NEED FOR WEIGHTED-INTEGRAL FORMULATIONS

IN THE NUMERICAL SOLUTION OF DIFF. EQUATIONS

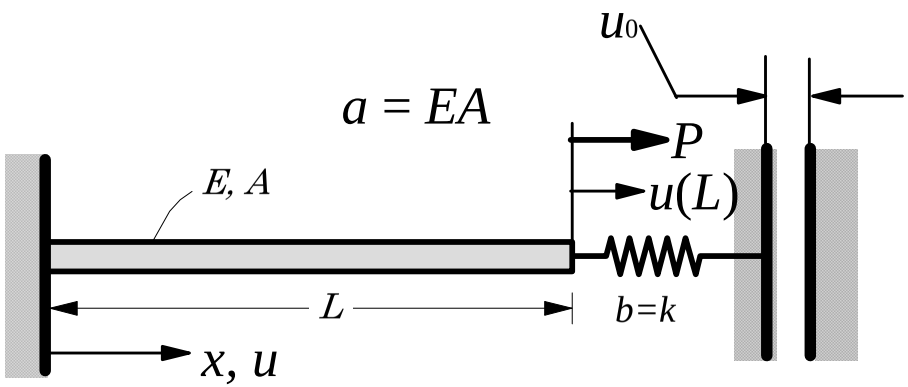
Discuss approximate solution of a differential equation using a variety of methods and thus introduce various traditional variational methods of approximation.

- Collocation method
- Least-squares method
- Weighted-residual methods
  - ♦ Petrov-Galerkin method
  - ♦ Galerkin Method

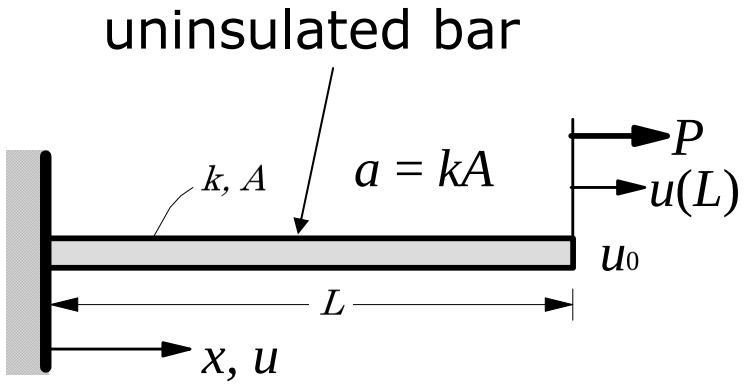
GOVERNING EQUATION

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$

$$a\frac{du}{dx}+b(u-u_0)=P \text{ at a boundary point}$$

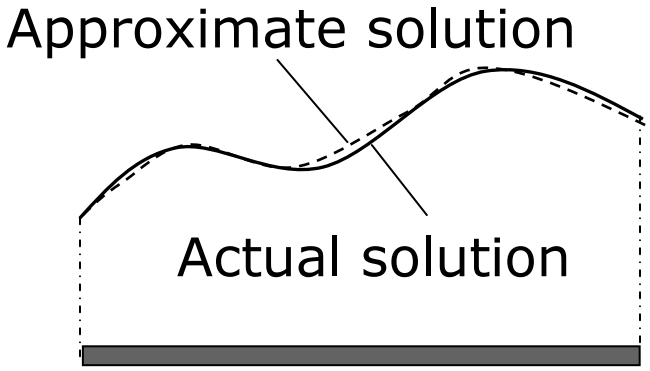


Elastic deformation of a bar



Heat transfer in a bar

APPROXIMATE SOLUTION



Approximation of the actual solution over the entire domain

An approximate solution must satisfy the differential equation as well as the boundary conditions in some "acceptable sense" (to be made clearer shortly).

Often, we seek the approximate solution as a linear combination of unknown parameters c_i and known functions ϕ_i

$$u(x) \approx U(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

DETERMINING APPROXIMATE SOLUTIONS

Note: The approximate solution must contain unknowns to be determined subject to the satisfaction of the differential equation or boundary conditions. Otherwise, there is nothing to be determined (It is impossible to select a solution, in general, that satisfies all the requirements).

Since $U(x)$ is an approximate solution, it will not satisfy either the differential equation or the boundary conditions exactly.

If $\phi_i(x)$ is selected to satisfy the differential equation exactly, then c_i are determined such that the boundary conditions are satisfied in some sense. The method is known as the *Trefftz method.*

If $\phi_i(x)$ is selected to satisfy the boundary conditions exactly, then c_i are determined such that the differential equation is satisfied in some sense. Most traditional variational methods use this approach, which is discussed next.

DETERMINING APPROXIMATE SOLUTIONS
(continued)

Suppose that $\phi_i(x)$ is selected to satisfy the boundary conditions exactly. Then substitution of $U(x)$ into the differential equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0$$

will result in a non-zero function on the left side of the equality.

$$-\frac{d}{dx}\left(a(x)\frac{dU}{dx}\right)+c(x)U-f(x)\equiv R(x)\neq 0$$

Then c_i are determined such that the residual, $R(x)$, is zero in some sense.

DETERMINING APPROXIMATE SOLUTIONS
(continued)

1. One sense in which the residual, $R(x)$, can be made zero is to require it to be zero at selected number of points. The number of points should be equal to the number of unknowns in the approximate solution

$$u(x) \approx U(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

This way of determining c_i is known as the *Collocation method*. We obtain N algebraic equations in N unknown C's

$$R(x_i) = 0, \quad i = 1, 2, \cdots, N$$

DETERMINING APPROXIMATE SOLUTIONS
(continued)

2. Another approach in which the residual, $R(x)$, can be made zero is in a least-squares sense; i.e., minimize the integral of the square of the residual with respect to C's.

$$\text{Minimize } J(c_1, c_2, \cdots, c_N) = \int_0^L R^2 \, dx$$

$$\text{or } \frac{\partial J}{\partial c_i} = 2 \int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

This method is known as the *Least-Squares method*. We obtain N algebraic equations in N unknown C's

$$\int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

DETERMINING APPROXIMATE SOLUTIONS
(continued)

3. Yet, another approach in which the residual, $R(x)$, can be made zero is in a weighted-residual sense

$$0 = \int_0^L \psi_i R\, dx, \;\; i = 1, 2, \cdots, N$$

where ψ_i are linearly independent set of functions

This method is known as the *Weighted-Residual method*. We obtain N algebraic equations in N unknown C's. In general $\boxed{\psi_i}$ are not the same as the approximation functions $\boxed{\phi_i}$

$$\text{Petrov}-\text{Galerkin Method:} \quad \psi_i \neq \phi_i$$

$$\text{Galerkin Method:} \quad \psi_i = \phi_i$$

DETERMINING APPROXIMATE SOLUTIONS
(continued)

4. The *Ritz method* is one in which an integral expression that is equivalent to the differential equation and natural boundary condition is minimized (because of a physical principle, such as the minimum of a total potential energy). For most problems of this course, the quadratic functional to be minimized is constructed from the so-called weak form.

$$I(u)=\tfrac{1}{2}\int_0^L\left[a\left(\frac{du}{dx}\right)^2+cu^2\right]dx-\int_0^L fu\,dx+b\left\{\tfrac{1}{2}\left[u(L)\right]^2-u(L)u_0\right\}$$

$$\frac{\partial I(U)}{\partial c_i}=0,\ \ i=1,2,\cdots,N$$

SUMMARY

We seek approximate solution in the form of a linear combination of known functions that satisfy the boundary conditions and uknown parameters that are determined to satisfy the differential equation in a some sense.

There are several ways one can make the residual (or error) in the approximation of the differential equation to zero. If the error is zero at every point of the domain, then we obtain the exact solution. If the error is made zero at selected points (N), it is the *Collocation Method*. The residual can be made zero in a least-squares sense or weighted-integral sense. These various ways of making the residual zero are known by different names, as discussed.

TERMINOLOGY

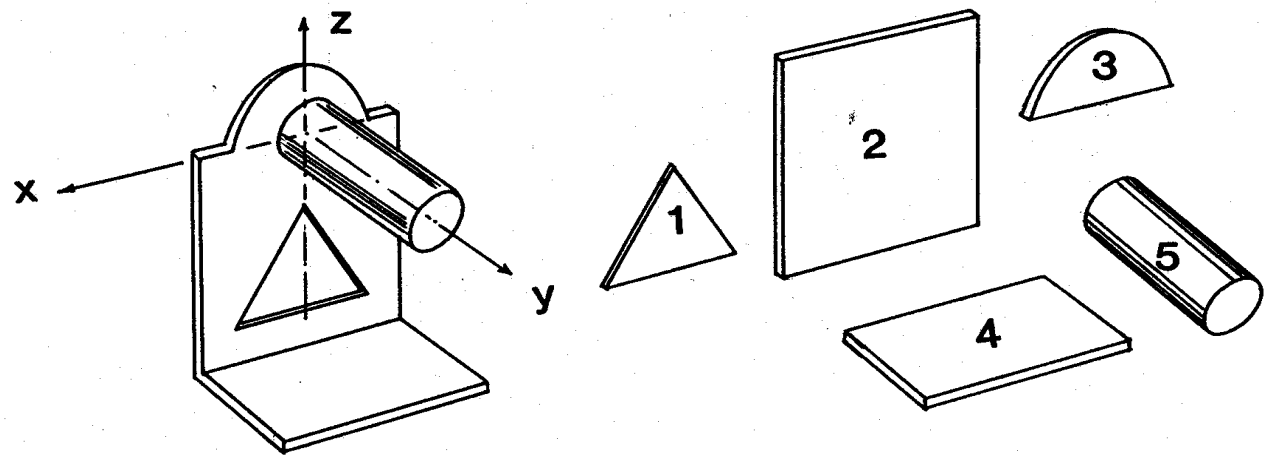
- *Mathematical Model* A set of algebraic, differential, and/or integral equations that govern the physical phenomenon of a particular system. The model is based on a set of assumptions and restrictions placed on the phenomenon and the laws of physics that govern it.
- *Numerical Method* An inexact procedure by which the governing equations can be solved for the dependent (unknowns) variables.
- *Computer* An electronic device that can be used to evaluate mathematical expressions and compute numbers using elementary operations of addition, multiplication, and division.

THREE BASIC IDEAS OF FEM

- Divide whole into parts (*finite element mesh*)
- Set up the `problem' over a typical part
  (derive a set of relationships between primary and secondary variables)
- Assemble the parts to obtain the solution to the whole

EXAMPLE 1: Determine the center of mass of a 3D machine part

THE BASIC IDEA OF THE FINITE ELEMENT METHOD



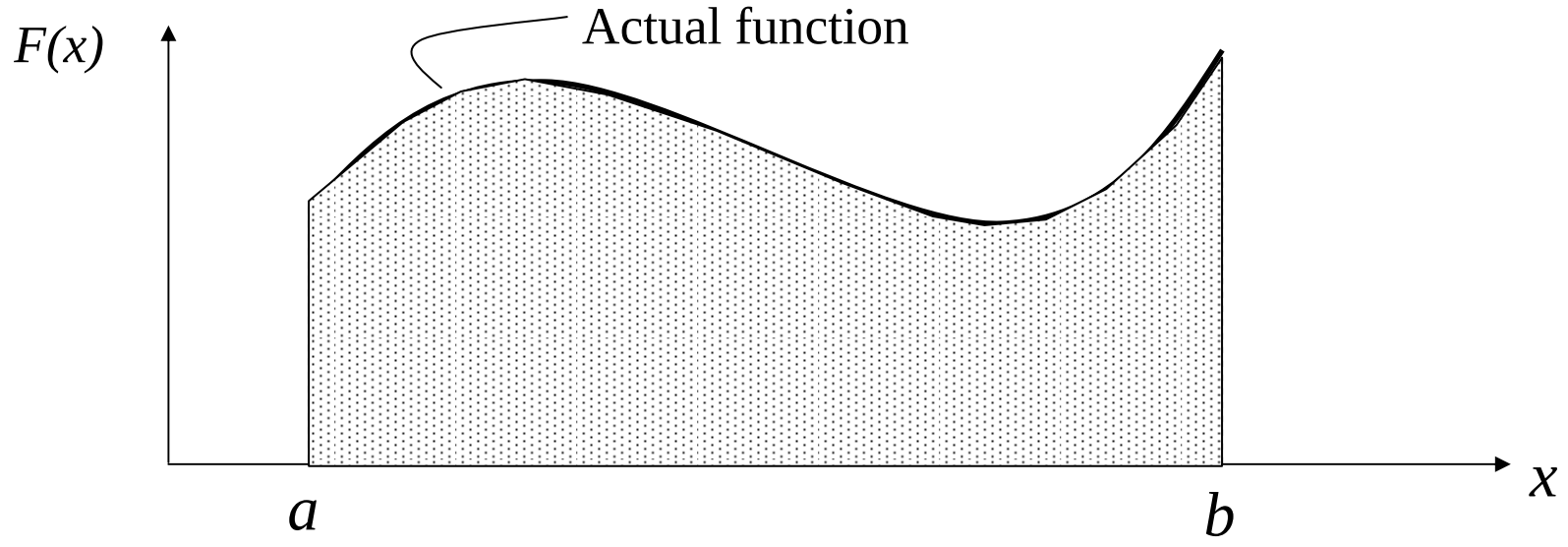
1. see simplicity in complicated and divide whole into parts

2. use a principle to set up the equations

$$X = \frac{\sum m\bar{x}}{\sum m} \qquad Y = \frac{\sum m\bar{y}}{\sum m} \qquad Z = \frac{\sum m\bar{z}}{\sum m}$$

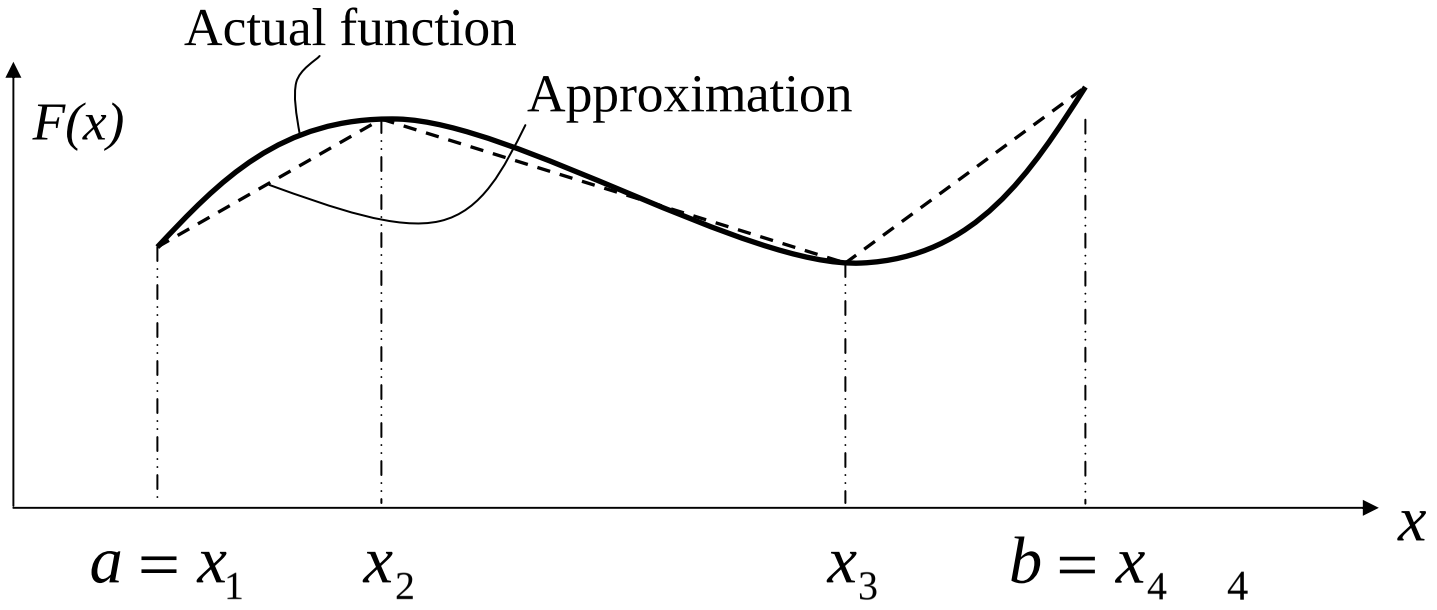
EXAMPLE 2: Determine the integral of a function

$$I = \int_a^b F(x)\,dx$$



EXAMPLE 2 (continued)

$$F(x) \approx \begin{cases} a_1 + b_1 x, & a \le x \le x_2 \\ a_2 + b_2 x, & x_2 \le x \le x_3 \\ a_3 + b_3 x, & x_3 \le x \le b \end{cases}$$



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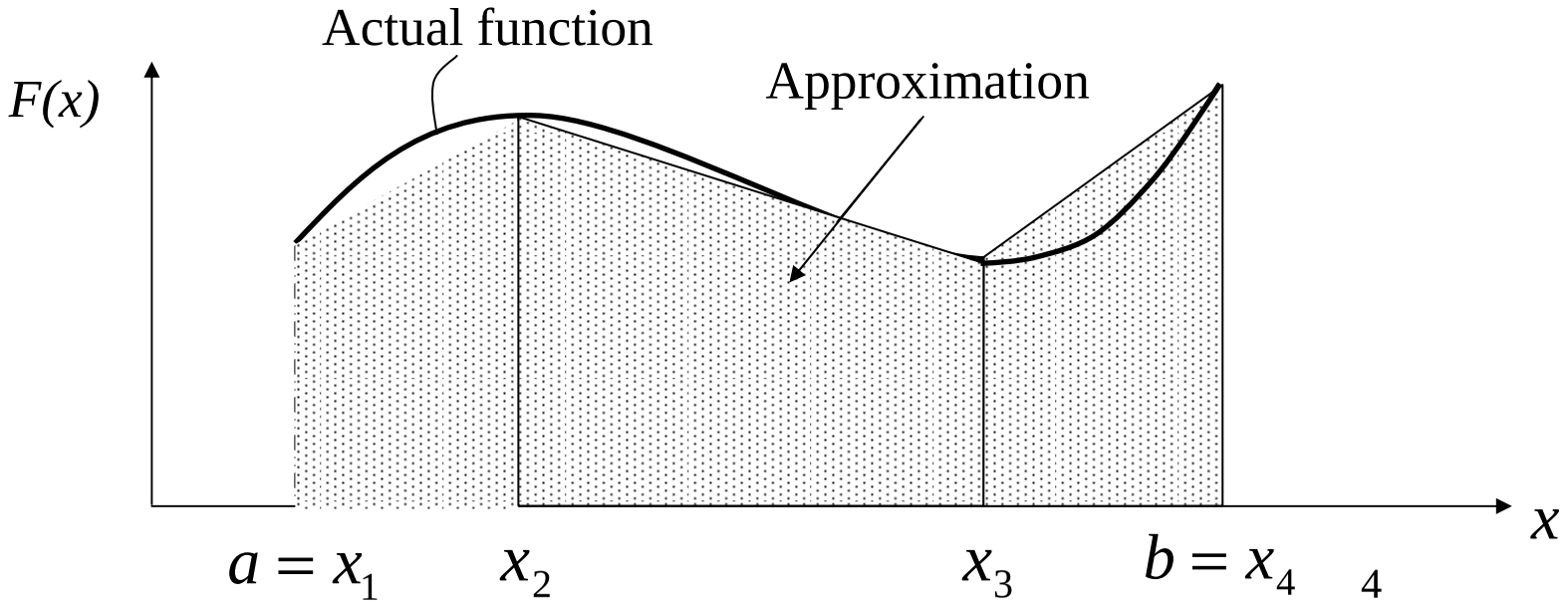
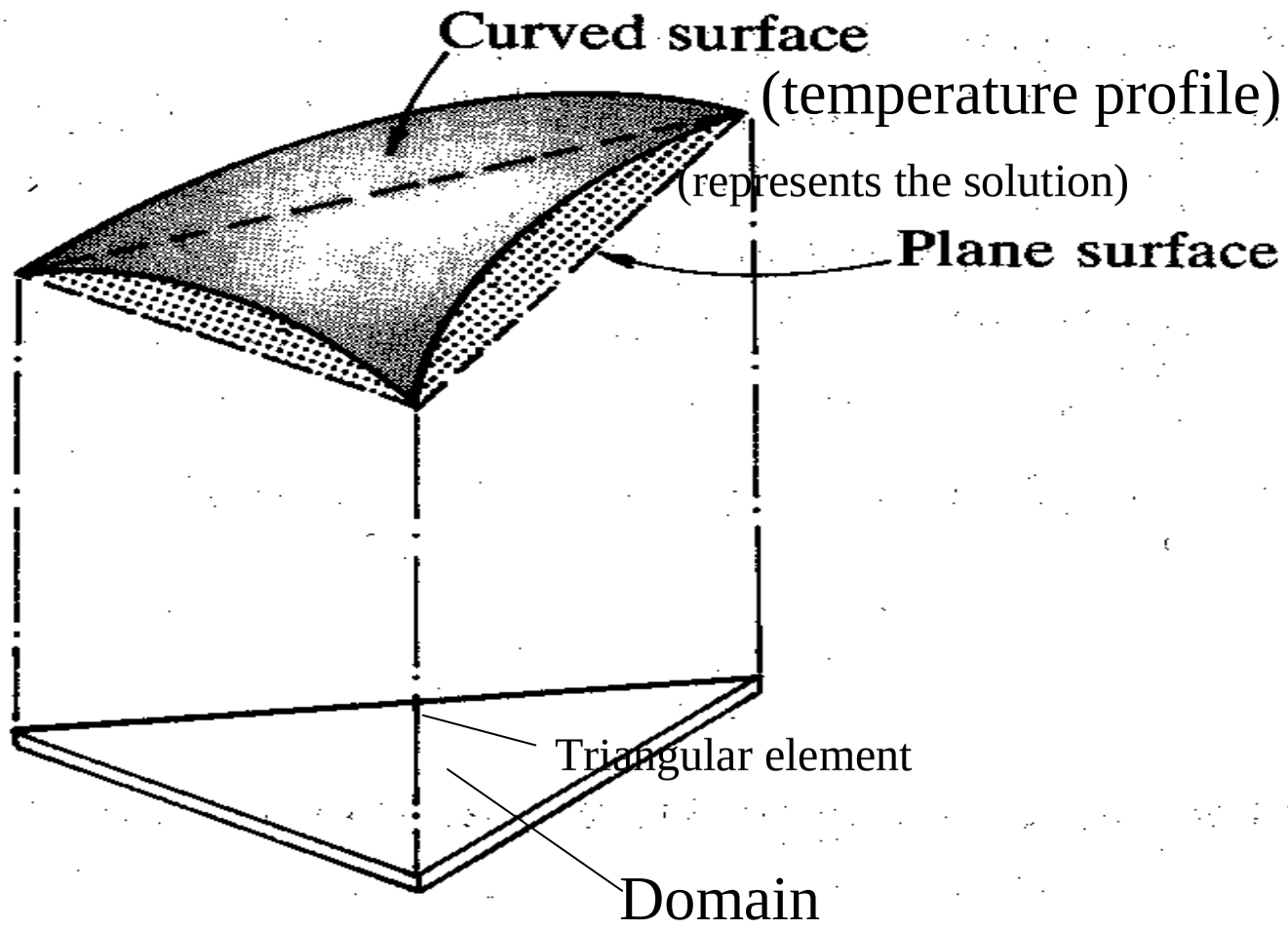
$$I_i = \int_{x_i}^{x_{i+1}} F_i(x)\,dx = \int_{x_i}^{x_{i+1}} (a_i + b_i x)\,dx$$

$$I \approx I_1 + I_2 + I_3$$



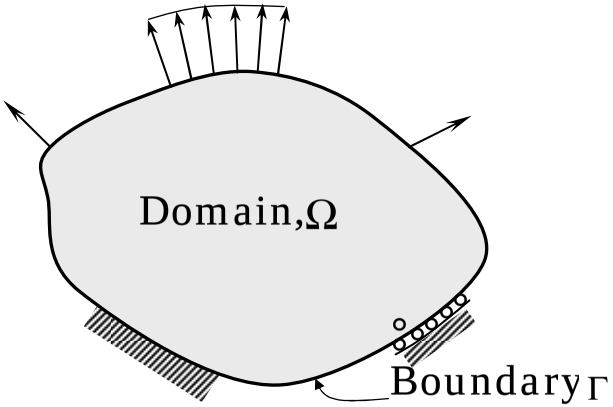
Approximation of a curved surface with a plane



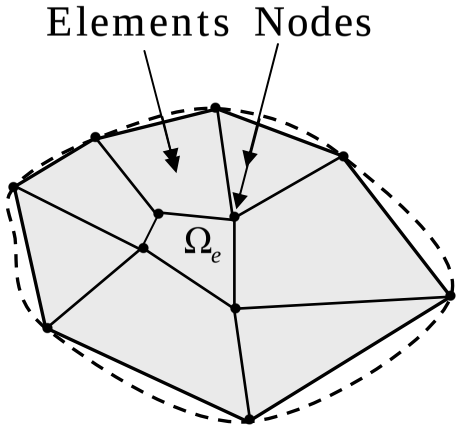
FEM TERMINOLOGY

- *Element* A geometric sub-domain of the region being simulated, with the property that it allows a unique derivation of the approximation (interpolation) functions.
- *Node* A geometric location in the element which plays a role in the derivation of the interpolation functions and it is the point at which solution is sought.
- *Mesh* A collection of elements (or nodes) that replaces the actual domain.
- *Weak Form* An integral statement equivalent to the governing equations and *natural* boundary conditions.

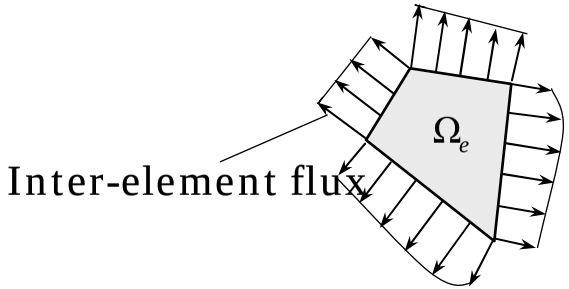
FINITE ELEMENT DISCRETIZATION



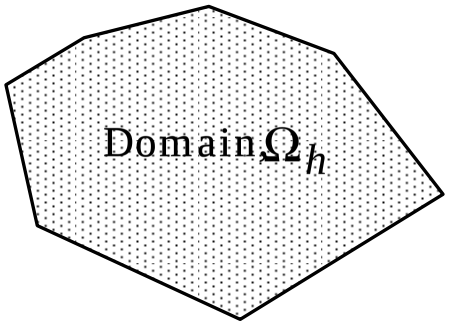
(*a*) Given domain



(*b*) Finite element mesh



(*c*) Typical element with boundary fluxes



(*d*) Discretized domain

FEM TERMINOLOGY

- *Finite Element Model* A set of algebraic equations relating the nodal values of the primary variables (e.g., displacements) to the nodal values of the secondary variables (e.g., forces) in an element.
- *Finite element model* is NOT the same as the *finite element method.* There is only one finite element method but there can be more than one finite element model of a problem (or mathematical model).
- *Numerical Simulation* Evaluation of the mathematical model (i.e., solution of the governing equations) using a numerical method and computer.

MAJOR STEPS OF FINITE ELEMENT MODEL DEVELOPMENT

- Begin with the *governing equations* of the problem
- Develop its *weak form* (weighted-integral statement) *over a typical element*
- *Approximate* the solution over each finite element
- Obtain relations among the *quantities of interest* over each finite element