SECTION 9.3

SPACE FIGURES
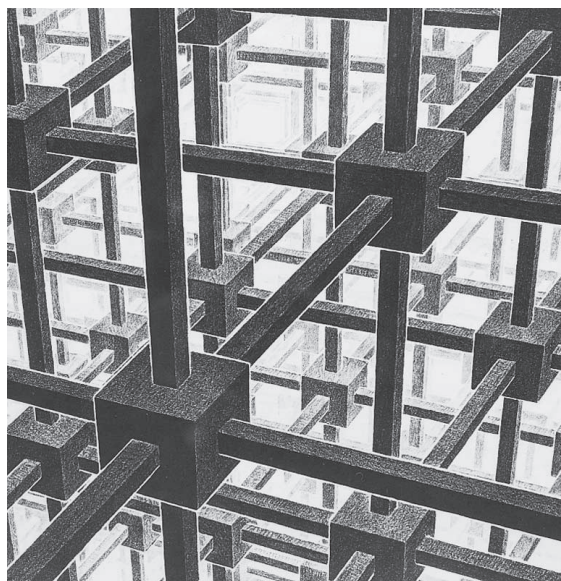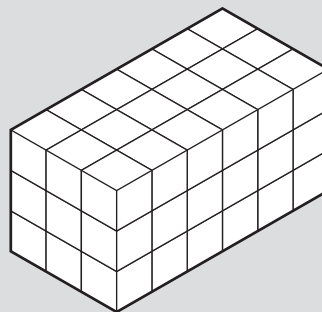

M. C. Escher's "Cubic Space Division"

PROBLEM OPENER

This is a sketch of a three-dimensional figure that contains 54 small cubes. If the outside of the figure is painted and then the figure is disassembled into 54 individual cubes, how many cubes will have paint on one face, two faces, three faces, and no faces?

In the above lithograph by the Dutch artist Maurits C. Escher (1898–1970), the girders intersect at right angles to form the edges of large cubes. The Canadian mathematician H. S. M. Coxeter calls it the *cubic honeycomb.* By representing space as being filled with cubes of the same size, Escher gives a wonderful sense of infinite space.

The notion of **space** in geometry is an undefined term, just as the ideas of point, line, and plane are undefined. We intuitively think of space as three-dimensional and of a plane as only two-dimensional. In his theory of relativity, Einstein tied together the three dimensions of space and the fourth dimension

of time. He showed that space and time affect each other and give us a four-dimensional universe.

In NCTM's K–4 Standard, *Geometry and Spatial Sense,* the importance of spatial understanding is discussed:

Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas, as well as other advanced mathematical topics.†

The Russian mathematician Sonya Kovalevsky is regarded as the greatest woman mathematician to have lived before 1900. Since women were barred by law from institutions of higher learning in Russia, Kovalevsky attended Heidelberg University in Germany. Later she was refused admission to the University of Berlin, which also barred women. Even the famous mathematician Karl Weierstrass, who claimed she had "the gift of intuitive genius," was unable to obtain permission for Kovalevsky to attend his lectures. She obtained her doctorate from the University of Göttingen but was without a teaching position for nine years, until the newly formed University of Stockholm broke tradition and appointed her to an academic position. Kovalevsky's prominence as a mathematician reached its peak in 1888, when she received the famous Prix Bordin from the French Académie des Sciences for her research paper "On the Rotation of a Solid about a Fixed Point." The selection committee "recognized in this work not only the power of an expansive and profound mind, but also a great spirit of invention."*

Sonya Kovalevsky
1850–1891

*D. M. Burton, *The History of Mathematics,* 4th ed. (New York: McGraw-Hill, 1999) pp. 557–560.

Planes

In two dimensions, the figures (lines, angles, polygons, etc.) all occur in a plane. In three dimensions, there are an infinite number of planes. Each plane partitions space into three disjoint sets: the points on the plane and two **half-spaces.** Portions of a few planes are shown in Figure 9.40. Any two planes either are **parallel,** as in part a, or **intersect** in a line, as in part b.

†*Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics 1989), p. 48.
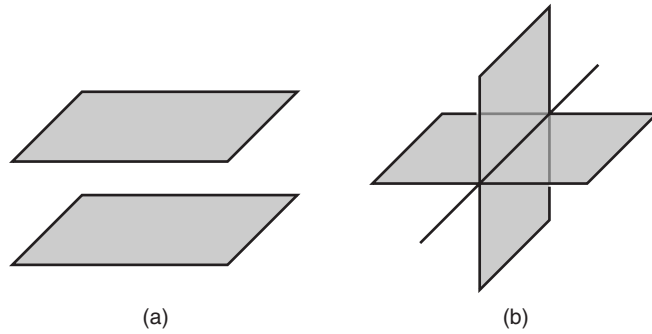
Figure 9.40

(a)　　　　　　　　　　　(b)

When two planes intersect, we call the angle between the planes a **dihedral angle.** Figure 9.41 shows three dihedral angles and their measures. A dihedral angle is measured by measuring the angle whose sides lie in the planes and are perpendicular to the line of intersection of the two planes. Parts a, b, and c of Figure 9.41 show examples of obtuse, right, and acute dihedral angles, respectively.
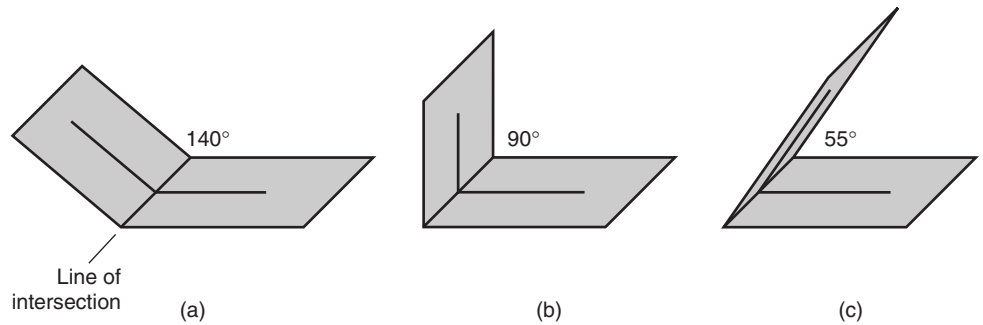

140°　　　　90°　　　　55°

Line of
intersection

Figure 9.41

(a)　　　　　　　　(b)　　　　　　　　(c)

When a line m in three-dimensional space does not intersect a plane P, it is **parallel to the plane,** as in Figure 9.42a. A line n is **perpendicular to a plane Q** at a point k if the line is perpendicular to every line in the plane that contains k, as in Figure 9.42b.

m

n

k

P　　　　　　　　　*Q*

Figure 9.42

(a)　　　　　　　　　　　(b)

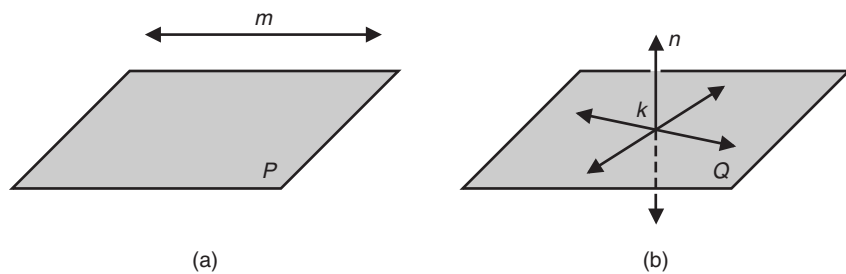## Polyhedra

The three-dimensional object with flat sides in Figure 9.43 is a crystal that is partially embedded in rock. Its flat pentagonal sides with their straight edges were not cut by people but were shaped by nature.
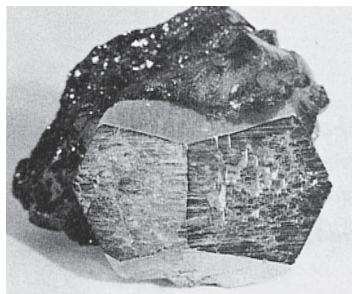

Figure 9.43

The surface of a figure in space whose sides are polygonal regions, such as the one in Figure 9.43, is called a **polyhedron** (*polyhedra* is the plural). The polygonal regions are called **faces,** and they intersect in the **edges** and **vertices** of the polyhedron. The union of a polyhedron and its interior is called a **solid.** Figure 9.44 shows examples of a polyhedron and two figures that are not polyhedra. The figure in part a is a polyhedron because its faces are polygonal regions. The figures in parts b and c are not polyhedra because one has a curved surface and the other has two faces that are not polygons.
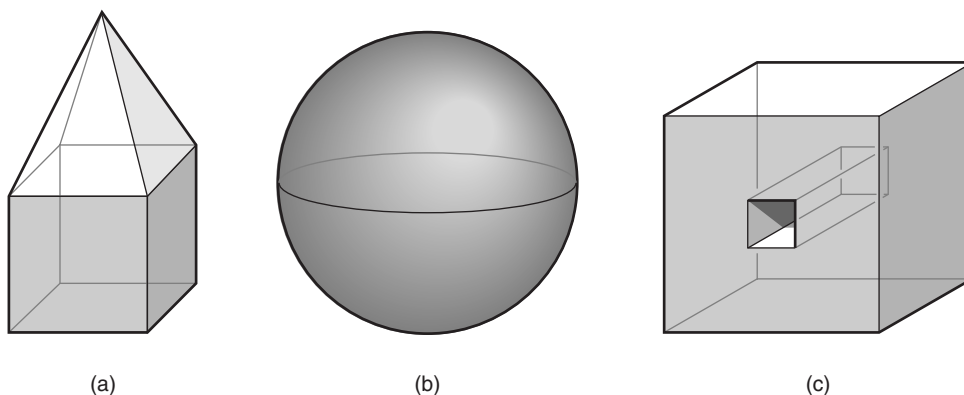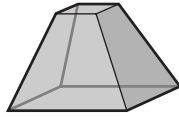

(a)   (b)   (c)

Figure 9.44

A polyhedron is **convex** if the line segment connecting any two of its points is contained inside the polyhedron or on its surface.
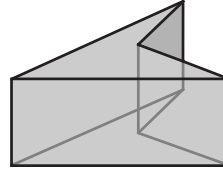
E x a m p l e A

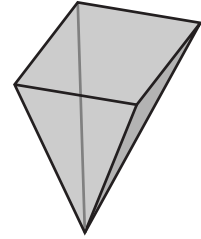Classify the following polyhedra as convex or nonconvex.

1.

2.

3.

Solution   Polyhedra 1 and 3 are convex; 2 is nonconvex.

Regular Polyhedra

The best known of all the polyhedra are the *regular polyhedra,* or *Platonic solids.* A **regular polyhedron** is a convex polyhedron whose faces are *congruent regular polygons,* the same number of which meet at each vertex. The ancient Greeks proved that there are only five regular polyhedra. Models of these polyhedra are shown in Figure 9.45. The **tetrahedron** has 4 triangles for faces; the **cube** has 6 square faces; the **octahedron** has 8 triangular faces; the **dodecahedron** has 12 pentagons for faces; and the **icosahedron** has 20 triangular faces.
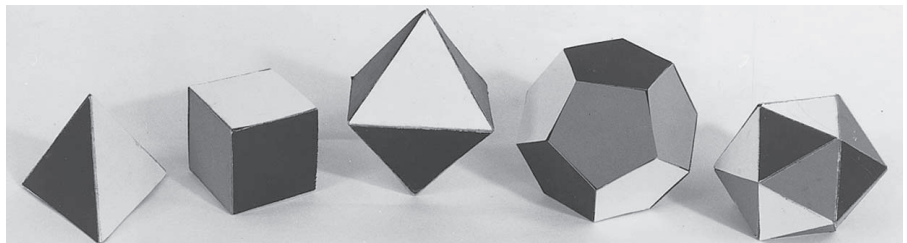

Figure 9.45
From left to right: tetrahedron, cube (hexahedron), octahedron; dodecahedron, icosahedron

The first three of the regular polyhedra shown in Figure 9.45 are found in nature as crystals. The cube and the octahedron occur in the common mineral pyrite, shown in Figure 9.46. The cube, which is embedded in rock, was found in Vermont, and the octahedron is from Peru. The other regular polyhedra, the dodecahedron and the icosahedron, do not occur as crystals but have been found in the skeletons of microscopic sea animals called *radiolarians.*

Figure 9.46
Crystals of pyrite

SEMIREGULAR POLYHEDRA   Some polyhedra have two or more different types of regular polygons for faces. The faces of the boracite crystal in Figure 9.47 are squares and equilateral triangles. This crystal, too, developed its flat, regularly shaped faces naturally, without the help of machines or people. Polyhedra whose faces are two or more regular polygons with the same arrangement of polygons around each vertex are called **semiregular polyhedra.** The boracite crystal is one of these. Each of its vertices is surrounded by three squares and one equilateral triangle.
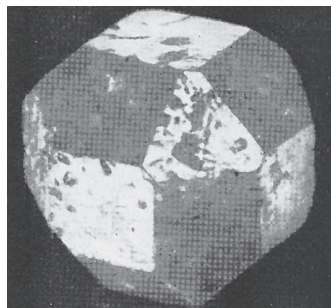

Figure 9.47
Crystal of boracite

    Several other semiregular polyhedra are shown in Figure 9.48. You may recognize the combination of hexagons and pentagons in part a as the pattern used on the surface of soccer balls.
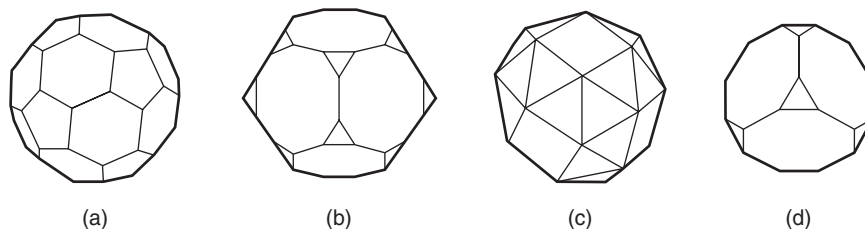

Figure 9.48   (a)   (b)   (c)   (d)

E x a m p l e B

For each semiregular polyhedron in Figure 9.48, list the polygons in the order in which they occur about any vertex.

Solution   Part a: hexagon, hexagon, pentagon; part b: dodecagon, dodecagon, triangle; part c: triangle, triangle, triangle, triangle, square; part d: octagon, octagon, triangle

---

Pyramids and Prisms

Chances are that when you hear the word *pyramid,* you think of the monuments built by the ancient Egyptians. Each of the Egyptian pyramids has a square base and triangular sides rising up to the vertex. This is just one type of pyramid. In general, the **base of a pyramid** can be any polygon, but its sides are always triangular. Pyramids are named according to the shape of their bases. Church spires are familiar examples of pyramids. They are usually square, hexagonal, or octagonal pyramids. The spire in the photograph in Figure 9.49 is an octagonal pyramid.

Figure 9.49
The Bruton Steeple,
Williamsburg, Virginia

Several pyramids with different bases are shown in the following example. Pyramids whose sides are isosceles triangles, as in Figures (1), (3), and (4) of Example C, are called **right pyramids.** Otherwise, as in Figure (2) of Example C, the pyramid is called an **oblique pyramid.** The vertex that is not contained in the pyramid's base is called the **apex.**
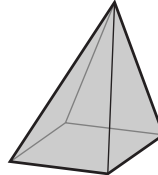
Example **C**
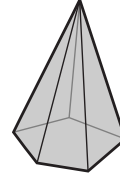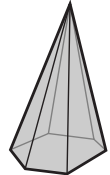
Determine the name of each pyramid.
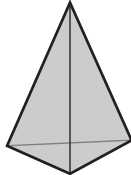
1.   **2.**   **3.**   **4.**

Solution   **1.** Triangular pyramid (also called a tetrahedron)   **2.** Oblique square pyramid   **3.** Pentagonal pyramid   **4.** Hexagonal pyramid

PRISMS   Prisms are another common type of polyhedron. You probably remember from your science classes that a prism is used to produce the spectrum of colors ranging from violet to red. Because of the angle between the vertical faces of a prism, light directed into one face will be bent when it passes out through the other face (Figure 9.50).
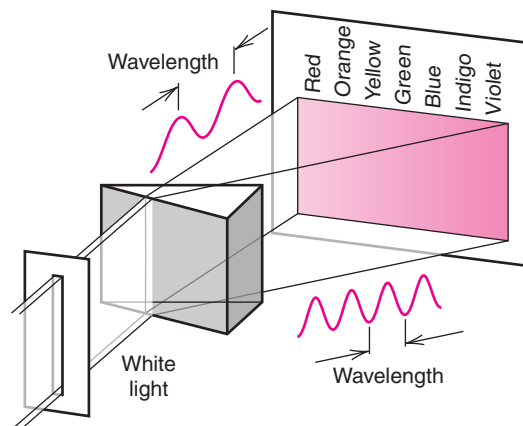

Wavelength

Red   Orange   Yellow   Green   Blue   Indigo   Violet

White
light

Wavelength

Figure 9.50

A **prism** has two parallel **bases,** upper and lower, which are congruent polygons. Like pyramids, prisms get their names from the shape of their bases. If the lateral sides of a prism are perpendicular to the bases, as in the case of the triangular, quadrilateral, hexagonal, and rectangular prisms in Figure 9.51, they are rectangles. Such a prism is called a **right prism.** A rectangular prism, which is modeled by a box, is the most common type of prism. If some of the lateral faces of a prism are parallelograms that are not rectangles, as in the pentagonal prism, the prism is called an **oblique prism.** The union of a prism and its interior is called a **solid prism.** A rectangular prism that is a solid is sometimes called a **rectangular solid.**
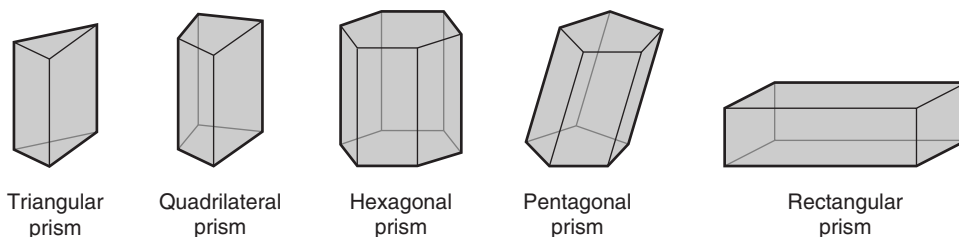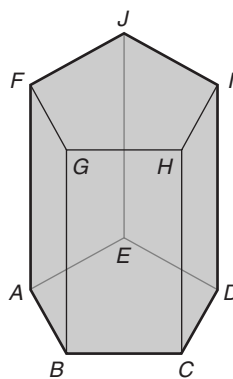

Figure 9.51

| Triangular prism | Quadrilateral prism | Hexagonal prism | Pentagonal prism | Rectangular prism |

Example D

The following figure is a right prism with bases that are regular pentagons.

Research Statement

In order to develop a conceptual understanding of geometry, students need to be placed in situations that allow them to apply deductive, inductive, and spatial reasoning.

Geddes and Fortunato 1993

1. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?

2. What is the measure of the dihedral angle between face *GHIJF* and face *CDIH*?

3. Name two faces that are in parallel planes.

Solution   1. It is the same as the measure of ∡*FGH*, which is 108°.   **2.** 90°. Since this is a right prism, the top base is perpendicular to each of the vertical sides.   **3.** *ABCDE* and *FGHIJ*

The two oblique hexagonal prisms in Figure 9.52 are crystals that grew with these flat, smooth faces and straight edges. Their lateral faces are parallelograms.

Figure 9.52
Prisms of the crystal orthoclase feldspar

Cones and Cylinders

Cones and cylinders are the circular counterparts of pyramids and prisms. Ice cream cones, paper cups, and party hats are common examples of cones. A cone has a circular region (disk) for a **base** and a lateral surface that slopes to the **vertex (apex).** If the vertex lies directly above the center of the base, the cone is called a **right cone** or usually just a cone; otherwise, it is an **oblique cone** (Figure 9.53).
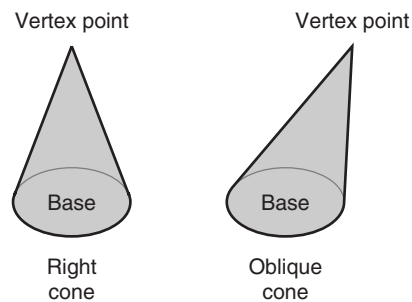
Figure 9.53

Ordinary cans are models of cylinders. A **cylinder** has two parallel circular **bases** (disks) of the same size and a lateral surface that rises from one base to the other. If the centers of the upper base and lower base lie on a line that is perpendicular to each base, the cylinder is called a **right cylinder** or simply a cylinder; otherwise, it is an **oblique cylinder** (Figure 9.54). Almost without exception, the cones and cylinders we use are right cones and right cylinders.
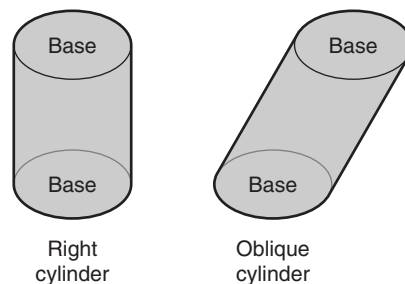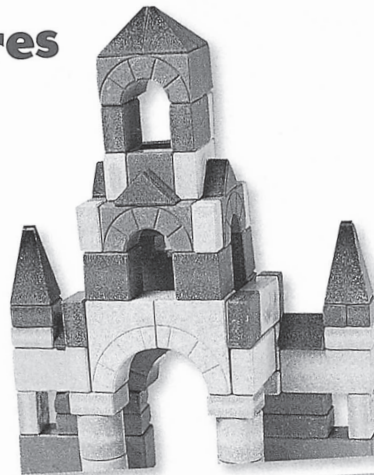

Figure 9.54

12·1 3-Dimensional Figures

Learn

Math Words

3-dimensional figure a figure in space

rectangular prism

cylinder

cube

cone

pyramid

sphere

base the flat face on which a 3-dimensional figure can rest

vertex the common point of the three or more edges of a 3-dimensional figure; plural form is **vertices**

face a flat side of a 3-dimensional figure

edge a line segment where two faces of a 3-dimensional figure meet

net a flat pattern that can be folded to make a 3-dimensional figure

Graham and Naomi built this model of a house for their social studies project. Which 3-dimensional figures did they use?

Example 1

You can compare the blocks used in the house with these **3-dimensional figures**.

rectangular prism — height, length, width

cylinder — curved edge, curved face

cube — vertex, flat face, straight edge

cone — base

pyramid

sphere

Graham and Naomi used cubes, rectangular prisms, pyramids, and cylinders to build their model house.

480 Cluster A

Spheres and Maps

The photograph in Figure 9.55 is a view of earth showing its almost perfect spherical shape. It was photographed from the *Apollo 17* spacecraft during its 1972 lunar mission. The dark regions are water. The Red Sea and the Gulf of Aden are near the top center, and the Arabian Sea and Indian Ocean are on the right.

Figure 9.55
Earth, as seen from Apollo 17 *during its 1972 lunar mission*

> **SPHERE**   A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** The union of a sphere and its interior is called a **solid sphere.**

A line segment joining the center of a sphere to a point on the sphere is called a **radius.** The length of such a line segment is also called the **radius of the sphere.** A line segment containing the center of the sphere and whose endpoints are on the sphere is called a **diameter,** and the length of such a line segment is called the **diameter of the sphere.**

The geometry of the sphere is especially important for navigating on the surface of the earth. You may have noticed that airline maps show curved paths between distant cities. This is because the shortest distance between two points on a sphere is along an arc of a **great circle.** In the drawing of the sphere in Figure 9.56, *G* (the color arc) is the arc of a great circle, because its center is also the center of the sphere, and arc *B* is not the arc of a great circle. The distance between points *X* and *Y* along arc *G* on a sphere is less than the distance between these points along arc *B*.
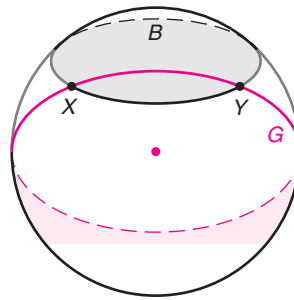
Figure 9.56

Locations on the earth's surface are often given by naming cities, streets, and buildings. A more general method of describing location uses two systems of circles (Figure 9.57). The circles that are parallel to the equator are called **parallels of latitude** and are shown in part a. Except for the equator, these circles are not great circles. Each parallel of latitude is specified by an angle from 0° to 90°, both north and south of the equator. For example, New York City is at a northern latitude of 41°, and Sydney, Australia, is at a southern latitude of 34°. The second system of circles is shown in part b. These circles pass through the north and south poles and are called **meridians of longitude.** These are great circles, and each is perpendicular to the equator. Since there is no natural point at which to begin numbering the meridians of longitude, the meridian that passes through Greenwich, England, was chosen as the zero meridian. Each meridian of longitude is given by an angle from 0° to 180°, both east and west of the zero meridian. The longitude of New York City is 74° west, and that of Sydney, Australia, is 151° east. These parallels of latitude and meridians of longitude shown together in part c, form a grid or coordinate system for locating any point on the earth.

Parallels of latitude	Meridians of longitude	Grid formed by both types of circles
(a)	(b)	(c)
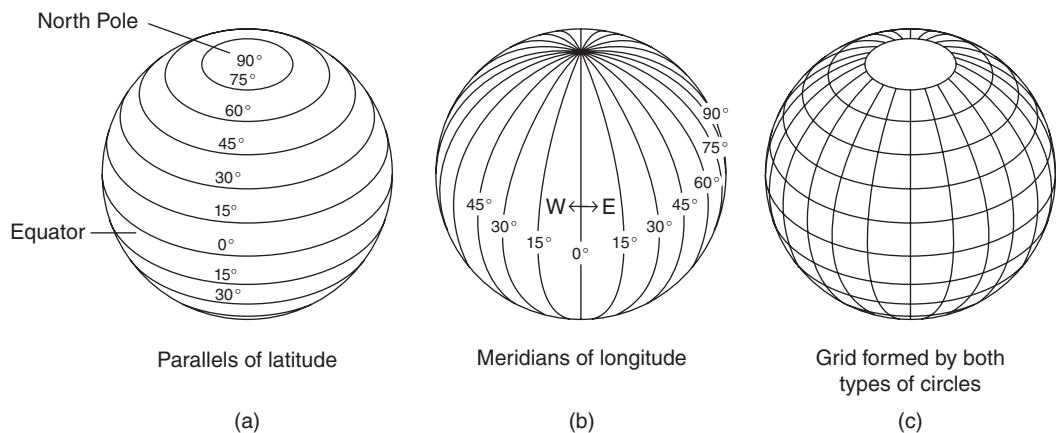
Figure 9.57

MAP PROJECTIONS   The globe is a spherical map of the earth. While such a map accurately represents the earth's shape and relative distances, we cannot see the whole globe at one time, nor can distances be measured easily. Maps on a flat surface are much more convenient. However, since a sphere cannot be placed flat on a plane without separating or overlapping some of its surface, making flat maps of the earth is a problem. There are three basic solutions: copying the earth's surface onto a cylinder, a cone, or a plane (Figure 9.58). These methods of copying are called **map projections.** In each case, some distortions of shapes and distances occur.
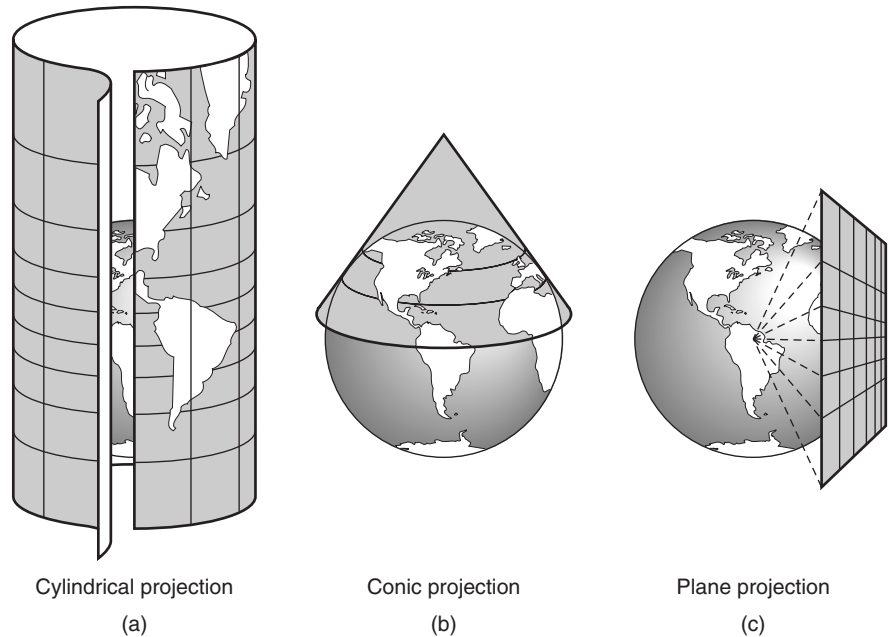
Cylindrical projection   Conic projection   Plane projection
(a)   (b)   (c)

Figure 9.58

A **cylindrical projection** (part a), also called a **mercator projection,** is obtained by placing a cylinder around a sphere and copying the surface of the sphere onto the cylinder. The cylinder is then cut to produce a flat map. Regions close to the equator are reproduced most accurately. The closer we get to the poles, the more the map is distorted.

A **conic projection** (part b) is produced by placing a cone with its apex over one of the poles and copying a portion of the surface of a sphere onto the cone. The cone is then cut and laid flat. This type of map construction is commonly used for countries that lie in an east-west direction and are middle latitude countries, as opposed to those near the poles or equator. The maps of the United States that are issued by the American Automobile Association are conical projections.

A **plane projection** (part c), also called an **azimuthal projection,** is made by placing a plane next to any point on a sphere and projecting the surface onto the plane. To visualize this process, imagine a light at the center of the sphere, and think of the boundary of a country as being pierced with small holes. The light shining through these holes, as shown by the dashed lines in part c, forms an image of the country on the plane. Less than one-half of the sphere's surface can be copied onto a plane projection, with the greatest distortion taking place at the outer edges of the plane. A plane projection, unlike cylindrical and conical projections, has the advantage that the distortion is uniform from the center of the map to its edges. Plane projections are used for hemispheres and maps of the Arctic and Antarctic. To map the polar regions, a plane is placed perpendicular to the earth's axis in contact with the north or south pole.

Problem-Solving Application

There is a remarkable formula that relates the numbers of vertices, edges, and faces of a polyhedron. This formula was first stated by René Descartes about 1635. In 1752 it was discovered again by Leonhard Euler and is now referred to

as **Euler's formula.** See if you can discover this formula, either before or as you read the parts of the solution presented below.

PROBLEM

What is the relationship among the numbers of faces, vertices, and edges of a polyhedron?

Understanding the Problem   Euler's formula holds for all polyhedra. Let's look at a specific example. A die is a cube which has six faces. **Question 1:** How many vertices and edges does it have?
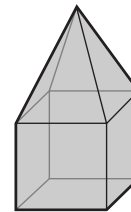
Devising a Plan   Let's *make a table;* list the numbers of faces, vertices, and edges for several polyhedra; and look for a relationship. **Question 2:** What are the numbers of faces, vertices, and edges for the polyhedra in figures (a), (b), and (c)?
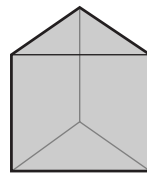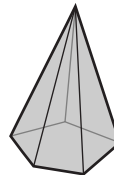
(a)   (b)   (c)

Carrying Out the Plan   The following table contains the numbers of faces, vertices, and edges for the cube above and the polyhedra in figures (a) through (c). Using F for the number of faces, V for the number of vertices, and E for the number of edges, we can construct Euler's formula from these data. **Question 3:** What is Euler's formula?

	F	V	E
Cube	6	8	12
Figure (a)	5	6	9
Figure (b)	6	6	10
Figure (c)	9	9	16

Looking Back   You may remember that an icosahedron has 20 triangular faces, but may not remember the number of edges or vertices. Altogether, 20 triangles have a total of 60 edges. Since every two edges of a triangle form one edge of an icosahedron, this polyhedron has $60 \div 2 = 30$ edges. Given the numbers of faces and edges for the icosahedron and Euler's formula $F + V - 2 = E$, we can determine the number of vertices. **Question 4:** How many vertices are there?

Answers to Questions 1–4   1. 8 vertices and 12 edges   **2.** Figure (a): 5 faces, 6 vertices, 9 edges; figure (b): 6 faces, 6 vertices, 10 edges; figure (c): 9 faces, 9 vertices, 16 edges   **3.** $F + V - 2 = E$ **4.** 12; $20 + V - 2 = 30$

Switzerland's Leonhard Euler is considered to be the most prolific writer in the history of mathematics. He published over 850 books and papers, and most branches of mathematics contain his theorems. After he became totally blind at the age of 60, he continued his amazing productivity for 17 years by dictating to a secretary and writing formulas in chalk on a large slate. On the 200th anniversary of his birthday in 1907, a Swiss publisher began reissuing Euler's entire collected works; the collection is expected to run to 75 volumes of about 60 pages each.*

Leonhard Euler
1707–1783

*H. W. Eves, *In Mathematical* Circles (Boston: Prindle, Weber, and Schmidt, 1969), pp. 46–49.

PUZZLER

How can four triangles be formed by using six matchsticks that touch only at their endpoints (do not cross)?

EXERCISES AND PROBLEMS 9.3
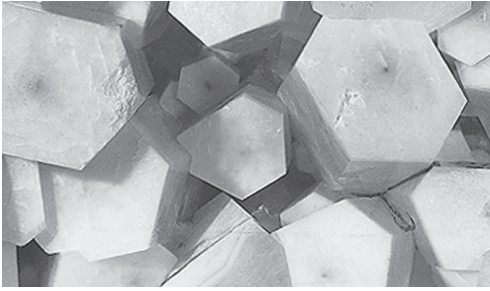
Crystals of calcite

1. The crystals crowded together in the above photograph are growing with flat polygonal faces.
   a. What type of polygon is the top face of these crystals?
   b. What type of polyhedron is formed by these crystals?

Which of the figures in exercises 2 and 3 are polyhedra?

2.

a.   b.   c.

3.

a.   b.   c.
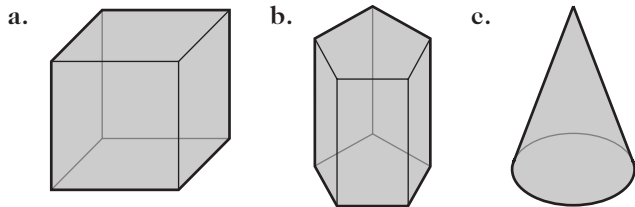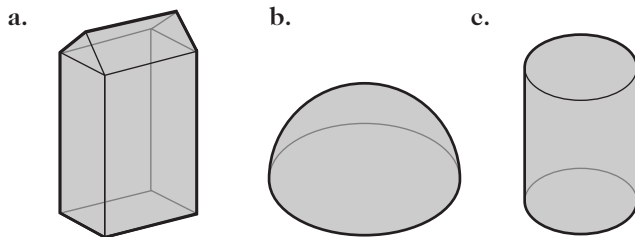
Classify the polyhedra in exercises 4 and 5 as convex or nonconvex.

4.

a.   b.

c.
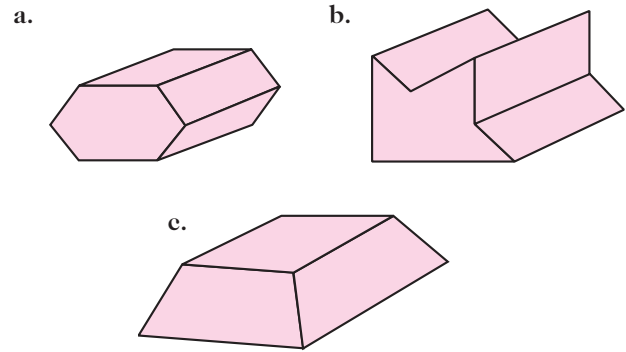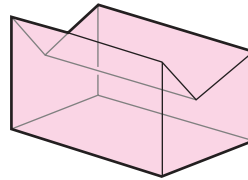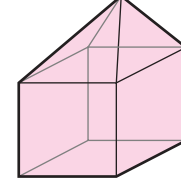
5. a.   b.   c.
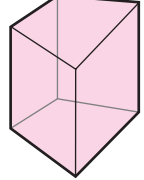
The semiregular polyhedra are classified according to the arrangement of regular polygons around each vertex. Proceeding counterclockwise, list the polygons about a vertex of each polyhedron in exercises 6 and 7.

6. a.   b.

20 hexagons   32 triangles
12 pentagons   6 squares

7 a.   b.
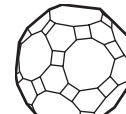
8 triangles   20 hexagons
6 squares   30 squares
               12 decagons

Name each of the figures in exercises 8 and 9.
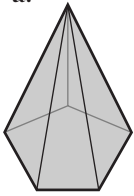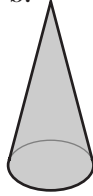
8. a.   **b.**   **c.**
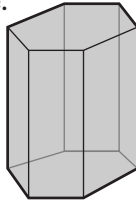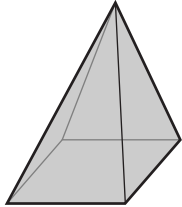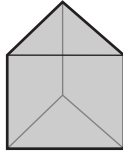

9.
a.   **b.**   **c.**
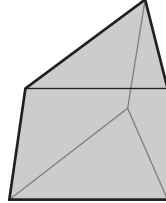
Name the figures in exercises 10 and 11, and state whether they are right or oblique.
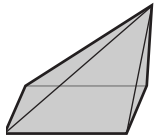
10.
a.   **b.**   **c.**
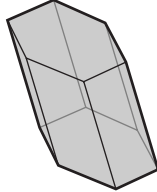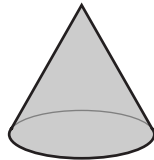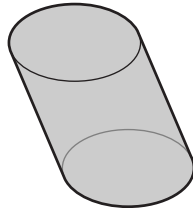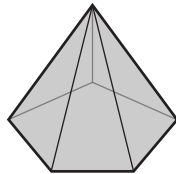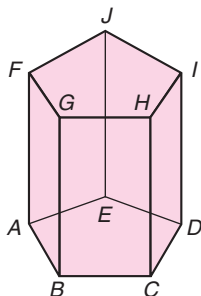

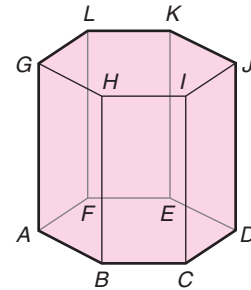
11.
a.   **b.**   **c.**

12. The polyhedron below is a right pentagonal prism whose bases are regular polygons.

a. What face is parallel to face *ABCDE*?
b. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?
c. What is the measure of the dihedral angle between face *FGHIJ* and face *EDIJ*?

13. The polyhedron below is a right prism, and its bases are regular hexagons.

a. What face is parallel to face *GHIJKL*?
b. What face is parallel to face *IJDC*?
c. What is the measure of the dihedral angle between face *ABHG* and face *ABCDEF*?
d. What is the measure of the dihedral angle between face *ABHG* and face *BCIH*?

Which of the three types of projections is best suited for making flat maps of the regions in exercises 14 and 15?

14. a. Australia
b. North, Central, and South America
c. The entire equatorial region between 30° north latitude and 30° south latitude

15. a. Arctic region
b. Western hemisphere between 20° north and 20° south
c. United States

Each of the geometric shapes listed in exercises 16 and 17 can be seen in the following photograph. Locate these objects.

Thompson Hall, University of New Hampshire

16. **a.** Cone   **b.** Pyramid   **c.** Cylinder
    d. Sphere   **e.** Circle

17. **a.** 30° angle   **b.** Rectangle   **c.** Semicircle
    d. Square   **e.** 45° angle

18. Use your knowledge of the spherical coordinate system to match each of the following cities with its approximate longitude and latitude.

Tokyo	38°N and 120°W
San Francisco	56°N and 4°W
Melbourne	35°N and 140°E
Glasgow	35°S and 20°E
Capetown	38°S and 145°E

19. Two points on the earth's surface that are on opposite ends of a line segment through the center of the earth are called **antipodal points.** The coordinates of such points are nicely related. The latitude of one point is as far above the equator as that of the other is below, and the longitudes are supplementary angles (in opposite hemispheres). For example, (30°N, 15°W) is off the west coast of Africa near the Canary Islands, and its antipodal point (30°S, 165°E) is off the eastern coast of Australia.

Babson College globe:
diameter 28 feet, weight 21 tons

   a. The globe in the accompanying photograph shows that (20°N, 120°W) is a point in the Pacific Ocean just west of Mexico. Its antipodal point is just east of Madagascar. What are the coordinates of this antipodal point?
   b. The point (30°S, 80°E) is in the Indian Ocean. What are the coordinates of its antipodal point? In what country is it located?

20. China is bounded by latitudes of 20°N and 55°N and by longitudes of 75°E and 135°E. It is playfully assumed that if you could dig a hole straight through the center of the earth, you would come out in China. For which of the following starting points is this true?
    a. Panama (9°N, 80°W)
    b. Buenos Aires (35°S, 58°W)
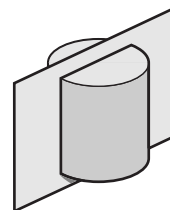   **c.** New York (41°N, 74°W)

The intersection of a plane and a three-dimensional figure is called a **cross section.** The cross section produced by the intersection of a plane and a right cylinder, where the plane is parallel to the base of the cylinder (see figure), is a circle. Determine the cross sections of the figures in exercises 21 and 22.
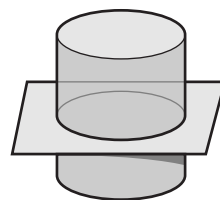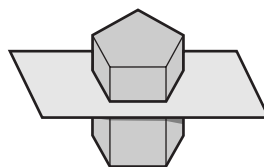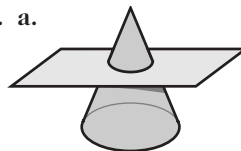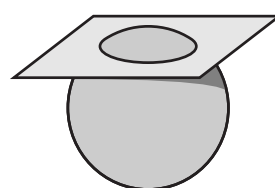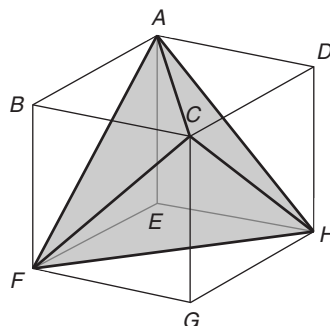
21. **a.**   **b.**

22. **a.**   **b.**

23. A cube can be divided into triangular pyramids in several ways. Pyramid *FHCA* divides this cube into five triangular pyramids. Name the four vertices of each of the other four pyramids.

24. $E, F, G, H,$ and C are the vertices of a square pyramid inside this cube. Name the five vertices of two more square pyramids that, together with the given pyramid, divide the cube into three pyramids.
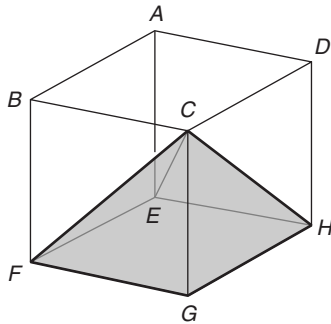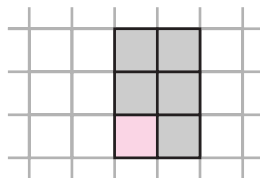

One method of describing a three-dimensional figure is to make a drawing of its different views. There are nine cubes in the following figure (two are hidden), and the top, right, and front views are shown.
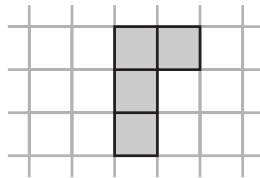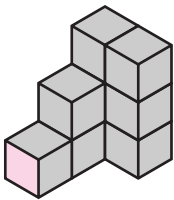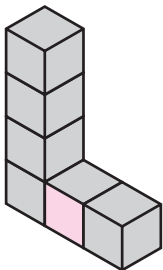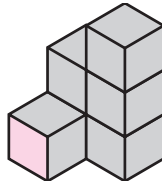

Top view
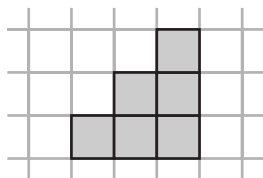
Side view (right)

Front view

Sketch the top, front, and side views of each of the figures in exercises 25 and 26. (*Note:* Figure 25b has one hidden cube beneath a cube that can be seen, and the color faces of the cubes are part of the front views of the figures.)
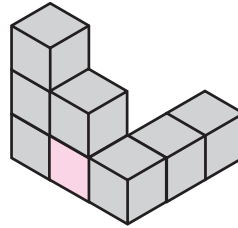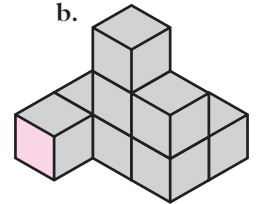
25. a.

b.

26. a.

b.

The table of polyhedra below illustrates some of the forms which crystals may take in nature. The polygons at the tops of the columns are the horizontal cross sections of the polyhedra in the columns. Use this table in exercises 27 and 28.

27. a. List the numbers of the polyhedra that are pyramids.
  b. Which of the polyhedra is most like a dodecahedron?

28. a. List the numbers of the polyhedra that are prisms.
  b. Which of the polyhedra is most like an octahedron?

Use Euler's formula in exercises 29 and 30 to determine the missing numbers for each polyhedron. For each set of conditions, find a polyhedron from those numbered from 1 to 21 in the table for exercises 27 and 28 which has the given number of faces, vertices, and edges.

29. a. 7 faces, 7 vertices, _____ edges
  b. 16 faces, _____ vertices, 24 edges
  c. _____ faces, 5 vertices, 8 edges

30. a. 6 faces, _____ vertices, 9 edges
  b. _____ faces, 8 vertices, 12 edges
  c. 14 faces, 24 vertices, _____ edges

Hurricane Ginger was christened on September 10, 1971, and became the longest-lived Atlantic hurricane on record. This tropical storm formed approximately 275 miles south of Bermuda and reached the U.S. mainland 20 days later. Use this map in exercises 31 and 32.
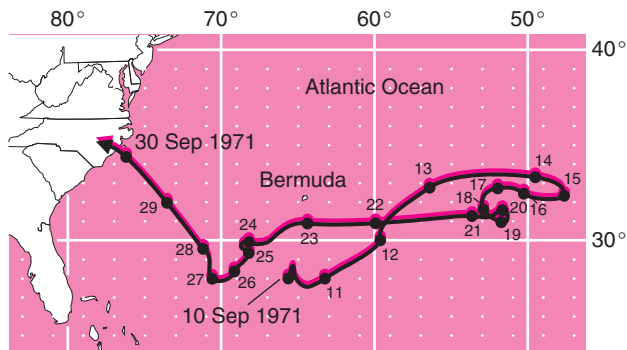

Erratic path of Hurricane Ginger

31. The storm's coordinates on September 10 were (28°N, 66°W). What were its coordinates on September 15, September 23, and September 30?

32. At this latitude on the earth's surface, each degree of longitude spans a distance of approximately 60 miles. About how many miles did this hurricane travel between September 10 and September 30? (*Hint:* Use a piece of string.)

REASONING AND PROBLEM SOLVING

33. Erica is designing a science experiment that requires two different three-dimensional figures such that one fits inside the other and both figures have at least one cross section that is the same for both figures (see exercises 21 and 22). Find such a pair of figures.

34. Here are the first three figures in a staircase pattern. These staircases are polyhedra.

1st   2d   3d
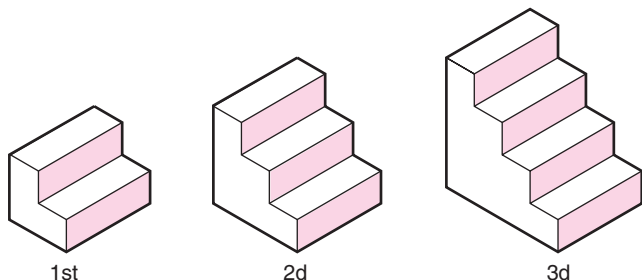
  a. The number of faces for the polyhedron in the first figure is 8. How many faces are there for the polyhedron in the 35th figure?

  b. The number of edges for the polyhedron in the first figure is 18. How many edges are there for the polyhedron in the 35th figure?

  c. The number of vertices for the polyhedron in the first figure is 12. How many vertices are there for the polyhedron in the 35th figure?

35. Sketch and describe how to form a piece of paper into the following figures (without bases).

a.   b.   c.

Right   Right   Oblique
circular cylinder   circular cone   circular cylinder
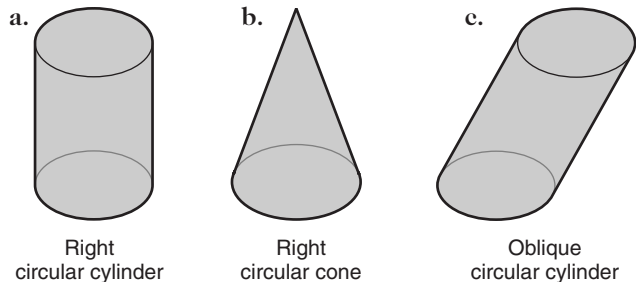
36. Featured Strategies: Making a Drawing and Using a Model   The five regular polyhedra and the numbers and shapes of their faces are shown in the following table. Determine the missing numbers of vertices and edges.

POLYHEDRON	VERTICES	FACES	EDGES
Tetrahedron		4 triangles	
Cube	8	6 squares	12
Octahedron		8 triangles	
Dodecahedron		12 pentagons	
Icosahedron		20 triangles	

  a. Understanding the Problem   The cube is the most familiar of the regular polyhedra. Its 6 faces meet in 12 edges, and its edges meet in 8 vertices (see figure i). How many vertices and edges does a tetrahedron have?


i.   ii.
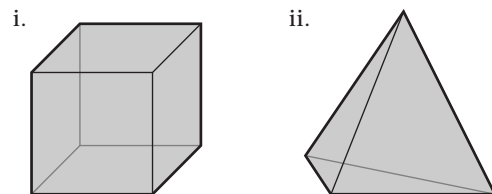
  b. Devising a Plan   One approach is to use a model or a sketch of the polyhedra and to count the numbers of vertices and edges. Or once we determine either the number of vertices or the number of edges, the missing number can be obtained by using Euler's formula $F + V - 2 = E$.

  Another approach that avoids counting is to use the fact that each pair of faces meets in exactly one edge. For example, since a dodecahedron has 12 pentagons for faces and each pair of

pentagons shares an edge, the number of edges is $(12 \times 5)/2 = 30$. Using Euler's formula, determine the number of vertices in a dodecahedron.

c. Carrying Out the Plan   Continue to find the numbers of edges by multiplying the number of faces by the number of sides on the face and dividing by 2. For example, what is the number of edges in an icosahedron? Fill in the rest of the table on the preceding page.

d. Looking Back   The number of vertices for each regular polyhedron can also be found directly from the number of edges that meet at each vertex. For example, three edges meet at each vertex of the dodecahedron, as shown in Figure iii below. Since there are 12 faces and each face has 5 vertex points, the dodecahedron has $(12 \times 5)/3 = 20$ vertex points. Use this approach to determine the number of vertices for the icosahedron in *iv*.
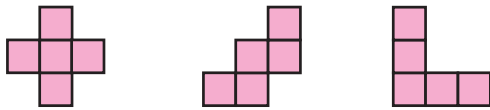
iii.

iv.
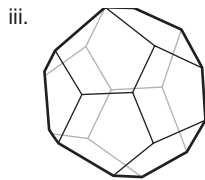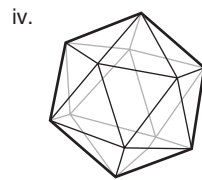
Dodecahedron   Icosahedron

37. Each of the following polygons contains five squares. There are only 12 such polygons that can be formed in the plane by joining five squares along their edges, and they are called **pentominoes.**

a. Which two of these pentominoes will fold into an open-top box, so that each face of the box is one of the squares?

b. Eight of the 12 pentominoes will fold into an open-top box. Find another one of these.

38. The polygons were formed by joining six squares along their edges. There are 35 such polygons, and they are called **hexominoes.**

a. Which two of these hexominoes will fold into a cube so that each face of the cube is one of the squares?

b. Eleven of the 35 hexominoes will fold into a cube. Find another such hexomino.

i.   ii.   iii.
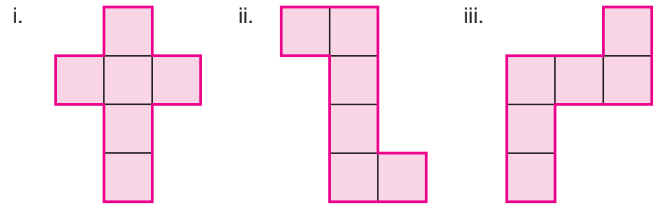
39. The centers of the faces of a cube can be connected to form a regular octahedron. Also, the centers of the faces of an octahedron can be connected to form a cube. Such pairs of polyhedra are called **duals.**
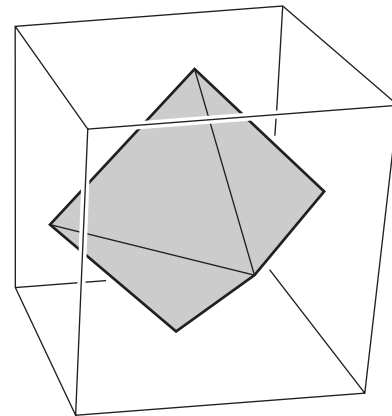
a. How is this dual relationship suggested by the table in exercises 36?

b. Find two other regular polyhedra that are duals of each other.

c. Which regular polyhedron is its own dual?

40. There are six categories of illusions.* One category, called *impossible objects,* is produced by drawing three-dimensional figures on two-dimensional surfaces. Find the impossible feature in each of these figures.

*P. A. Rainey, *Illusions* (Hamden, CT: The Shoe String Press, 1973), pp. 18–43.
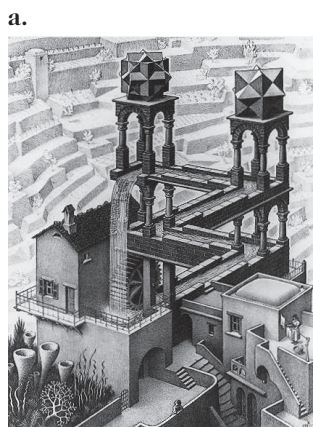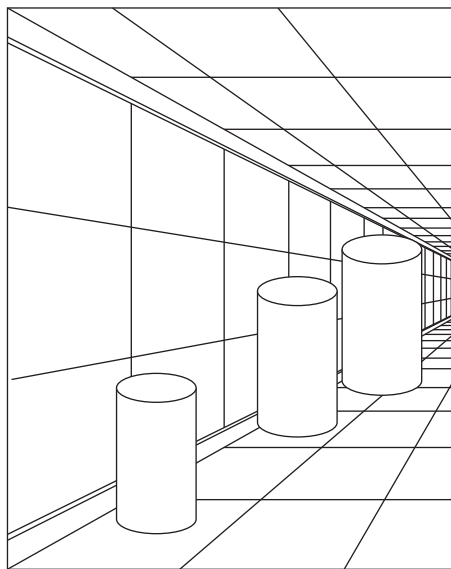
**a.**

M. C. Escher's "Waterfall"

b.

c.
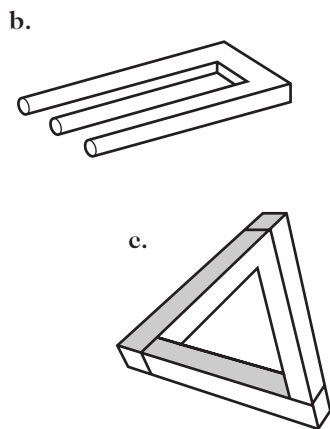
41. A second type of illusion involves depth perception.
We have accustomed our eyes to see depth when three-dimensional objects are drawn on two-dimensional surfaces. Answer questions a and b by disregarding the depth illusions.

a. Is one of these cylinders larger than the others?

b. Which of the four numbered angles below is the largest? Which are right angles? (*Hint:* Use a corner of a piece of paper.)

• •
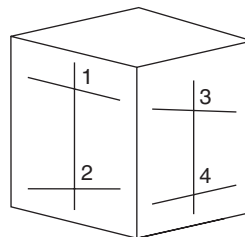
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• Math Investigation 9.3   *Pyramid Patterns*
Section-Related: • Links   • Writing/Discussion Problems   • Bibliography

• •

PUZZLER

Some wildlife researchers, having pitched camp, set out on an exploratory trip. They walked 15 miles due south, then 15 miles due east, where they saw a bear. Walking 15 miles due north, they returned to their camp. What was the color of the bear?