Chapter 2

Second-Order Linear Equations

2.1 Second-Order Linear Equations with Constant Coefficients

1. Find the general solution. See Table 2.1.

3. The associated polynomial, r^2+Pr+Q, has roots $r_{1,2} = \dfrac{-P \pm \sqrt{P^2-4Q}}{2}$. Suppose that P and Q are both positive. Then $P^2 - 4Q \geq 0$ implies that the roots are real and negative so $y \to 0$ as $x \to \infty$ because both exponential terms in the solution have negative exponents. If $P^2 - 4Q < 0$, then the roots are complex with negative real part. Consequently, the solutions are of the form $y = Ae^{-Px/2}\cos\omega x + Be^{-Px/2}\sin\omega x$ and will oscillate towards 0 as $x \to \infty$. The other cases are handled similarly.

5. **Euler's Equidimensional Equation** Changing the independent variable using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{1}{x}\dot{y}$ where the dot indicates differentiation with respect to the new independent variable, z. Similarly, $y'' = \dfrac{d}{dx}\left(\dfrac{1}{x}\dot{y}\right) = \dfrac{1}{x} \cdot \dfrac{1}{x}\ddot{y} - \dfrac{1}{x^2}\dot{y} = \dfrac{1}{x^2}(\ddot{y} - \dot{y})$. Making these substitutions into $x^2y''+pxy'+qy = 0$ yields $x^2 \cdot \dfrac{1}{x^2}(\ddot{y}-\dot{y})+$

Table 2.1: The general solutions for Exercise 1.

	Assoc Poly	Roots	General Solution
(a)	$r^2 + r - 6$	$2, -3$	$Ae^{2x} + Be^{-3x}$
(b)	$r^2 + 2r + 1$	$-1, -1$	$Ae^{-x} + Bxe^{-x}$
(c)	$r^2 + 8$	$\pm 2\sqrt{2}i$	$A\cos(2\sqrt{2}x) + B\sin(2\sqrt{2}x)$
(d)	$2r^2 - 4r + 8$	$1 \pm \sqrt{3}i$	$Ae^x \cos(\sqrt{3}x) + Be^x \sin(\sqrt{3}x)$
(e)	$r^2 - 4r + 4$	$2, 2$	$Ae^{2x} + Bxe^{2x}$
(f)	$r^2 - 9r + 20$	$4, 5$	$Ae^{4x} + Be^{5x}$
(g)	$2r^2 + 2r + 3$	$-\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$	$Ae^{-x/2}\cos(\sqrt{5}x/2) + Be^{-x/2}\sin(\sqrt{5}x/2)$
(h)	$4r^2 - 12r + 9$	$3/2, 3/2$	$Ae^{3x/2} + Bxe^{3x/2}$
(i)	$r^2 + r$	$-1, 0$	$Ae^{-x} + B$
(j)	$r^2 - 6r + 25$	$3 \pm 4i$	$e^{3x}(A\cos 4x + B\sin 4x)$
(k)	$4r^2 + 20r + 25$	$\pm 5/2$	$Ae^{5x/2} + Be^{-5x/2}$
(l)	$r^2 + 2r + 3$	$-1 \pm \sqrt{2}i$	$Ae^{-x}\cos(\sqrt{2}x) + Be^{-x}\sin(\sqrt{2}x)$
(m)	$r^2 - 4$	± 2	$Ae^{2x} + Be^{-2x}$
(n)	$4r^2 - 8r + 7$	$1 \pm \frac{\sqrt{3}}{2}i$	$Ae^x\cos(\sqrt{3}x/2) + Be^x\sin(\sqrt{3}x/2)$
(o)	$2r^2 + r - 1$	$-1, 1/2$	$Ae^{-x} + Be^{x/2}$
(p)	$16r^2 - 8r + 1$	$\pm 1/4$	$Ae^{x/4} + Bxe^{x/4}$
(q)	$r^2 + 4r + 5$	$-2 \pm i$	$Ae^{-2x}\cos x + Be^{-2x}\sin x$
(r)	$r^2 + 4r - 5$	$-5, 1$	$Ae^{-5x} + Be^x$

equation. Note that the solution is only valid for $x > 0$.

(a) The z equation is $\ddot{y} + 2\dot{y} + 10y = 0$ with solution $y = Ae^{-z}\cos(3z) + Be^{-z}\sin(3z)$.
The x equation has the solution $y = Ax^{-1}\cos(3\ln x) + Bx^{-1}\sin(3\ln x)$.

(b) First divide the equation by 2. The z equation is then $\ddot{y} + 4\dot{y} + 4y = 0$ with solution $y = Ae^{-2z} + Bze^{-2z}$. The x equation has the solution $y = Ax^{-2} + Bx^{-2}\ln x$.

(c) The z equation is $\ddot{y} + \dot{y} - 12y = 0$ with solution $y = Ae^{-4z} + Be^{3z}$.
The x equation has the solution $y = Ax^{-4} + Bx^{3}$.

(d) Divide the equation by 4. The z equation is then $\ddot{y} - \dot{y} - \dfrac{3}{4}y = 0$ with solution $y = Ae^{-z/2} + Be^{3z/2}$.
The x equation has the solution $y = Ax^{-1/2} + Bx^{3/2}$.

(e) The z equation is $\ddot{y} - 4\dot{y} + 4y = 0$ with solution $y = Ae^{2z} + Bze^{2z}$.
The x equation has the solution $y = Ax^{2} + Bx^{2}\ln x$.

(f) The z equation is $\ddot{y} + \dot{y} - 6y = 0$ with solution $y = Ae^{-3z} + Be^{2z}$.
The x equation has the solution $y = Ax^{-3} + Bx^{2}$.

(g) The z equation is $\ddot{y} + \dot{y} + 3y = 0$ with solution $y = Ax^{-z/2}\cos(\sqrt{11}z/2) + Bz^{-z/2}\sin(\sqrt{11}z/2)$.
The x equation has the solution $y = Ax^{-1/2}\cos(\sqrt{11}\ln x/2) + Bx^{-1/2}\sin(\sqrt{11}\ln x/2)$.

(h) The z equation is $\ddot{y} - 2y = 0$ with solution $y = Ae^{-\sqrt{2}z} + Be^{\sqrt{2}z}$.
The x equation has the solution $y = Ax^{-\sqrt{2}} + Bx^{\sqrt{2}}$.

(i) The z equation is $\ddot{y} - 16y = 0$ with solution $y = Ae^{-4z} + Be^{4z}$.
The x equation has the solution $y = Ax^{-4} + Bx^{4}$.

2.2 The Method of Undetermined Coefficients

1. Find the general solution to the following equations.

(a) The auxiliary roots are -5 and 2 so the homogeneous equation has the solution $y = Ae^{-5x} + Be^{2x}$. Try $y = \alpha e^{4x}$ as a particular solution. Substitute and simplify to obtain $18\alpha e^{4x} = 6e^{4x}$ which implies that $\alpha = 1/3$. The general solution is $y = Ae^{-5x} + Be^{2x} + \frac{1}{3}e^{4x}$.

$3\sin x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos 2x + B\sin 2x + \sin x$.

(c) The auxiliary roots are $-5, -5$ so the homogeneous equation has the solution $y = Ae^{-5x} + Bxe^{-5x}$. Neither $y = \alpha e^{-5x}$ nor $y = \alpha xe^{-5x}$ can be a particular solution because they are solutions to the homogeneous equation. Try $y = \alpha x^2 e^{-5x}$ instead. Substitute and simplify to obtain $2\alpha e^{-5x} = 14e^{-5x}$ which implies that $\alpha = 7$. The general solution is $y = Ae^{-5x} + Bxe^{-5x} + 7x^2e^{-5x}$.

(d) The auxiliary roots are $1 \pm 2\imath$ so the homogeneous equation has the solution $y = Ae^x\cos 2x + Be^x\sin x$. Try $y = \alpha x^2 + \beta x + \gamma$ as a particular solution. Substitute and simplify to obtain $5\alpha x^2 + (5\beta - 4\alpha)x + 2\alpha - 2\beta + 5\gamma = 25x^2 + 12$ which implies that $\alpha = 5, \beta = 4, \gamma = 2$. The general solution is $y = Ae^x\cos 2x + Be^x\sin x + 5x^2 + 4x + 2$.

(e) The auxiliary roots are $-2, 3$ so the homogeneous equation has the solution $y = Ae^{-2x} + Be^{3x}$. The function $y = \alpha e^{-2x}$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha xe^{-2x}$ instead. Substitute and simplify to obtain $-5\alpha xe^{-2x} = 20e^{-2x}$ which implies that $\alpha = -4$. The general solution is $y = Ae^{-2x} + Be^{3x} - 4xe^{-2x}$.

(f) The auxiliary roots are $1, 2$ so the homogeneous equation has the solution $y = Ae^x + Be^{2x}$. Try $y = \alpha\cos 2x + \beta\sin 2x$ as a particular solution. Substitute and simplify to obtain $(-6\alpha - 2\beta)\cos 2x + (-2\alpha + 6\beta)\sin 2x = 14\sin 2x - 18\cos 2x$ which implies that $\alpha = 3$ and $\beta = 2$. The general solution is $y = Ae^x + Be^{2x} + 3\cos 2x + 2\sin 2x$.

(g) The auxiliary roots are $\pm\imath$ so the homogeneous equation has the solution $y = A\cos x + B\sin x$. The function $y = \alpha\cos x + \beta\sin x$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha x\cos x + \beta x\sin x$ instead. Substitute and simplify to obtain $2\beta\cos x - 2\alpha\sin x = 2\cos x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos x + B\sin x + x\sin x$.

(h) The auxiliary roots are $0, 2$ so the homogeneous equation has the solution $y = A + Be^{2x}$. The function $y = \alpha x + \beta$ is not a particular solution because part of it is a solution to the homogeneous equation. Try $y = \alpha x^2 + \beta x$ instead. Substitute and simplify to obtain $-4\alpha x + 2\alpha - 2\beta = 12x - 10$ which implies that $\alpha = -3$ and $\beta = 2$. The general solution is $y = A + Be^{2x} - 3x^2 + 2x$.

(i) The auxiliary roots are $1, 1$ so the homogeneous equation has the solution $y = Ae^x + Bxe^x$. Neither $y = \alpha e^x$ nor $y = \alpha xe^x$ is a particular solution because they are both solutions to the homogeneous

(j) The auxiliary roots are $1 \pm i$ so the homogeneous equation has the solution $y = Ae^x \cos x + Be^x \sin x$. This means that $y = \alpha e^x \cos x + \beta e^x \sin x$ can not be a particular solution. Try $y = \alpha x e^x \cos x + \beta x e^x \sin x$ instead. Substitute and simplify to obtain $2\beta e^x \cos x - 2\alpha e^x \sin x = e^x \sin x$ which implies that $\alpha = -1/2$ and $\beta = 0$. The general solution is $y = Ae^x \cos x + Be^x \sin x - \frac{1}{2} x e^x \cos x$.

(k) The auxiliary roots are $-1, 0$ so the homogeneous equation has the solution $y = Ae^{-x} + B$. This means that $y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$ can not be a particular solution. Try $y = \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x$ instead. Substitute and simplify to obtain $5\alpha x^4 + (20\alpha + 4\beta)x^3 + (12\beta + 3\gamma)x^2 + (6\gamma + 2\delta)x + 2\delta + \epsilon = 10x^4 + 2$ which implies that $\alpha = 2$, $\beta = -10$, $\gamma = 40$, $\delta = -120$ and $\epsilon = 242$. The general solution is $y = Ae^{-x} + B + 2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x$.

3. Substitute $y = y_1 + y_2$ into the left side of the differential equation to get

$$\begin{aligned}(y_1 + y_2)'' + P(y_1 + y_2)' + Q(y_1 + y_2) &= y_1'' + Py_1' + Qy_1 + \\ &\qquad y_2'' + Py_2' + Qy_2 \\ &= R_1 + R_2\end{aligned}$$

(a) The auxiliary roots are $\pm 2\imath$ so the homogeneous equation has the solution $y = A\cos 2x + B\sin 2x$.

A particular solution for $y'' + 4y = 4\cos 2x$ has the form $y_1 = \alpha x \cos 2x + \beta x \sin 2x$. Substitute to obtain $y_1 = x \sin 2x$.

A particular solution for $y'' + 4y = 6\cos x$ has the form $y_2 = \alpha \cos x + \beta \sin x$. Substitute to obtain $y_2 = 2\cos x$.

A particular solution for $y'' + 4y = 8x^2 - 4x$ has the form $y_3 = \alpha x^2 + \beta x + \gamma$. Substitute to obtain $y_3 = 2x^2 - x - 1$.

By the superposition principal $y = y_1 + y_2 + y_3$ is a particular solution to the given equation and $y = A\cos 2x + B\sin 2x + x\sin 2x + 2\cos x + 2x^2 - x - 1$ is a general solution.

(b) The auxiliary roots are $\pm 3\imath$ so the homogeneous equation has the solution $y = A\cos 3x + B\sin 3x$.

A particular solution for $y'' + 9y = 2\sin 3x$ has the form $y_1 = \alpha x \cos 3x + \beta x \sin 3x$. Substitute to obtain $y_1 = -\frac{1}{3} x \cos 3x$.

A particular solution for $y'' + 4y = 4\sin x$ has the form $y_2 = \alpha \cos x + \beta \sin x$. Substitute to obtain $y_2 = \frac{1}{2}\sin x$.

$\alpha x^3 + \beta x^2 + \gamma x + \delta$. Substitute to obtain $y_4 = 3x^3 - 2x$.

By the superposition principal $y = y_1 + y_2 + y_3 + y_4$ is a particular solution to the given equation and $y = A\cos 3x + B\sin 3x - \frac{1}{3}x\cos 3x + \frac{1}{2}\sin x - 2e^{-2x} + 3x^2 - 2x$ is a general solution.

2.3 The Method of Variation of Parameters

1. Find a particular solution.

    (a) The homogeneous solution is $y = A\sin 2x + B\cos 2x$ so the particular solution has the form $y = v_1 \sin 2x + v_2 \cos 2x$ where v_1 and v_2 satisfy the system

    $$\begin{aligned} v_1' \sin 2x + v_2' \cos 2x &= 0 \\ 2v_1' \cos 2x - 2v_2' \sin 2x &= \tan 2x \end{aligned}$$

    Therefore, $v_1' = \frac{1}{2}\sin x$ and $v_2' = -\frac{1}{2}\sin 2x \tan 2x$. Integrate to get $v_1 = -\frac{1}{4}\cos 2x$ and $v_2 = \frac{1}{4}\sin 2x - \frac{1}{4}\ln(\sec 2x + \tan 2x)$. Therefore, $y = -\frac{1}{4}\cos 2x \ln(\sec 2x + \tan 2x)$.

    (b) The homogeneous solution is $y = Ae^{-x} + Bxe^{-x}$ so the particular solution has the form $y = v_1 e^{-x} + v_2 xe^{-x}$ where v_1 and v_2 satisfy the system

    $$\begin{aligned} v_1' e^{-x} + v_2' xe^{-x} &= 0 \\ -v_1' e^{-x} + v_2'(e^{-x} - xe^{-x}) &= e^{-x}\ln x \end{aligned}$$

    Therefore, $v_1' = -x\ln x$ and $v_2' = \ln x$. Integrate to get $v_1 = -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2$ and $v_2 = x\ln x - x$. Therefore, $y = \frac{1}{4}x^2 e^{-x}(2\ln x - 3)$.

    (c) The homogeneous solution is $y = Ae^{3x} + Be^{-x}$ so the particular solution has the form $y = v_1 e^{3x} + v_2 e^{-x}$ where v_1 and v_2 satisfy the system

    $$\begin{aligned} v_1' e^{3x} + v_2' e^{-x} &= 0 \\ 3v_1' e^{-x} - v_2' e^{-x} &= 64xe^{-x} \end{aligned}$$

    Therefore, $v_1' = 16xe^{-4x}$ and $v_2' = -16x$. Integrate to get $v_1 = -(4x+1)e^{-4x}$ and $v_2 = -8x^2$. Therefore, $y = -e^{-x}(8x^2 + 4x + 1)$. The last term can be dropped since $-e^{-x}$ is a solution to the homogeneous equation.

$$
\begin{aligned}
v_1' e^{-x} \sin 2x + v_2' e^{-x} \cos 2x &= 0 \\
v_1'(-e^{-x} \sin 2x + 2e^{-x} \cos 2x) &+ \\
v_2'(e^{-x} \cos 2x + 2e^{-x} \sin 2x) &= e^{-x} \sec 2x
\end{aligned}
$$

Therefore, $v_1' = \frac{1}{2}$ and $v_2' = -\frac{1}{2} \tan 2x$. Integrate to get $v_1 = \frac{1}{2} x$ and $v_2 = \frac{1}{4} \ln(\cos 2x)$. Therefore, $y = \frac{1}{2} x e^{-x} \sin 2x + \frac{1}{4} e^{-x} \cos 2x \ln(\cos 2x)$.

(e) The homogeneous solution is $y = Ae^{-x/2} + Be^{-x}$ so the particular solution has the form $y = v_1 e^{-x/2} + v_2 e^{-x}$ where v_1 and v_2 satisfy the system

$$
\begin{aligned}
v_1' e^{-x/2} + v_2' e^{-x} &= 0 \\
-\frac{1}{2} v_1' e^{-x/2} - v_2' e^{-x} &= \frac{1}{2} e^{-3*x}
\end{aligned}
$$

Warning The equation must be divided by 2 before the algorithm can be applied.

Therefore, $v_1' = e^{-5x/2}$ and $v_2' = -e^{-2x}$. Integrate to get $v_1 = -\frac{2}{5} e^{-5x/2}$ and $v_2 = \frac{1}{2} e^{-x}$. Therefore, $y = \frac{1}{10} e^{-3x}$.

(f) The homogeneous solution is $y = Ae^{x} + Be^{2x}$ so the particular solution has the form $y = v_1 e^{x} + v_2 e^{2x}$ where v_1 and v_2 satisfy the system

$$
\begin{aligned}
v_1' e^{x} + v_2' e^{2x} &= 0 \\
v_1' e^{x} + 2v_2' e^{2x} &= (1 + e^{-x})^{-1}
\end{aligned}
$$

Therefore, $v_1' = -\dfrac{e^{-x}}{1 + e^{-x}}$ and $v_2' = \dfrac{e^{-2x}}{1 + e^{-x}}$. Integrate to get $v_1 = \ln(1 + e^{-x})$ and $v_2 = \ln(1 + e^{-x}) - e^{-x}$. Consequently, the particular solution is $y = (e^{x} + e^{2x}) \ln(1 + e^{-x}) - e^{x}$.

3. By Inspection The auxiliary polynomial is $r^2 - 2r + 1$ with roots $1, 1$ so the homogeneous solution is $y = Ae^{x} + Bxe^{x}$. Therefore, there is a particular solution of the form $y = \alpha x + \beta$. Substitute to find that $y = 2x + 4$ is a particular solution.

By Variation of Parameters The particular solution has the form $y = v_1 e^{x} + v_2 x e^{x}$ where v_1 and v_2 satisfy the system

$$
\begin{aligned}
v_1' e^{x} + v_2' x e^{x} &= 0 \\
v_1' e^{x} + v_2'(x e^{x} + e^{x}) &= 2x
\end{aligned}
$$

Therefore, $v_1' = -2x^2 e^{-x}$ and $v_2' = 2x e^{-x}$. Integrate to get $v_1 = (2x^2 + 4x + 4)e^{-x}$ and $v_2 = -(2x + 2)e^{-x}$. Therefore, $y = 2x + 4$.

Warning Write the equation in the form

$$y'' + P(x)y' + Q(x)y = R(x)$$

before applying the variation of parameters algorithm.

(a) $y_h = Ax + B(x^2+1)$; $y_p = x^4/6 - x^2/2$.

(b) $y_h = Ax^{-1} + Be^x$; $y_p = -\frac{1}{3}x^2 - x - 1$.

(c) $y_h = Ax + Be^x$; $y_p = x^2 + x + 1$.

(d) $y_h = A(1+x) + Be^x$; $y_p = \frac{1}{2}e^{2x}(x-1)$.

(e) $y_h = Ax^2 + Bx$; $y_p = -xe^{-x} - (x^2+x)\int \frac{e^{-x}}{x}\,dx$.

To obtain this solution formula you will have to apply integration by parts to $\int \frac{e^{-x}}{x^2}\,dx$. The remaining integral does not evaluate to an elementary function.

2.4 The Use of a Known Solution to Find Another

1. Find y_2 and the general solution, given y_1.

   (a) Since $p(x) = 0$, $e^{-\int p(x)\,dx} = e^0 = 1$ and $y_2(x) = \sin(x)v(x)$ where $v(x) = \int \frac{1}{\sin^2 x}\,dx = \int \csc^2 x\,dx = -\cot x$. Therefore, $y_2(x) = -\cos x$. The general solution is $y = A\sin x + B\cos x$.

   (b) Once more, $p(x) = 0$ and $e^{-\int p(x)\,dx} = e^0 = 1$. Therefore, $y_2(x) = e^x v(x)$ where $v(x) = \int \frac{1}{e^{2x}}\,dx = -\frac{1}{2}e^{-2x}$. Therefore, $y_2(x) = e^x \cdot -\frac{1}{2}e^{-2x} = -\frac{1}{2}e^{-x}$. The general solution is $y = Ae^x + Be^{-x}$.

3. If $y = y_1 = x^2$, then $x^2y'' + xy' - 4y = x^2 \cdot 2 + x \cdot 2x - 4 \cdot x^2 = 0$. To find y_2 observe that $p(x) = 1/x$ and $e^{-\int 1/x\,dx} = e^{-\ln x} = 1/x$. Therefore, $y_2 = x^2 \int \frac{1}{x^4} \cdot \frac{1}{x}\,dx = x^2 \cdot \dfrac{-4}{x^4} = -4/x^2$. The general solution is $y = Ax^2 + Bx^{-2}$.

5. If $y = y_1 = x^{-1/2}\sin x$, then

$$y' = x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x$$
$$y'' = -x^{-1/2}\sin x - x^{-3/2}\cos x + \frac{3}{4}x^{-5/2}\sin x$$

$x^{-1/2}\sin x(-\cot x) = -x^{-1/2}\cos x$. Therefore, the general solution is $y = Ax^{-1/2}\sin x + Bx^{-1/2}\cos x$.

7. By inspection, $y = y_1 = x$ is one solution. Since $p(x) = -xf(x)$, the second solution has the form $y_2 = x\int \frac{1}{x^2}e^{\int xf(x)\,dx}\,dx$. The general solution has the form $y = Ax + Bx\int x^{-2}e^{\int xf(x)\,dx}\,dx$.

9. If y_1 and y_2 are linearly dependent, then the function $v(x)$ is constant and has a derivative that is identically 0. However, $v'(x) = \dfrac{1}{y_1^2}e^{-\int P\,dx}$, which is never 0 (exponentials cannot vanish).

2.5 Vibrations and Oscillations

1. The amplitude $A = \dfrac{F_0}{\sqrt{(k-\omega^2M)^2+\omega^2c^2}}$ attains its maximum at the ω value that *minimizes* the polynomial $\phi(\omega) = (k-\omega^2M)^2+\omega^2c^2$. A simple calculation will show that $\phi'(\omega) = 0$ when $\omega = 0$ or $\omega = \pm\sqrt{\dfrac{k}{M} - \dfrac{c^2}{2M^2}}$. Thus if $\dfrac{k}{M} \le \dfrac{c^2}{2M^2}$, i.e. $c \ge \sqrt{2kM}$, then there is no resonance frequency and as ω increases from 0, the amplitude A will steadily decrease to 0. On the other hand, if $0 < c < \sqrt{2kM}$, then A will increase as ω increases reaching its maximum value at the $\omega^* = \sqrt{\dfrac{k}{M} - \dfrac{c^2}{2M^2}}$ and decrease to 0 thereafter. The resonance frequency is $\dfrac{1}{2\pi}\sqrt{\dfrac{k}{M} - \dfrac{c^2}{2M^2}}$. This frequency is clearly less than the natural frequency $\dfrac{1}{2\pi}\sqrt{k/M}$.

3. Let b denote the density of the buoy (weight per unit volume) and w the density of water. The volume of the buoy is $V = \frac{4}{3}\pi r^3$.

   The volume of a slice of the buoy from its center to a point y units from center is $\pi r^2 y - \frac{1}{3}y^3$ (exercise).

   Since the buoy floats half-submerged, $b = w/2$. As it bobs up and down let y be the distance from its center to the surface of the water (up is positive). If $y > 0$, then the net force on the buoy is negative given by the difference between the upward buoyant force of the water:

$$w \cdot \left(\frac{1}{2}V - \pi r^2 y + \frac{1}{3}\pi y^3\right),$$

constant)

$$\frac{b\cdot V}{g}y'' = -\pi w r^2 y + \frac{1}{3}w\pi y^3.$$

This is a second-order *non-linear* differential equation. However, if the buoy is only "slightly" depressed, then the linearized version (ignore the y^3 term) provides an excellent model for the motion. The linearized equation simplifies to $y'' + a^2 y = 0$ were $a = \sqrt{\frac{3g}{2r}}$. The period of the motion is $2\pi\sqrt{\frac{2r}{3g}}$ seconds.

5. Recall, Section 1.9 problem 4, that inside the Earth the force of gravity on an object is proportional to its distance from the center. Let x be the distance from the train to the center of a tunnel of length $2L$. Draw a picture to see that the distance from the train to the center of the Earth is $\sqrt{x^2+R^2-L^2}$ where R is the radius of the Earth. The magnitude of the force on the train, in the direction of the center of the Earth, is then $F_c = k\sqrt{x^2+R^2-L^2}$. The value of k can be found from this equation when the train is at the surface of the Earth: $mg = kR$, so $k = mg/R$.

   The magnitude of the force on the train parallel to the tracks is the component of F_c in that direction: $F_c \cdot \cos\theta = F_c \cdot \dfrac{x}{\sqrt{x^2+R^2-L^2}} = kx$. When x is positive, the force is negative. Applying Newton's Second Law we have $mx'' = -kx = -\frac{mg}{R}x$. Thus $x'' + \frac{g}{R}x = 0$, and the period of motion is independent of L: $T = 2\pi\sqrt{\frac{R}{g}}$ seconds; this is approximately 90 minutes. The equation of motion for a particular L value is found from the initial conditions: $x(0) = L$ and $x'(0) = 0$. This yields $x(t) = L\cos\sqrt{\frac{g}{R}}t$. The greatest speed is $|x'(T/4)| = \sqrt{\frac{g}{R}}L \approx 4.43L$ miles per hour.

7. Neither `Mathematica` nor `Maple` can obtain complete solutions to any of these problems. The `Maple` code for a partial solution to problem (a) appears below. Only the final output is displayed.

```
> DE:=diff(x(t),t,t)+exp(t)*diff(x(t),t)=sin(t)+cos(2*t);
> dsolve( DE );
```

$$x(t) = \int e^{(-e^t)}\left(\int e^{(e^t)}(\sin(t)+\cos(2t)\,dt + \_C1\right)dt + \_C2$$

1. Kepler's Third

   (a) In astronomy the semi-major axis of the orbit is called the *mean distance* to the Sun because it is the average of the least and greatest values of r. Let a_u and T_u denote the semi-major axis and period of Uranus. These are known from Example 2.12. According to Kepler's Third Law, $T_u^2/a_u^3 = T_m^2/a_m^3$ where a_m and T_m are Mercury's semi-major axis and period. Consequently, being careful with the units—see Example 2.12—we have

$$\begin{aligned} a_m = \left(\frac{T_m}{T_u}\right)^{2/3} \cdot a_u &= \left(\frac{\frac{88}{365} \cdot (3.16 \times 10^7)}{26.516 \times 10^8}\right)^{2/3} \cdot (2.87 \times 10^{14}) \\ &= 5.800 \times 10^{12} \text{ centimeters.} \end{aligned}$$

   This is 5.800×10^7 kilometers or approximately 36,000,000 miles.

   (b) When distance is measured in astronomical units and time in years, then $\frac{4\pi^2}{GM} = 1$ (verify). Therefore, in this system of units, $T^2 = a^3$. For example, the value of a_m calculated above can also be found (in astronomical units) using $a_m = T_m^{2/3} = (88/365)^{2/3} = 0.3874$ au. Multiply by 93,000,000 to obtain $a_m = 36,000,000$ miles.

   Regarding Saturn, $T_s = a_s^{3/2} = (9.54)^{3/2} = 29.5$ years.

3. According to Exercise 2, in the instant after the explosion, the motion of every particle that moves into an elliptical orbit about the Sun obeys the equation $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$. Consequently all of these particles move in an orbit with the same semi-major axis, a astronomical units, and (according to Kepler's Third Law) the same period, $T = a^{3/2}$ years. This means that T years later all of them will return to their original positions.

5. See Exercise 1, part (b).

   (a) $T = 2^{3/2} = 2.83$ years.

   (b) $T = 3^{3/2} = 5.20$ years.

   (c) $T = 25^{3/2} = 125$ years.

1–15. Find the general solution

	Associated Polynomial	General Solution
1.	$r(r-1)(r-2)$	$y = A + Be^x + Ce^{2x}$
3.	$(r-1)(r^2+r+1)$	$y = Ae^x + e^{-x/2}(B\cos(\sqrt{3}\,x/2) + C\sin(\sqrt{3}\,x/2))$
5.	$(r+1)^3$	$y = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x}$
7.	$(r^2-1)(r^2+1)$	$y = Ae^x + Be^{-x} + C\cos x + D\sin x$
9.	$(r-a)^2(r+a)^2$	$y = Ae^{ax} + Bxe^{ax} + Ce^{-ax} + Cxe^{-ax}$
11.	$(r+1)^2(r^2+1)$	$y = Ae^{-x} + Bxe^{-x} + C\cos x + D\sin x$
13.	$(r-1)(r-2)(r-3)$	$y = Ae^x + Be^{2x} + Ce^{3x}$
15.	$(r-6)(r-2)^2(r+2)^2$	$y = Ae^{6x} + Be^{2x} + Cxe^{2x} + De^{-2x} + Exe^{-2x}$

17. The associated polynomial is $r(r-1)(r-2)$ so the general solution to the homogeneous equation is $y_g = A + Be^x + Ce^{2x}$. Based on the forcing function our first choice for y_p is $y = A + Be^{3x}$. However, this will not work because $y = A$ is a solution to the homogeneous equation. Try $y = Ax + Be^{3x}$ instead. Substitute this into the forced equation to see that $A = 5$ and $B = 7$. The general solution is $y = A + Be^x + Ce^{2x} + 5x + 7e^{3x}$.

19. **The Euler Equidimensional Equation (order 3)** Using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \frac{1}{x}\dot{y}$ and $y'' = \frac{1}{x^2}(\ddot{y} - \dot{y})$. The dot indicates differentiation with respect to the new independent variable, z. See Section 1.1 problem 5. For the third derivative,

$$\begin{aligned} y''' = \frac{d}{dx}\left(\frac{1}{x^2}(\ddot{y} - \dot{y})\right) &= \frac{1}{x^2}(\dddot{y} - \ddot{y})\frac{1}{x} - \frac{2}{x^3}(\ddot{y} - \dot{y}) \\ &= \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}). \end{aligned}$$

Making these substitutions into $x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0$ yields $x^3 \cdot \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}) + a_2x^2 \cdot \frac{1}{x^2}(\ddot{y} - \dot{y}) + a_1x \cdot \frac{1}{x}\dot{y} + a_0y = 0$ which simplifies to

$$\dddot{y} + (a_2 - 3)\ddot{y} + (a_1 - a_2 + 2)\dot{y} + a_0y = 0,$$

an equation with constant coefficients. If $y = \phi(z)$ is the general solution to this equation, then $y = \phi(\ln x)$ will be the general solution to the Euler equidimensional equation. Note that this solution is only valid for $x > 0$.

(b) The z equation is $\dddot{y} - 2\ddot{y} - \dot{y} + 2y = 0$ with associated polynomial $r^3 - 2r^2 - r + 2 = (r^2 - 1)(r - 2)$. The solution is $y = Ae^z + Be^{-z} + Ce^{2z}$ so the solution to the original equation is $y = Ax + Bx^{-1} + Cx^2$.

(c) The z equation is $\dddot{y} - \ddot{y} + \dot{y} - y = 0$ with associated polynomial $r^3 - r^2 + r - 1 = (r - 1)(r^2 + 1)$. The solution is $y = Ae^z + B\cos z + C\sin z$ so the solution to the original equation is $y = Ax + B\cos(\ln x) + C\sin(\ln x)$.

21. The equation is

$$m_1 m_2 \frac{d^4 x_1}{dt^4} + (m_1(k_2 + k_3) + m_2(k_1 + k_3)) \frac{d^2 x_1}{dt^2} + (k_1 k_2 + k_1 k_3 + k_2 k_3) x_1 = 0.$$