

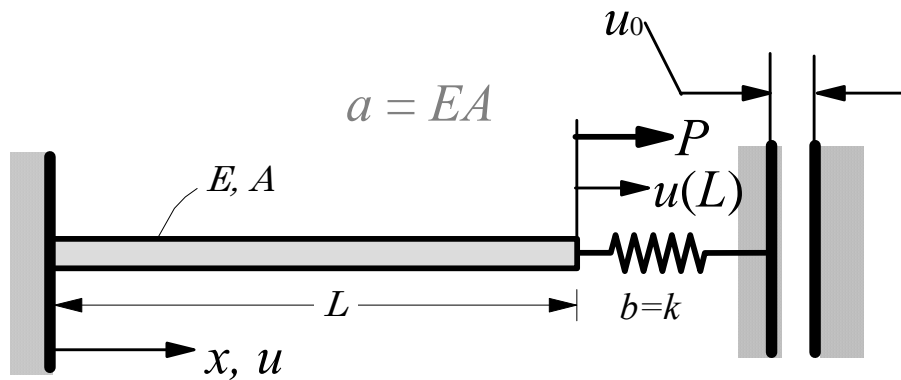
FINITE ELEMENT ANALYSIS OF A 1D MODEL PROBLEM WITH A SINGLE VARIABLE

Finite element model development of a linear 1D model differential equation involving a single dependent unknown (governing equations, FE model development weak form).

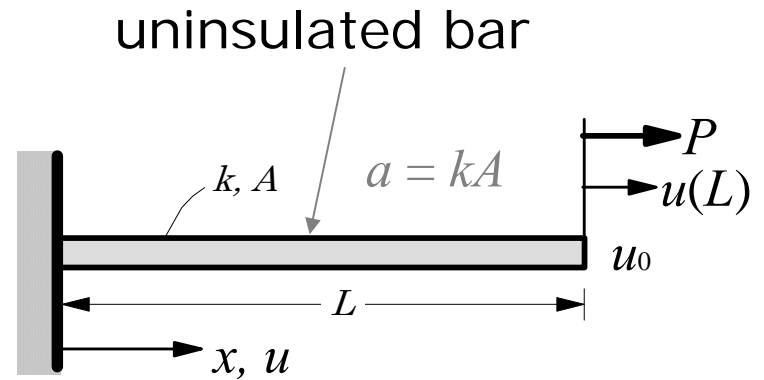
GOVERNING EQUATION

$$-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + c(x)u = f(x) \text{ in } \Omega = (0, L)$$

$$a \frac{du}{dx} + b(u - u_0) = P \text{ at a boundary point}$$



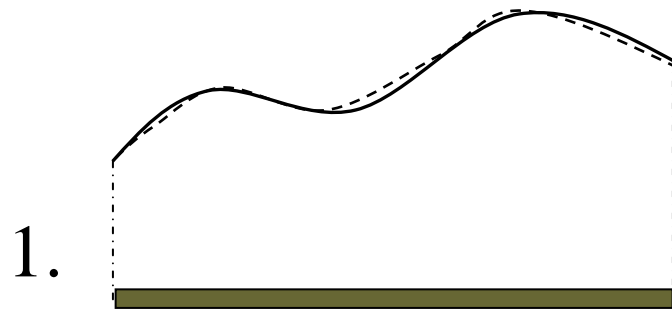
Elastic deformation of a bar



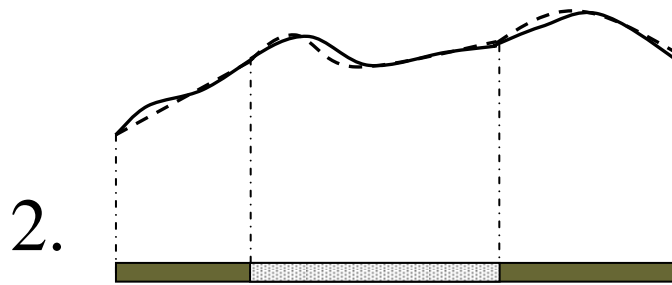
Heat transfer in a bar

FINITE ELEMENT APPROXIMATION

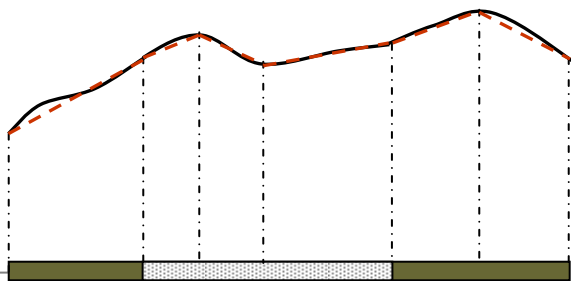
(NEED FOR SEEKING SOLUTION ON SUB-INTERVALS)



Approximation of the actual solution over the entire domain requires higher-order approximation

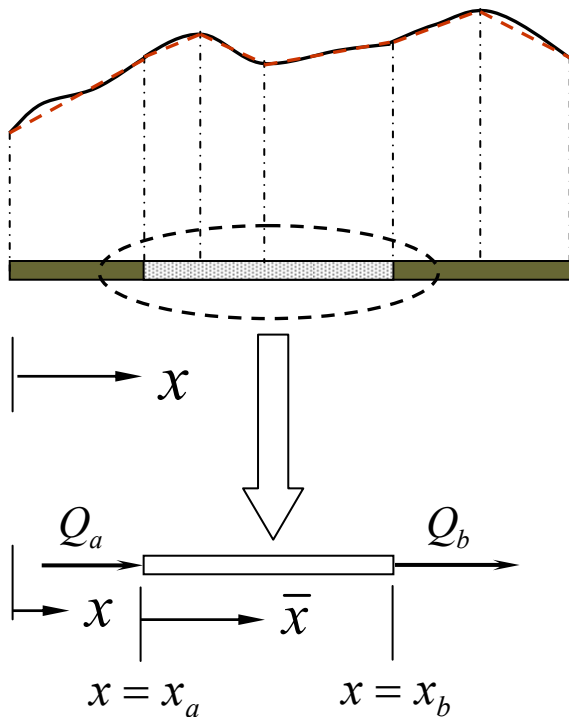


Actual solution may be defined by Sub-intervals because of discontinuity of the data $a(x)$.



Approximation over sub-intervals subintervals allows lower-order Approximation of the actual solution

FINITE ELEMENT DISCRETIZATION



Approximation over sub-intervals
subintervals allows lower-order
Approximation of the actual solution

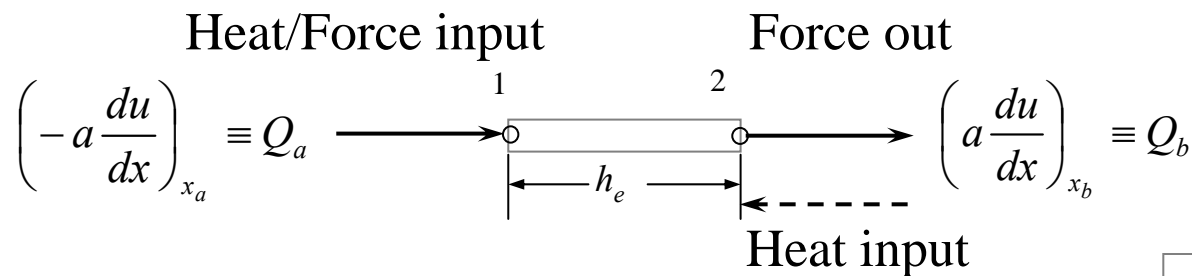
A typical element
(geometry and ‘forces’)

Q_a, Q_b end forces or heats

$h = x_b - x_a = \text{element length}$

WEAK FORM OVER AN ELEMENT

$$\begin{aligned}
 0 &= \int_{x_a}^{x_b} w \left[-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + c(x)u - f(x) \right] dx \\
 &= \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right] dx - \left[w \cdot a \frac{du}{dx} \right]_{x_a}^{x_b} \\
 &= \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right] dx - w(x_a) \cdot \left(-a \frac{du}{dx} \right)_{x_a} - w(x_b) \cdot \left(a \frac{du}{dx} \right)_{x_b} \\
 &= \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right] dx - w(x_a) Q_a - w(x_b) \cdot Q_b
 \end{aligned}$$



LINEAR AND BILINEAR FORMS AND THE VARIATIONAL PROBLEM

Weak Form

$$\begin{aligned} 0 &= \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right] dx - w(x_a)Q_a - w(x_b) \cdot Q_b \\ &= \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu \right] dx - \left[\int_{x_b}^{x_a} wf dx + w(x_a)Q_a + w(x_b) \cdot Q_b \right] \\ &= B(w,u) - l(w) \end{aligned}$$

Variational Problem: Find u such that

$$B(w,u) = l(w) \quad \text{holds for all } w$$

Bilinear Form and Linear Form

$$B(w,u) = \int_{x_a}^{x_b} \left[a \frac{dw}{dx} \frac{du}{dx} + cwu \right] dx, \quad l(w) = \left[\int_{x_b}^{x_a} wf dx + w(x_a)Q_a + w(x_b) \cdot Q_b \right]$$

EQUIVALENCE BETWEEN MINIMUM OF A QUADRATIC FUNCTIONAL AND WEAK FORM

Quadratic Functional: Strain energy Work done by applied forces

$$\begin{aligned}
 I(u) &\equiv \frac{1}{2} B(u,u) - l(u) \\
 &= \frac{1}{2} \int_{x_a}^{x_b} \left[a \left(\frac{du}{dx} \right)^2 + cu^2 \right] dx - \left[\int_{x_b}^{x_a} uf \, dx + u(x_a)Q_a + u(x_b) \cdot Q_b \right]
 \end{aligned}$$

Variational Problem: Find u such that $I(u)$ is a minimum:

$$\delta I = 0 \Rightarrow B(\delta u, u) - l(\delta u) = 0 \quad \text{for all } \delta u$$

which is the same as the weak form or the variational problem with $\delta u = w$

FINITE ELEMENT MODEL

Finite element approximation (to be derived later)

$$u(x) \approx U^e(x) = \sum_{j=1}^n u_j^e \psi_j^e(x)$$

Finite element model

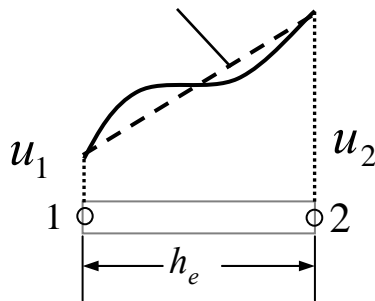
$$[K^e] \{u^e\} = \{F^e\}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c \psi_i \psi_j \right) dx,$$

$$F_i^e = \int_{x_a}^{x_b} f \psi_i dx + \psi_i(x_a) Q_a + \psi_i(x_b) Q_b$$

APPROXIMATION FUNCTIONS FOR LINEAR ELEMENT

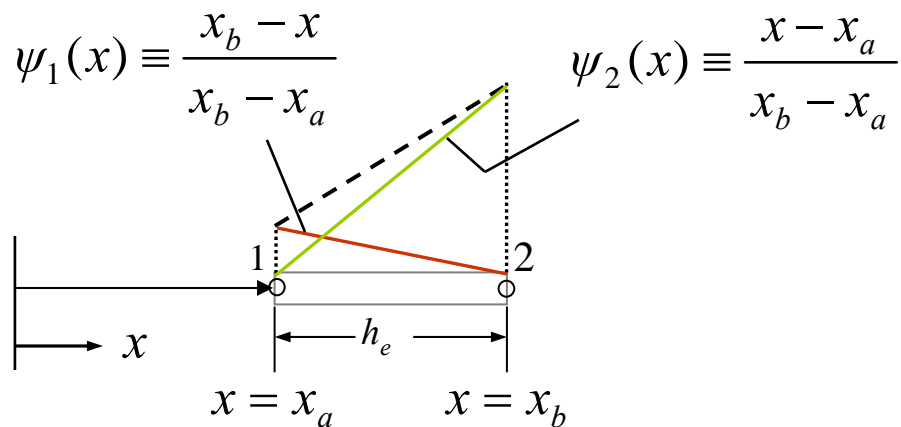
$$U(x) = c_1 + c_2x$$



$$u(x) \approx U(x) = c_1 + c_2x$$

$$U(x_b) \equiv u_b = u_2 \quad U(x_a) \equiv u_a = u_1$$

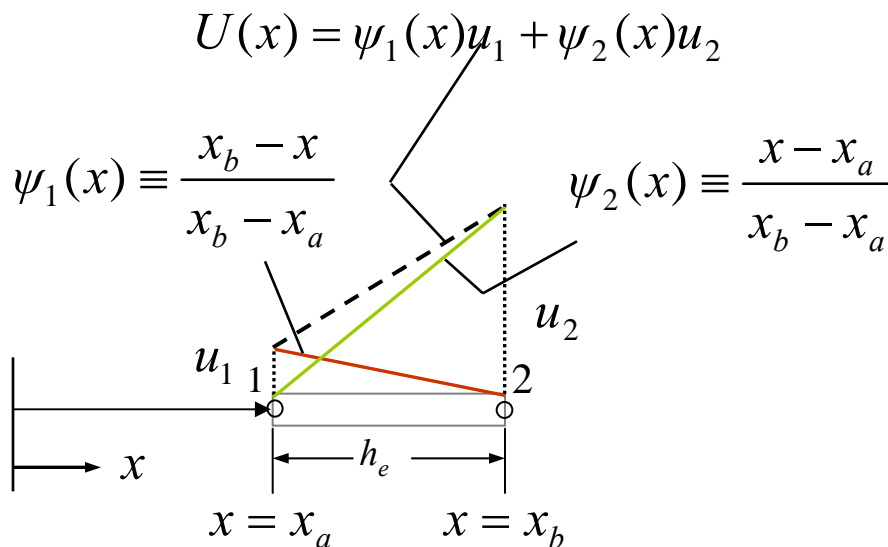
$$\begin{aligned} U(x_a) = u_1 = c_1 + c_2x_a \\ U(x_b) = u_2 = c_1 + c_2x_b \end{aligned} \rightarrow c_1 = \frac{u_1x_b - u_2x_a}{x_b - x_a}, \quad c_2 = \frac{u_2 - u_1}{x_b - x_a}$$



$$\begin{aligned} u(x) \approx U(x) &= c_1 + c_2x \\ &= \frac{u_1x_b - u_2x_a}{x_b - x_a} + \frac{u_2 - u_1}{x_b - x_a}x \\ &= \psi_1(x)u_1 + \psi_2(x)u_2 \end{aligned}$$

ALTERNATE DERIVATION OF APPROXIMATION FUNCTIONS (Linear Element)

$$\psi_i(x_j) \equiv \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad x_1 \equiv x_a, \quad x_2 \equiv x_b \quad (\text{interpolation functions})$$



Alternate derivation using the interpolation property

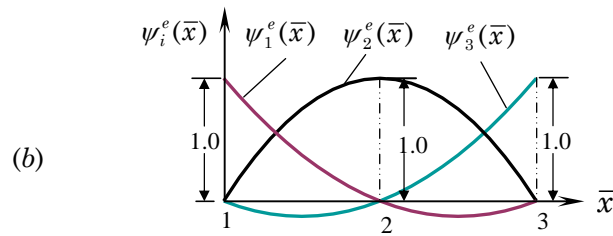
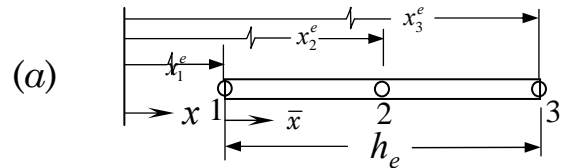
$$\psi_1(x) = \alpha_1(x_b - x)$$

$$\text{and } \psi_1(x_a) = 1 \rightarrow \alpha_1 = (x_b - x_a)^{-1}$$

$$\psi_2(x) = \alpha_2(x - x_a)$$

$$\text{and } \psi_2(x_b) = 1 \rightarrow \alpha_2 = (x_b - x_a)^{-1}$$

ALTERNATE DERIVATION OF APPROXIMATION FUNCTIONS (Quadratic Element)



Alternate derivation using the interpolation property

$$\psi_1(x) = \alpha_1(x_a + h - x)(x_a + 0.5h - x)$$

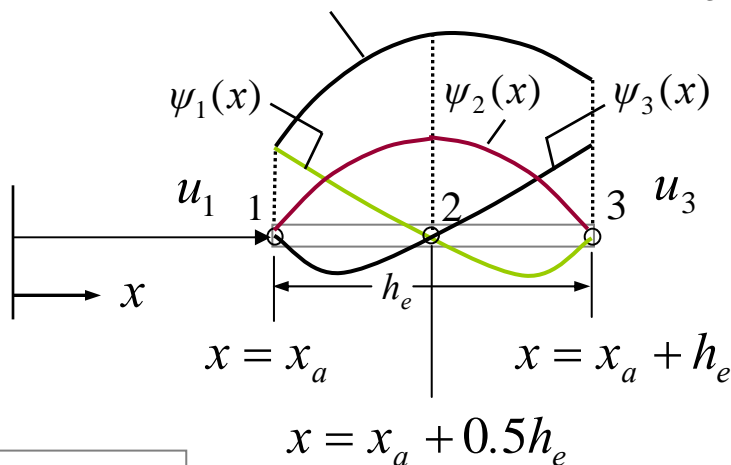
$$\text{and } \psi_1(x_a) = 1 \rightarrow \alpha_1 = \frac{2}{h^2}$$

$$U(x) = \psi_1(x)u_1 + \psi_2(x)u_2 + \psi_3(x)u_3 \quad \psi_2(x) = \alpha_2(x - x_a)(x_a + h - x)$$

$$\text{and } \psi_2(x_a + 0.5h) = 1 \rightarrow \alpha_2 = \frac{4}{h^2}$$

$$\psi_3(x) = \alpha_3(x - x_a)(x_a + 0.5h - x)$$

$$\text{and } \psi_3(x_a + h) = 1 \rightarrow \alpha_3 = -\frac{2}{h^2}$$

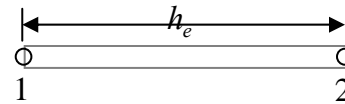


NUMERICAL EVALUATION OF COEFFICIENTS for element-wise constant data

For constant *data* : $a = a_e, c = c_e, f = f_e$

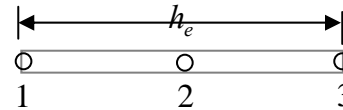
$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c \psi_i \psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f \psi_i dx + Q_i$$

Linear element:



$$[K^e] = \frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

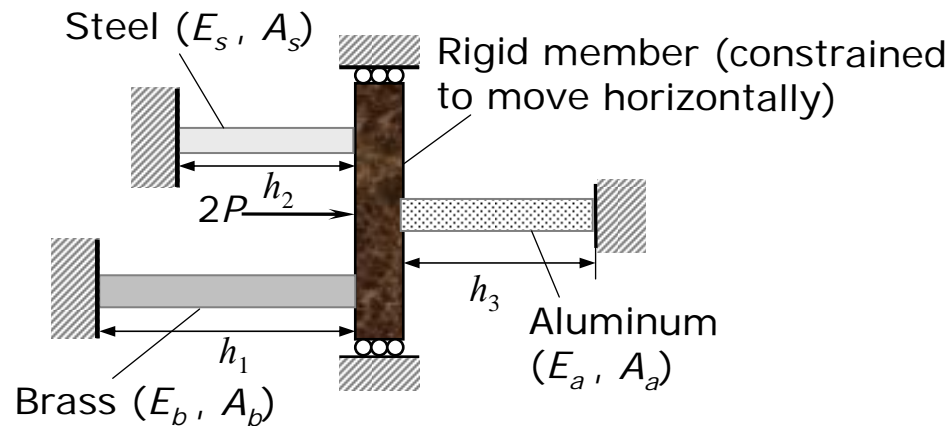
Quadratic element:



$$[K^e] = \frac{a_e}{3h_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

NUMERICAL EXAMPLE - 1

Problem: Wish to determine the deformation and stresses in the three members of the structure.



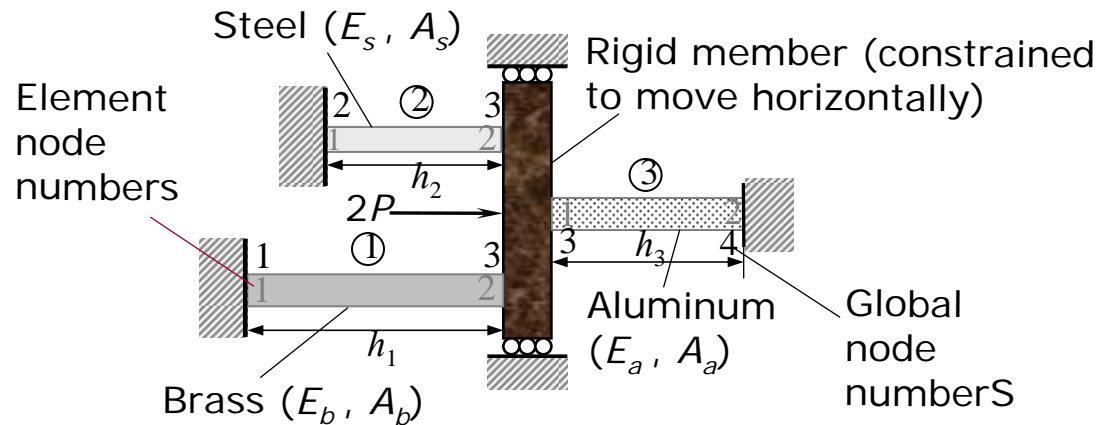
$$\begin{aligned}
 h_1 &= 8 \text{ ft.}, \quad h_2 = 4 \text{ ft.}, \quad h_3 = 6 \text{ ft.}, \\
 d_s &= 0.5 \text{ in.}, \quad d_b = 0.875 \text{ in.}, \\
 d_a &= 1.0 \text{ in.}, \quad E_s = 30 \times 10^6 \text{ psi}, \\
 E_b &= 14 \times 10^6 \text{ psi}, \quad E_a = 10 \times 10^6 \text{ psi}, \\
 P &= 3000 \text{ lb}
 \end{aligned}$$

Solution: (1) Note that the governing equation is

$$-\frac{d}{dx} \left(EA \frac{du}{dx} \right) = 0 \text{ in each element}$$

NUMERICAL EXAMPLE – 1 (continued)

- (2) We use linear elements to represent the members of the structure and label the elements, element nodes and global nodes.



- (3) The element equations for a typical element are
 $(a_e = E_e A_e, c_e = 0, f_e = 0)$

$$\frac{E_e A_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

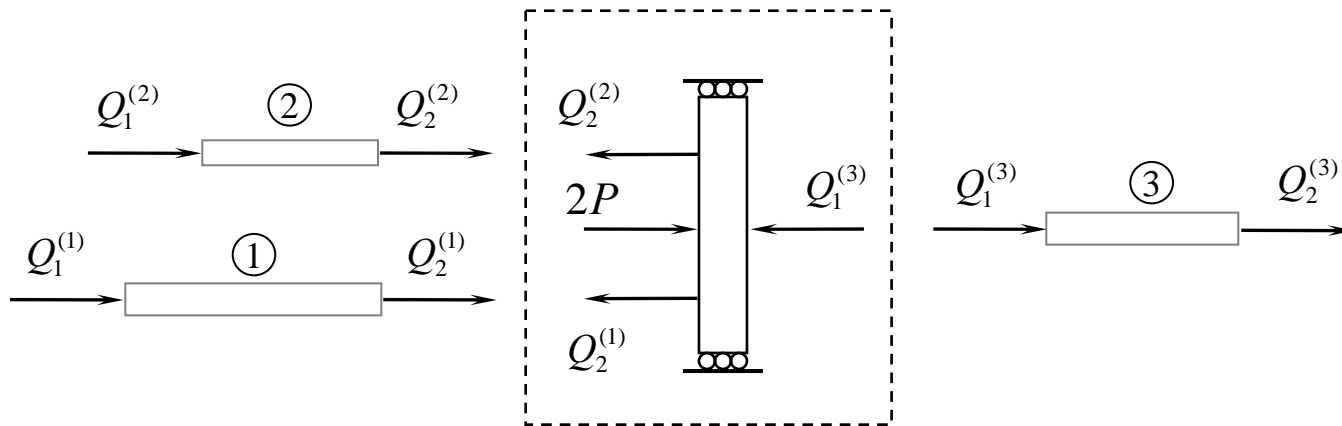
NUMERICAL EXAMPLE – 1 (continued)

(4) The displacement boundary conditions expressed in terms of the global displacements are

$$U_1 = 0, U_2 = 0, U_4 = 0$$

(5) The force equilibrium conditions are obtained by looking at the equilibrium of forces on the rigid bar. We have

$$-Q_2^{(1)} + 2P - Q_2^{(2)} - Q_1^{(3)} = 0 \rightarrow Q_2^{(1)} + Q_2^{(2)} + Q_1^{(3)} = 2P$$



NUMERICAL EXAMPLE – 1 (continued)

- (6) The equilibrium conditions suggest that we must add the second equation of element 1, the second equation of element 2, and the first equation of element 3 so that we can replace the sum of the Q 's with $2P$. The element equations in terms of the global displacements are

$$\frac{E_1 A_1}{h_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^1 = \cancel{U_1} \\ u_2^1 = U_3 \end{Bmatrix} \overset{0}{=} \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} \quad \frac{E_2 A_2}{h_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^2 = \cancel{U_2} \\ u_2^2 = U_3 \end{Bmatrix} \overset{0}{=} \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

$$\frac{E_3 A_3}{h_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^3 = U_3 \\ u_2^3 = \cancel{U_4} \end{Bmatrix} \overset{0}{=} \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

Using the fact that $U_1 = U_2 = U_4 = 0$, we obtain

$$\left(\frac{E_1 A_1}{h_1} + \frac{E_2 A_2}{h_2} + \frac{E_3 A_3}{h_3} \right) U_3 = Q_2^1 + Q_2^2 + Q_1^3 = 2P$$

NUMERICAL EXAMPLE – 1 (continued)

Thus we can compute U_3 , which represents the elongation in elements 1 and 2 and compression in element 3.

(7) We can then determine the member forces $Q_2^{(e)}$ using

$$Q_2^{(1)} = \frac{E_1 A_1}{h_1} U_3, \quad Q_2^{(2)} = \frac{E_2 A_2}{h_2} U_3, \quad Q_2^{(3)} = -\frac{E_3 A_3}{h_3} U_3$$

The member stresses are then can be computed from

$$\sigma^{(1)} = \frac{Q_2^{(1)}}{A_1} = \frac{E_1}{h_1} U_3, \quad \sigma^{(2)} = \frac{Q_2^{(2)}}{A_2} = \frac{E_2}{h_2} U_3, \quad \sigma^{(3)} = \frac{Q_2^{(3)}}{A_3} = -\frac{E_3}{h_3} U_3$$

This completes Example 1 (one can substitute the given data to obtain numerical values of the displacements, forces, and stresses in the members).

NUMERICAL EXAMPLE – 2

Problem: Wish to determine the numerical solution of the differential equation

$$-\frac{d^2u}{dx^2} - u = -x^2 \quad \text{in } 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0$$

Solution: We have the following correspondence compared to the model equation:

$$a = 1, \quad c = -1, \quad f = -x^2, \quad f_i^e = -\int_{x_a}^{x_b} x^2 \psi_i \, dx$$

(1) We wish to use a mesh of linear elements to solve the problem. The equations of a typical element are

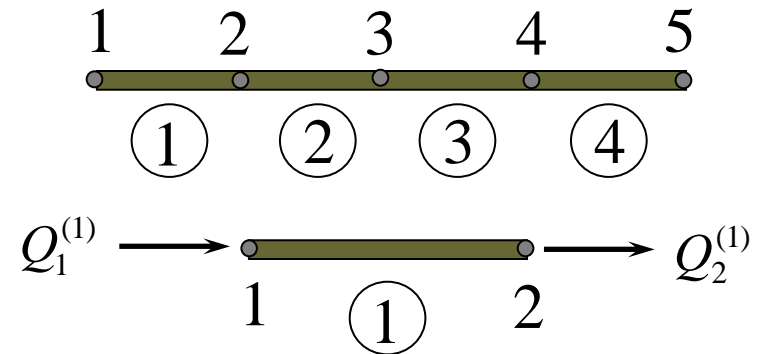
$$\left(\frac{1}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = -\frac{1}{h_e} \begin{Bmatrix} \frac{x_b}{3}(x_b^3 - x_a^3) - \frac{1}{4}(x_b^4 - x_a^4) \\ -\frac{x_a}{3}(x_b^3 - x_a^3) + \frac{1}{4}(x_b^4 - x_a^4) \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

NUMERICAL EXAMPLE – 2 (continued)

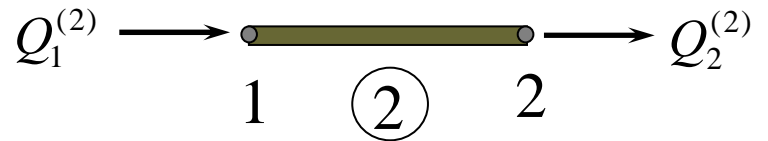
(2) We consider a mesh of 4 linear elements ($h = 0.25$).

The element equations are

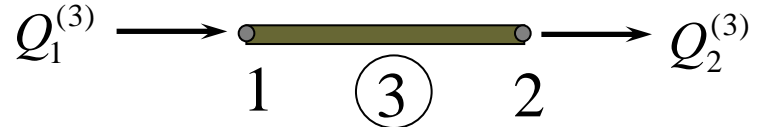
$$\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = - \begin{Bmatrix} 0.00130 \\ 0.00391 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$



$$\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = - \begin{Bmatrix} 0.01432 \\ 0.02232 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$



$$\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = - \begin{Bmatrix} 0.04297 \\ 0.05599 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$



$$\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix} = - \begin{Bmatrix} 0.08724 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

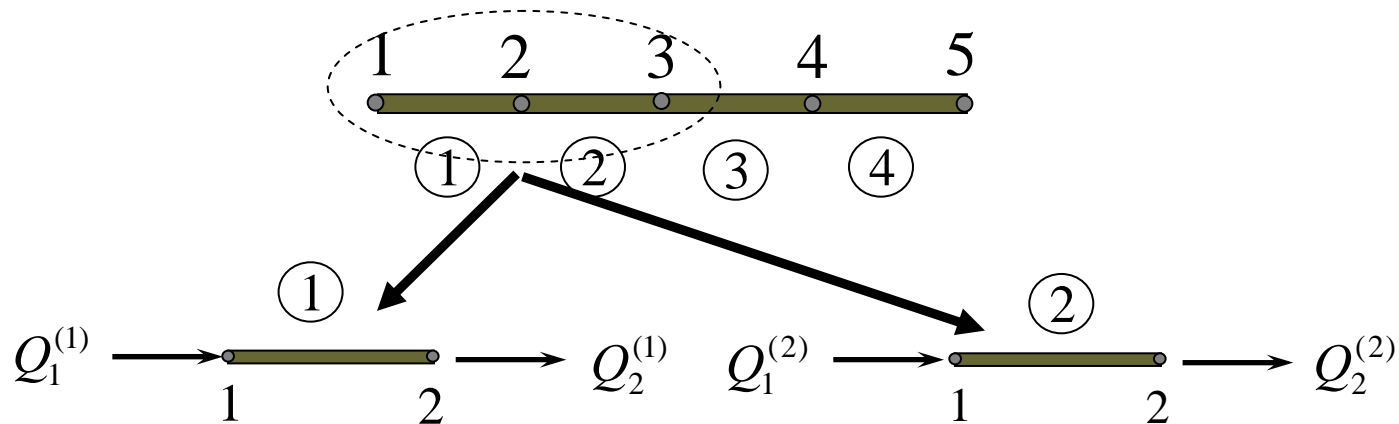


NUMERICAL EXAMPLE – 2 (continued)

(3) The boundary conditions are

$$U_1 = 0, \quad U_5 = 0$$

The equilibrium conditions are



$$Q_2^{(1)} + Q_1^{(2)} = 0, \quad Q_2^{(2)} + Q_1^{(3)} = 0, \quad Q_2^{(3)} + Q_1^{(4)} = 0, \quad Q_2^{(4)} + Q_1^{(5)} = 0$$

NUMERICAL EXAMPLE – 2 (continued)

(4) The assembled equations are

$$\frac{1}{24} \begin{bmatrix} 94 & -97 & 0 & 0 & 0 \\ -97 & 188 & -97 & 0 & 0 \\ 0 & -97 & 188 & -97 & 0 \\ 0 & 0 & -97 & 188 & -97 \\ 0 & 0 & 0 & -97 & 94 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = - \begin{Bmatrix} 0.001302 \\ 0.018229 \\ 0.065104 \\ 0.143230 \\ 0.105470 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

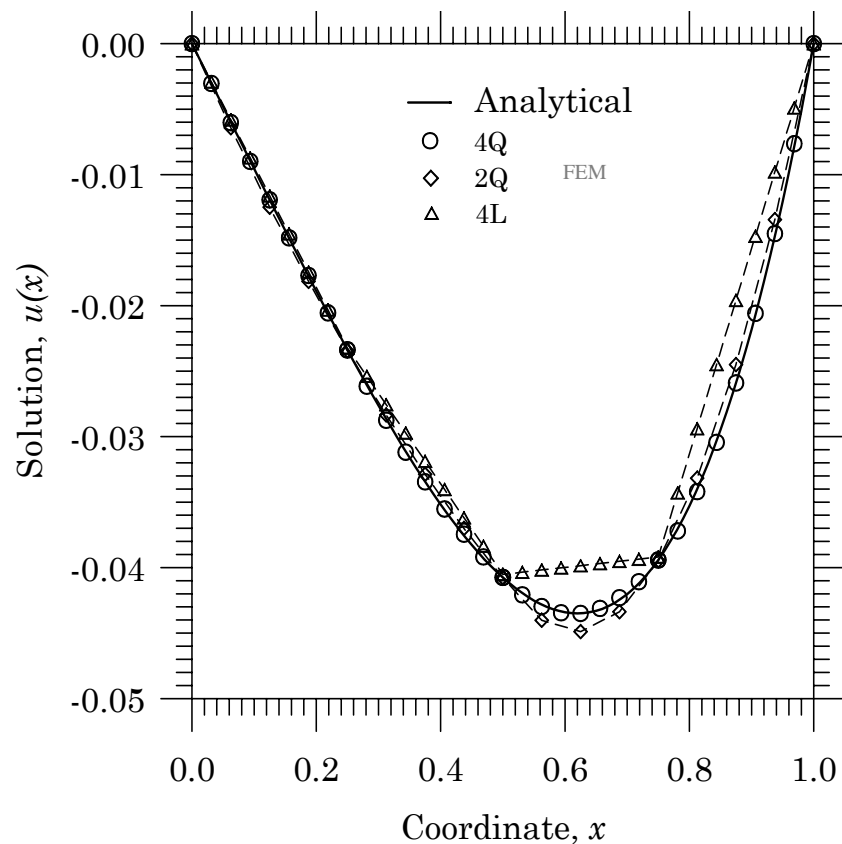
(5) The condensed equations for the unknown U 's and Q 's are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = - \begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^1 \\ Q_2^4 \end{Bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 & 0 \\ 0 & -4.0417 & 3.9167 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} 0.001302 \\ 0.10547 \end{Bmatrix} = \begin{Bmatrix} 0.09520 \\ 0.26386 \end{Bmatrix}$$

NUMERICAL EXAMPLE – 2 (continued)

Plot of the solution



Plot of the derivative of the solution

