value of the annuity at the end of the thirtieth year is the sum of the finite geometric series

$$1000(1.06) + 1000(1.06)^2 + 1000(1.06)^3 + \cdots + 1000(1.06)^{30}.$$

Use the formula for the sum of 30 terms of a finite geometric series with $a_1 = 1000(1.06)$ and $r = 1.06$:

$$S_{30} = \frac{1000(1.06)(1 - (1.06)^{30})}{1 - 1.06} = \$83{,}801.68$$

So 30 annual deposits of \$1000 each amount to \$83,801.68. ■

WARM-UPS

True or false? Explain your answer.

1. The sequence 2, 6, 24, 120, . . . is a geometric sequence.
2. For $a_n = 2^n$ there is a common difference between adjacent terms.
3. The common ratio for the geometric sequence $a_n = 3(0.5)^{n-1}$ is 0.5.
4. If $a_n = 3(2)^{-n+3}$, then $a_1 = 12$.
5. In the geometric sequence $a_n = 3(2)^{-n+3}$ we have $r = \frac{1}{2}$.
6. The terms of a geometric series are the terms of a geometric sequence.

WARM-UPS

(continued)

7. To evaluate $\sum_{i=1}^{10} 2^i$, we must list all of the terms.

8. $\sum_{i=1}^{5} 6\left(\frac{3}{4}\right)^{i-1} = \dfrac{9\left[1 - \left(\frac{3}{4}\right)^5\right]}{1 - \frac{3}{4}}$

9. $10 + 5 + \frac{5}{2} + \cdots = \dfrac{10}{1 - \frac{1}{2}}$

10. $2 + 4 + 8 + 16 + \cdots = \dfrac{2}{1 - 2}$

14.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a geometric sequence?

2. What is the nth term of a geometric sequence?

3. What is a geometric series?

4. What is the formula for the sum of the first n terms of a geometric series?

5. What is the approximate value of r^n when n is large and $|r| < 1$?

6. What is the formula for the sum of an infinite geometric series?

Write a formula for the nth term of each geometric sequence. See Examples 1 and 2.

7. $\frac{1}{3}, 1, 3, 9, \ldots$

8. $\frac{1}{4}, 2, 16, \ldots$

9. $64, 8, 1, \ldots$

10. $100, 10, 1, \ldots$

11. $8, -4, 2, -1, \ldots$

12. $-9, 3, -1, \ldots$

13. $2, -4, 8, -16, \ldots$

14. $-\frac{1}{2}, 2, -8, 32, \ldots$

15. $-\frac{1}{3}, -\frac{1}{4}, -\frac{3}{16}, \ldots$

16. $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}, \ldots$

Write the first five terms of the geometric sequence with the given nth term. See Example 3.

17. $a_n = 2\left(\frac{1}{3}\right)^{n-1}$

18. $a_n = -5\left(\frac{1}{2}\right)^{n-1}$

19. $a_n = (-2)^{n-1}$

20. $a_n = \left(-\frac{1}{3}\right)^{n-1}$

21. $a_n = 2^{-n}$

22. $a_n = 3^{-n}$

23. $a_n = (0.78)^n$

24. $a_n = (-0.23)^n$

Find the required part of each geometric sequence. See Example 4.

25. Find the first term of the geometric sequence that has fourth term 40 and common ratio 2.

26. Find the first term of the geometric sequence that has fifth term 4 and common ratio $\frac{1}{2}$.

27. Find r for the geometric sequence that has $a_1 = 6$ and $a_4 = \frac{2}{9}$.

28. Find r for the geometric sequence that has $a_1 = 1$ and $a_4 = -27$.

29. Find a_4 for the geometric sequence that has $a_1 = -3$ and $r = \frac{1}{3}$.

30. Find a_5 for the geometric sequence that has $a_1 = -\frac{2}{3}$ and $r = -\frac{2}{3}$.

Find the sum of each geometric series. See Examples 5 and 6.

31. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512}$

32. $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{81}$

33. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}$

34. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81}$

35. $30 + 20 + \frac{40}{3} + \cdots + \frac{1280}{729}$

36. $9 - 6 + 4 - \cdots - \frac{128}{243}$

37. $\sum_{i=1}^{10} 5(2)^{i-1}$

38. $\sum_{i=1}^{7} (10{,}000)(0.1)^{i-1}$

39. $\sum_{i=1}^{6} (0.1)^i$

40. $\sum_{i=1}^{5} (0.2)^i$

41. $\sum_{i=1}^{6} 100(0.3)^i$

42. $\sum_{i=1}^{7} 36(0.5)^i$

Find the sum of each infinite geometric series. See Examples 7 and 8.

43. $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

44. $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

45. $3 + 2 + \frac{4}{3} + \cdots$

46. $2 + 1 + \frac{1}{2} + \cdots$

47. $4 - 2 + 1 - \frac{1}{2} + \cdots$

48. $16 - 12 + 9 - \frac{27}{4} + \cdots$

49. $\sum_{i=1}^{\infty} (0.3)^i$

50. $\sum_{i=1}^{\infty} (0.2)^i$

51. $\sum_{i=1}^{\infty} 3(0.5)^{i-1}$

52. $\sum_{i=1}^{\infty} 7(0.4)^{i-1}$

53. $\sum_{i=1}^{\infty} 3(0.1)^i$

54. $\sum_{i=1}^{\infty} 6(0.1)^i$

55. $\sum_{i=1}^{\infty} 12(0.01)^i$

56. $\sum_{i=1}^{\infty} 72(0.01)^i$

Use the ideas of geometric series to solve each problem. See Examples 9 and 10.

57. ***Retirement fund.*** Suppose a deposit of \$2000 is made at the beginning of each year for 45 years into an account paying 12% compounded annually. What is the amount of this annuity at the end of the forty-fifth year?

58. ***World's largest mutual fund.*** If you had invested \$5000 at the beginning of each year for the last 10 years in Fidelity's Magellan fund you would have earned 18.97% compounded annually (Fidelity Investments, www.fidelity.com). Find the amount of this annuity at the end of the tenth year.

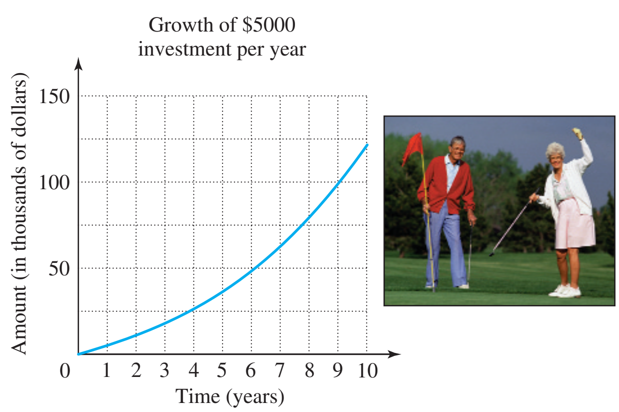

FIGURE FOR EXERCISE 58

59. ***Big saver.*** Suppose you deposit one cent into your piggy bank on the first day of December and, on each day of December after that, you deposit twice as much as on the previous day. How much will you have in the bank after the last deposit?

60. ***Big family.*** Consider yourself, your parents, your grandparents, your great-grandparents, your great-great-grandparents, and so on, back to your grandparents with the word "great" used in front 40 times. What is the total number of people you are considering?

61. ***Total economic impact.*** In Exercise 43 of Section 14.1 we described a factory that spends \$1 million annually in a community in which 80% of the money received is respent in the community. Economists assume the money is respent again and again at the 80% rate. The total economic impact of the factory is the total of all of this spending. Find an approximation for the total by using the formula for the sum of an infinite geometric series with a rate of 80%.

62. ***Less impact.*** Repeat Exercise 61, assuming money is respent again and again at the 50% rate.

GETTING MORE INVOLVED

63. ***Discussion.*** Which of the following sequences is not a geometric sequence? Explain your answer.

a) 1, 2, 4, . . .

b) 0.1, 0.01, 0.001, . . .

c) $-1, 2, -4, \ldots$

d) 2, 4, 6, . . .

64. ***Discussion.*** The repeating decimal number 0.44444 . . . can be written as

$$\frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \cdots,$$

an infinite geometric series. Find the sum of this geometric series.

65. ***Discussion.*** Write the repeating decimal number 0.24242424 . . . as an infinite geometric series. Find the sum of the geometric series.

14.5 BINOMIAL EXPANSIONS

In this section

- Some Examples
- Obtaining the Coefficients
- The Binomial Theorem

In Chapter 5 you learned how to square a binomial. In this section you will study higher powers of binomials.

Some Examples

We know that $(x + y)^2 = x^2 + 2xy + y^2$. To find $(x + y)^3$, we multiply $(x + y)^2$ by $x + y$:

$$\begin{aligned}(x + y)^3 &= (x^2 + 2xy + y^2)(x + y)\\ &= (x^2 + 2xy + y^2)x + (x^2 + 2xy + y^2)y\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

The sum $x^3 + 3x^2y + 3xy^2 + y^3$ is called the **binomial expansion** of $(x + y)^3$. If we again multiply by $x + y$, we will get the binomial expansion of $(x + y)^4$. This method is rather tedious. However, if we examine these expansions, we can find a pattern and learn how to find binomial expansions without multiplying.