Exercises

To derive the maximum benefit from these exercises, work through them sequentially. This is especially true for new users who need time and practice to become accustomed to *Mathematica* syntax and the idiosyncrasies of the *Mathematica* interface. Solutions to the Exercises marked with an asterisk will be found in the Solutions section. An important reminder for PC users: Any reference to the Command key should be replaced with the Control key on a PC.

Part I. The *Mathematica* Notebook

■ Section 1. Cell structure: Input/Output

0. Pull down the Help menu, choose Tutorial and work through it.

Open a new *Mathematica* Notebook and do the following.

1. Reduce the fraction 546/1001 to lowest terms by typing

546/1001

and pressing [Enter]. This is referred to as executing the entry.

2.* Simplify the square root of 19,220 by entering and executing

Sqrt[19220]

Then obtain the integer factorization of 19220 by entering and executing

FactorInteger[19220]

The output means that $19220 = 2^2 * 5 * 31^2$

3. * Obtain the integral of the expression $e^{3x} \cos(2x) \sin(4x)$. You may enter it into the **Integrate** function as follows

Integrate[Exp[3x] Cos[2x] Sin[4x], x]

4. Pull down the Format menu. Choose Style Sheet/Natural Color. Experiment with some of the other styles. Note. This Manual was written using the TutorialBook style.

5. Restore the Style to the original TutorialBook style. Save it with the name "MyFirstMathematicaNB".

■ Section 2. Mixing Text and Mathematics

1. Open a new *Mathematica* Notebook. Pull down the **Format** menu, choose **Style/Text** (Command-7 on a Macintosh, Control-7 on a PC) and type your name. Press the return key and type the date. Press the down arrow, then Command-7, and type Mixing Text and Mathematics. Note that this appears in a new text cell. Now press the down arrow,/Command-7 combination one more time, and type the following sentence:

This Notebook contains examples of text cells like these and input/output cells like the following.

2, Press the down arrow to get an **Input** cell and define the variable *y* as a function of *x* by typing **y = Sin[x]** and pressing [enter]. **Sin[x]** appears as the output.

3. Before continuing, use the mouse to select the third text line at the top of the worksheet, "Mixing Text and ....". Then pop down the menu of paragraph and text styles on the left side of the Context bar and choose the paragraph style named Title.

4. Go back down the Notebook and click the mouse in the white area below the last entry and define z as the derivative of y with respect to x by entering and executing **z = D[y,x]**

5. When the new output cell appears there will be a line across the page and the paragraph style should be Input. Press Command-7 to change it into a text cell and type the following:

z is the derivative of y. Both z and y are plotted below. Which one is which? How can you tell?

6. Press the down arrow to get an input cell and graph y and z as functions of x by executing the entry

Plot[{y,z}, {x,0,6}]

Note that both curves are black.

7. Click the mouse after **{x,0,6}** and type a comma followed by

PlotStyle -> {RGBColor[1,0,0],RGBColor[0,0,1]}

Press [enter] to execute the Plot function again. The y-curve is red and the z-curve is blue.

8. Click in the white space below the graph if necessary, press Command-7 and type in your answers to the questions asked above.

9. Save this Notebook with the title "My2ndNotebook".

10. Quit *Mathematica*.

11. Open the Notebook you just saved by double clicking on its icon.

12. Pull down the **Kernel** menu and choose **Delete All Output**.

13. Pull down the **Kernel** menu again and choose **Evaluation/Evaluate Notebook**. This puts the information in the Notebook back into the *Mathematica* kernel.

14. With the mouse, click in the white region below the last cell and define the variable Y as an antiderivative of z with the following entry **Y = Integrate(z,x)** .

15. Press the down arrow and Command-7. Then type the following:

Well, that wasn't hard. Now I now how to enter text and mathematics in *Mathematica*.

16.* Save your Notebook (Command-S) then print it.

Part II. Calculations and Calculus with Mathematica

■ Section 1. Getting Started: *Mathematica* as a Calculator

The following exercises provide practice using *Mathematica* to make simple calculations like the ones in the manual. Compare the output to what you can get from your calculator.

Open a new *Mathematica* worksheet and to the following.

1. Enter the following list of square roots.

$$\{\sqrt{24},\ \sqrt{864},\ \sqrt{555}\ \}$$

Enter and execute **N[%]** . Now execute **N[%%, 5]**. Now execute **Sum[%%%[[k]], {k,3}]**.

Read the Help page for **Sum** and comment on what *Mathematica* did to get the last output.

2. Execute the following entry

x = 4/(1+Sqrt[2]) + Exp[3] - Log[4]

Execute **N[%]**. Now execute **x** and note that x is not a free variable. This illustrates the fact that equations assign values in *Mathematica*.

3. Enter **x/(x+x^2)** (x is defined in 2). Then enter **Simplify[%]**. Compare the output to the output for the new entry **1/(1+x)**. Which is simpler?

4. Execute the following entry:

y = (1 + Sqrt[3])/(1 - Sqrt[3])

Enter **Simplify[y]** and then **Expand[y]**. Comment on what *Mathematica* did to get the two outputs. Now enter **FullSimplify[y]**. Comments? (Remember what "rationalize the denominator" means from high school algebra.)

5. Use *Mathematica* to simplify $\frac{x\,(x^2-1)}{x-1}$, where x is the expression entered in problem 2.

6. Use the **Table** function to make the following lists.

a. { 2, 4, 6, 8, 10, 12 }

b. { 20, 40, 60, ... , 260 }

c. The 30 prime numbers starting with 11 and ending with 139. Hint. Use **Prime[n]** whose output is the nth prime number. Name the list P.

7.* Add the numbers in the sequence P found in 6 c. Hint. Execute **Sum[P[[k]], {k,30}]** .

8. Multiply the numbers in the list P (6 c). (Use **Product**.) Then execute **IntegerDigits[%]** and **Length[%]** to count the number of digits in the product.

9. Make the list C of the cubes of the integers 2, 5, 6, 9, 12, 44. Hint: Enter the integers in a list named L, Then enter C = L^3. Add the numbers in the list of cubes and then display the prime factorization of the sum. What do you notice about the prime factorization? (Hint. The integer 87990 is called "square free".)

10.* Obtain the prime factorization of the product of the integers in the list C of cubes described in Exercise 9.

11. Use the **Factor** function to factor the following polynomial expressions. (Note. Begin by entering **x = .** to free the x variable.)

a. $x^3 - x^2 + x - 1$

b. $x^7 - x^6 + x^5 - \ldots - 1$ Hint. Enter this as **P = -Sum[(-x)^k, {k,0,7}**] then execute **Factor[P]**.

12.* Use **Solve** to obtain the zeros of the polynomials in Exercise 11.

13.* Add the zeros of the polynomial in 11 a and the zeros of the polynomial in 11 b. Hint. For example, if P is the polynomial in 11b, enter **Z = Solve[P==0, x]** and then **Sum[x/.Z[[k]], {k,3}]**

14. Multiply the zeros of the polynomial in 11 a and the zeros of the polynomial in 11 b.

■ Section 2. Symbolics: Equations and Assignments

Solving equations is the bread and butter of mathematics. *Mathematica* does it in a natural way. It is always a good idea to assign a name to the equation and the solution.

0. Pull down the **Help** menu, choose **Help Browser...** , Go to **Built-in Functions/Algebraic Computation/Equation Solving** and read each of the help pages that appear.

Open a new Maple worksheet and do the following.

1. Make the following entries, **x = y - z; y = 3; z = 4;** Then enter **x** and explain the output. When you are done enter **Clear[x,y,z]**. Why?

2. Enter the equation $x^3 - x^2 + x - 1 == 0$ with the name **eqn**. Solve the equation and name the solutions solns with the entry **solns = Solve[eqn, x]** .

a. Check the third solution with the entry **eqn/.solns[[3]]**.

b. Check all three solutions with the entry **Table[eqn/.solns[[k]], {k,3}]**.

3. Use **NSolve** to obtain an approximate real solutions to the equation $x^3 - 0.9\,x^2 + x - 1 = 0$. Name the solutions **soln**. Hint. Name the polynomial **P** and enter **soln = NSolve[P==0, x]**. Check the second approximate solution using **P/.soln[[2]]**.

4.* Plot the expression $x^3 - x^2 + 0.005\,x + \cos(x) - 0.7$ with the entries **y = x^3 - x^2 + 0.05 x + Cos[x] - 0.7** and

Plot[y, {x,-2,2}]

Then enter **FindRoot[y==0, {x,0}]** to see which zero **FindRoot** finds.

Using the graph as a guide, obtain an approximation to the largest positive zero using **FindRoot**.

5. Find the first three positive solutions to the equation cos(x) = x tan(x). Hint. Define the function y = cos(x) - x tan(x) with the entry **y = Cos[x] - x Tan[x]**. (Put a space between x and Tan[x].) Plot y using appropriate domain and range settings, then use **FindRoot** .

6. Graph the function $y = \cos(x^2) - \sin(2\,x)$ over the interval from $x = 0$ to $x = 2$. Name the Plot **Gy**. Find the derivative function and call it yp using yp = D[y,x]. Find the root of yp near $x = 1.5$. Name it **xmin**. Use the following entry to plot the graph of y and the low point.

Show[Gy, ListPlot[{{x,y}/.xmin}, PlotStyle->PointSize[0.02]]]

Start everything with **Clear[x,y]**.

7.* Find the area of the region between the graph of y in Exercise 6 and the x-axis. Hint: Numerically integrate the absolute value of y from $x = 0$ to $x = 2$ via the entry **NIntegrate[Abs[y], {x,0,2}]** .

■ Section 3. Functions as Transformations

Functions play a key role in many applications of mathematics. *Mathematica* makes functions in a natural way. The **Table** function can be used to make tables of data. Use **MatrixForm** to display the data in an array.

0. Pull down the **Help** menu, choose **Help Browser...** , Choose **The Mathematica Book/A Practical Introduction to Mathematica/Functions and Programs**. Read the Help pages listed in the last column.

Open a new Maple worksheet and do the following

1. Define the function $f(x) = \cos(x) - x\tan(x)$ Then use the entry **f ' [x]** to obtain the derivative formula.

2. Continuing 1. Plot f over the interval x = -1 to x = 1 using **Plot[f[x], {x,-1,1}]**. Then find the area of the region below the graph of f and above the x axis. Hint. This will require the values b where f(b) = 0. Find the positive value using **b = FindRoot[f[x]==0, {x,0,1}][[1,2]]**.

Hint. By symmetry, the negative zero is at $x = -b$. Check this is true by computing $f(-b)$.

3. Continuing 2. Plot the graph of the function *f* from -1 to 1 and the tangent line segment to the graph at the point x = 0.5, y = f(0.5) over the interval x = 0 to x = 1. Hint. Define the tangent line function using T[x_] := f[0.5] + f'[0.5]((x - 0.5). Then execute

Show[Plot[f[x], {x,-1,1}], Plot[T[x], {x,0,1}]]

Now jazz up the plot by making the curve red and the tangent line blue. (Just edit the Show entry.)

4. Continuing 3. Find the length of the curve plotted in Exercise 2. Hint. Do the integration numerically using

NIntegrate[Sqrt[1 + f'[x]^2], {x,-1,1}]

5.* An animation. The tangent line plot in Exercise 3 can be animated as follows. First clear T and a with **Clear[-T,a]** and define the function T(a,x) whose value at (a,x) is the formula for the tangent line to the graph of f at (a,f(a)): T[a_,x_] := f[a] + f'[a](x - a). Then load the Animation package and apply **Animate** as shown below.

```
<<Graphics`Animation`
Animate[ Show[ Plot[ f[x], {x,-1,1}, PlotStyle->RGBColor[1,0,0]],
               Plot[ T[a,x], {x,a-0.5,a+0.5}, PlotStyle->RGBColor[0,0,1]],
               PlotRange->{{-1,1},{-2,2}}
             ], {a,-1,1} ]
```

Mathematica will make 24 plots over the specified range of a values. Once this is done, collapse the 24 output cells into one by double clicking on the single blue bracket that encloses them all. Then select the collapsed bracket (it will have a down arrow indicating that there are cells collapsed inside), pull down the **Cell** menu, and choose **Animate Selected Graphics**. This should show the plots in sequence. Animation controls will appear at the bottom left of the Notebook window.

Read the Help page for animated graphics. Click on The *Mathematica* Book and type 1.9.11 into the search field.

6. Solving a differential equation. First enter **Clear[x,y]**. The **DSolve** function solves differential equations. The syntax is

DSolve[DE, y[x], x]

where DE is a differential equation for y(t) (or the name of one). Define a simple first order differential equation as follows

DE = y'[x] + x y[x] == x

Obtain the general solution to DE using the **DSolve** function as above. Then obtain the solution satisfying y(0) = 0 using

soln = DSolve({DE, y[0]==0}, y[x], x)

7.* Plot the solution in Exercise 6 using **Plot[y[x]/.soln, {x,-2,2}]**

8.* Use **Table** and **MatrixForm** to make a display of the values of the second solution in Exercise 6 at x = 0, 0.2, 0.4, ... , 1.0. Use

Table[y[x]/.soln, {x,0,1,0.2}]//MatrixForm

Now make an array displaying the values x and y(x) with the following entry

Table[{x,y[x]}/.soln[[1]], {x,0,1,0.2}]//MatrixForm

Part III. First Order Ordinary Differential Equations

■ Section 1. Entering, Solving, Plotting

Unevaluated derivatives are used to enter a differential equation, **DSolve** solves it. The **Table** function can generate and plot families of solutions satisfying specified initial conditions.

0. Pull down the **Help** menu, choose **Help Browser...** ,Go to **Built-in Functions/Algebraic Computation/Equation Solving** and read the help page for **DSolve**.

Open a new *Mathematica* worksheet and do the following

1. Enter the differential equation y' + y = sin(t). Name it DE. Obtain the general solution and the solution satisfying y(0) = 0.

2. Continuing 1. Plot the solutions to DE satisfying this initial conditions y(0) = -2, -1, 0, 1, 2. Do it by first creating a list of solutions, then plotting them as shown below.

```
 solns = Table[ DSolve[ {DE, y[0]==y0}, y[t], t], {y0,-2,2}]
Plot[ Evaluate[Table[y[t]/.solns[[k]], {k,5}]], {t,0,12}]
```

3. Obtain the general solution to the equation $y' = \frac{t}{\cos(y)}$ by entering the equation with the name DE and using **DSolve**.

4. Continuing 3. Obtain the solution to DE satisfying y(0) = 1. Call it soln. Note that there are two solution formulas. Plot both solutions over the interval [-0.2,0.2]. Are they the same? Can you tell from the solution formulas that they are the same? Hint. To make the first plot use the entry

```
Plot[ y[t]/.soln[[1]], {t,-0.2,0.2} ]
```

5. Obtain the general solution to the equation

$$y'(t) = \frac{y^2+1}{t^2+1}$$

Call is soln. Plot the solutions corresponding to C[1] = -2, -1, 0, 1, 2. Hint. Experiment with the horizontal plot range until you get nice picture displaying all 5 curves near t = 0. Use the option **Framed->True**.

Hint. *Mathematica* will not permit substitution for C[1]. Get around that by using the following initial plot entry.

```
Plot[ Evaluate[Table[ y[t]/.soln/.C[1]->c, {c,-2,2}]], {t,-1,1} ]
```

6. Continuing 5. Obtain the solution to DE satisfying y(1) = 1. Plot it over a reasonable interval containing t = 1. Use **Table** and **MatrixForm** to generate an array of solution values for $t = 0$, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, and 2.0. Hint. See Exercise 8 in Part II, Section 3.

7.* Enter the differential equation y' + y = cos(t) with the name DE. Obtain the general solution.

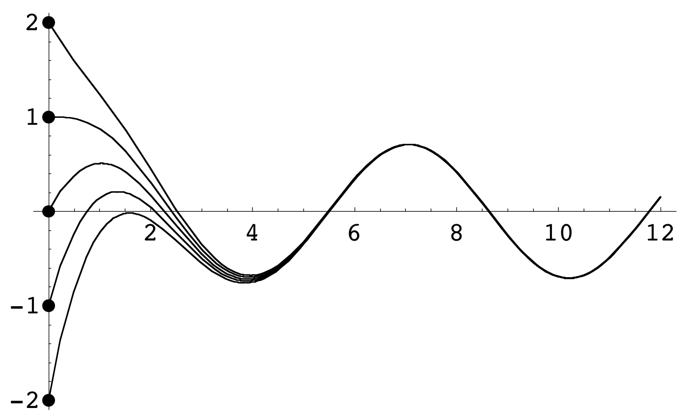
a. Plot some solutions starting at points evenly spaced on the y axis.

b. Use plot some solutions starting at points evenly spaced on the t axis.

c. Plot solutions starting at points evenly spaced around the unit circle in the style of the two plots on page 35 of the manual. That is, one picture runs time forward, another runs time backward.

Hint. Use **Table** to generate a list of solutions, then plot them. See the example below.

```
In[28]:= DE = y'[t] + y[t] == Cos[t];
         solna = Table[ DSolve[{DE,y[0]==y0},y[t],t], {y0,-2,2}];
         Show[ ListPlot[ Table[{0,k},{k,-2,2}], PlotStyle->PointSize[0.02]],
               Plot[ Evaluate[Table[y[t]/.solna[[k]],{k,5}]], {t,0,12}]]
```



8. Look before you leap. Consider the following differential equation $y'(t) = \frac{y}{t-1}$

a. What is the formula for the function f such that this equation is equivalent to $y'(t) = f(t, y)$

b. Based upon the statement of the Unique Solution Theorem at what points do you expect that solutions will fail to exist and/or fail to be unique?

c. Plot the solutions described in parts a, b, and c of Exercise 7 except replace the unit circle with the circle of radius 0.5.

9. Obtain an informative picture of the family of solutions to $y'(t) = \frac{y}{t^2-t}$ over the interval t > 1.

■ Section 2. Working with Solutions: Modeling

■ Section 3. Slope Fields

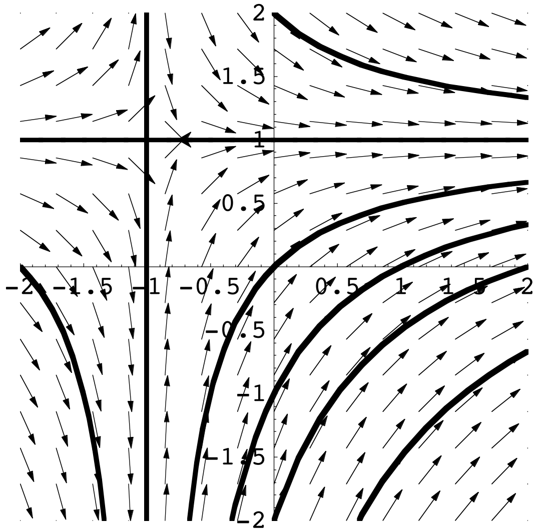
The **PlotVectorField** function plots vector fields. Normalize the vectors to obtain direction fields (or slope fields). It must be loaded from the Graphics package.

```
In[9]:= <<Graphics`PlotField`
```

Open a *Mathematica* notebook and do the following.

1. Consider the first order differential equation $(t+1)\,y'(t) = 1 - y(t)$.

a. Use **DSolve** to obtain the general solution and also the solution satisfying y(0) = 0. Plot the second solution using in the window -2 ≤ t ≤ 2, -2 ≤ y ≤ 2 and compare the picture to the solution curves shown below.



b. Comment on the existence and uniqueness of solutions using the statement of the Unique Solution Theorem. Your comments should be based upon the properties of the function f (t,y) where y' = f (t,y).

2.* Use **PlotVectorField** and to make a nice-looking slope field for the autonomous equation y' = sin(y). (Use the window -6 ≤ t ≤ 6, -6 ≤ y ≤ 6).

a. Put some solution curves into the plot using **DSolve** and **Table**. Comment on the relationship between one curve and the next.

b. Obtain the general solution formula.

c. Use **DSolve** to obtain the solution satisfying the initial condition y(0) = 1. What is the value of this solution when t = 1? Get the exact value and an approximation.

d. Comment on the long-term behavior of solutions to this differential equation.

3. Repeat Exercise 3 using the differential equation $y'(t) = \frac{1}{\sin(y(t))}$. Use the same plot window. Note that the symbolic solutions are much simpler than the ones in Exercise 3. Explain why.

■ Section 4. Approximate Solutions

Iterated functions and user defined procedures are featured in this set of exercises.

0. Pull down the **Help** menu, choose **Help Browser...** . Then choose **The Mathematica Book/Principals of Mathematica/Modularity and the Naming of Things**. Read the pages explaining Modules and local variables.

Open a new *Mathematica* notebook and do the following.

1. Create the modular function called euler defined on in Part 3, Section 4 of the manual. Test it on the differential equation y' = cos(t) y with the initial condition y(0) = 1 (as in the manual).

2. Use euler to obtain a tabular display of Euler approximations to the initial value problem (IVP)

$$y'(t) = \frac{8 e^{-t}}{3+y(t)}, \; y(0) = 1$$

for t = 0, 0.2, 0.4, 0.6, 0.8, 1.0. (I.e. h = 0.2). Plot these points and the line segments connecting them along with the actual solution. Name the plot P1. Note. You will have to create the solution using **DSolve**.

3. (Continuing 2) Make a similar plot named P2, displaying the solution and the approximation for h = 0.1.

Use the **Show** function to display plots P1 and P2 together. Comment on the error displayed in the picture.

4. (Continuing 3) Use your calculator to check that the Error has been cut in half (approximately) when going from h = 0.2 to h = 0.1.

5. Create the modular function called impeuler and test it on y' = cos(t) y with the initial condition y(0) = 1 (as in the manual).

6. Repeat 2 - 5 using impeuler in place of euler. Comment on the graphs and the reduction of error in the Matrix. Hint. Use Copy and Paste, then make minor changes in the code.

7.* Modify the impeuler procedure to make a modular function that implements the classical Runge-Kutta algorithm (see Simmons/Krantz, Chapter 9, Section 5). Test in by applying it to the problems described in Exercises 2 - 6. (Copy and Paste). Is the error cut in half in problem 4?

Part IV. Linear Differential Equations

■ Section 1. Linear Oscillators

The harmonic oscillator is the fundamental model for the analysis of oscillating systems. Phase plane trajectories are constructed. PlotVectorField draws direction fields.

1. Obtain the solution to the following initial value problem. Call it soln.

$$y'' + 4y = 0 \;, \; y(0) = 2 \;, \; y'(0) = -3$$

From the form of the solution decide if the system is undamped, underdamped, critically damped, or overdamped. What is the period of the oscillations?

2, (Continuing 1) Plot the solution to the IVP in Exercise 1. Then create a plot showing the cosine term, the sine term and their sum (the solution curve). Make the solution red, the cosine blue, and the sine green. Hint. The cosine term is the solution using the initial conditions y(0) = 2, y'(0) = 0.

a. What IVP does the sine term solve?

b. Determine the amplitude of the oscillations by solving y'(t) = 0 and substituting the time value into the solution. Compare the answer to the amplitude calculated using the standard formula for converting the solution into amplitude/phase angle form.

c. Assuming this is the model of a mass spring system, determine the speed of the mass as it passes through equilibrium.

d. Convert the solution into the function g. Use g to plot the phase plane trajectory. What type of curve is this trajectory?

e. Add to the trajectory the points corresponding to t = 0, 0.25, 0.5, 0.75, ... , 2.0.

2.* Consider now the following damped system. Obtain the solution.

$$y'' + y' + 4y = 0 \;,\; y(0) = 2 \;,\; y'(0) = -3$$

From the form of the solution decide if the system is underdamped, critically damped, or overdamped.

a. What is the pseudo-period of the oscillations?

b. What is the time constant?

c. Based upon your answer to part b estimate the time interval required for the oscillations to disappear from view.

d. Plot the solution curve over the interval you named in part c.

e. Add to the curve in part d the curves defined by $A\,e^{-t/2}$ and $-A\,e^{-t/2}$ where $A = \sqrt{4 + \frac{16}{15}}$. Make them blue. What is the significance of these curves? Where did the formula for *A* come from?

f. Convert the solution into the function g. Use g to plot the phase plane trajectory.

g. Add to the trajectory the points corresponding to t = 0, 0.25, 0.5, 0.75, ... , 2.0.

■ Section 2. State Space

The forced oscillator is modeled with a non-autonomous equation. Solution trajectories are best viewed in state space.

1. Consider the driven IVP

$$y'' + 4y = \cos(1.8\,t) \;,\; y(0) = 2 \;,\; y'(0) = -3$$

Obtain the solution, call it soln and convert it into a function g.

a. Plot the solution curve over the interval $0 \le t \le 120$. What you witness in the plot is the phenomenon called "beats". The output pulsates like this when the driver frequency is very close to the natural frequency of the system.

b. Obtain the phase plane trajectory for this system. Use the same time interval.

c. Obtain the state space trajectory for this system. Use the same time interval.

2.* Damp the system slightly by changing the IVP to the following

$$y'' + 0.1y' + 4y = \cos(1.8\ t)\ ,\ y(0) = 2\ ,\ y'(0) = -3$$

Obtain the solution, call it soln and convert it into a function h.

a. Based upon the solution formula determine the time constant for the beats. That is, how long will it take (approximately) for the beats to disappear from the solution curve as it settles down to its steady-state mode?

b. Plot the solution curve to verify your answer to part a.

c. Use h to obtain the phase plane and state space trajectories for this system.

3. Use **PlotVectorField** to obtain the direction field and a plot of the solution trajectory in Exercise 1 of Section 1 in Part IV. (The direction field can be drawn in phase space because the equation is autonomous.)

4.* Obtain a numeric solution for the IVP in Exercise 2. Sketch the time series for position and the time series for velocity. Then plot the phase plane trajectory and the state space trajectory.

■ Section 3. Two Dimensional Systems

A two dimensional system defines a two dimensional vector field and a vector flow. The **NDSolve** function outputs approximate solutions. Linear systems can be solved using standard methods by reduction to one second order equation or by matrix methods (featured in the next section).

Load the **PlotField** package.

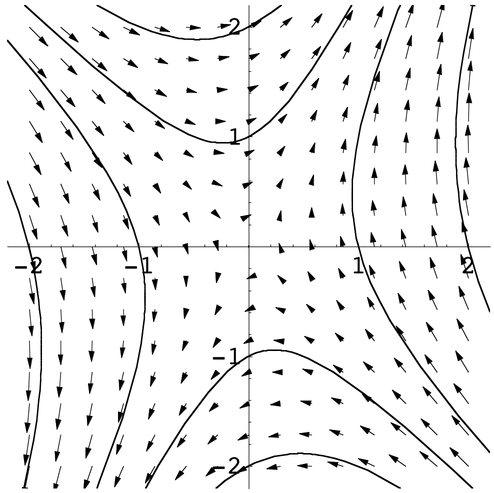
```
<<Graphics`PlotField`
```

1. Obtain the phase portrait for the linear system

$$x' = -x + 2y$$

$$y' = 4x + y$$

Note. The phase portrait is a sketch of some solution curves and the direction field when the system is autonomous. Compare the picture you get to the following picture.



Hint. Make the vector field using **PlotVectorField**. Use **Table** to generate lists of solutions satisfying initial conditions specified along the x and the y axes.

2. Continuing 1. Use **DSolve** to obtain the general solution to the system in Exercise 1. Call the solution soln.

3. Add the two nullclines to the phase portrait in Exercise 1. (What is the stationary point?) Hint. Go back and name the plot PP. Make the nullclines using **ImplicitPlot** (both can be drawn at the same time). Call the nullcline plot NP and then use Show(PP, NP);

4.* The model

$$x' = x\,(1 - y - x/a)$$

$$y' = y\,(1 - x - y/b)$$

is used to study competing species. Use **PlotVectorField** and **DSolve** to do the following.

a. Draw the direction field when $1/a = 1.9$ and $1/b = 1.5$. Use the window $0 \le x \le 1$, $0 \le y \le 1$.

b. Find the stationary point and add the nullclines to the direction field. Discuss the solutions based upon the picture you see.

c. Add solution curves corresponding to the following set of initial conditions (a circle of points around the stationary point) $x(0) = 0.3 + 0.2\cos(k\,\pi/6)$, $y(0) = 0.5 + 0.2\sin(k\,\pi/6)$, $k = 1, 2, \ldots, 12$

5. Use **PlotVectorField** to obtain the phase portrait for the system

$$x' = x - 2\,y - 1$$

$$y' = x - y - 2$$

Add the stationary point and nullclines. Find the solution formulas using **DSolve**. Display the solutions in simplified form.

This is a linear system with periodic solutions (closed curves). Use the solution formulas to determine the period of the trajectories.

■ Section 4. Matrix Methods

Matrix and vector manipulation exercises provide practice via eigenvalue and eigenvector calculations. The matrix exponential is a key tool for solving constant coefficient linear systems of differential equations.

0. Pull down the **Help** menu, choose **Help Browser...** . Choose **The Mathematica Book/Advanced Mathematics in Mathematica/Linear Algebra**. Read the help pages.

Use *Mathematica* to do the following.

1. Enter the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ with the name A.

a. Find the determinant and the characteristic polynomial of A.

b. Enter the vector v = {2,3,5}. Calculate the vector w = Av using the entry w = A.v.

c. Calculate the dot product of v and w using v.w. The product will be a scalar. Verify that the entry w.v yields the same scalar.

2.* Define the matrix B having the columns {0,1,-1}, {1,1,0}, {-1,0,1}. Hint.

B = Transpose[{ {0,1,-1}, {1,1,0}, {-1,0,1} }]

a. Find the characteristic polynomial of B and factor it.

b. Find B's eigenvalues with **Eigenvalues[B]**. Name them lambda.

c. Find B's eigenvectors using **Eigenvectors[B]**. Name them V and define P to be the *transpose* of V.

d. Calculate **Inverse[P].B.P**.

3.* Continuing 2. Obtain the solution to **v' = Bv** satisfying **v**(0) = {1,2,3}}.

a. Do if first using **DSolve** applied to the system of linear differential equations defined by v' = Bv with the appropriate initial conditions.

b. Do it second by making a fundamental matrix solution X(t) defined as the matrix with the eigenvector solutions in the columns then computing

$$v(t) = X(t)\, X(0)^{-1}\, v(0)$$

c. Do it third by using the Matrix Exponential, Mexp. Once you have it, the solution is

$$v(t) = \text{Mexp } v(0)$$

■ Section 5. The Laplace Transform

The Laplace transform and the inverse Laplace transform are used to solve linear differential equations and systems. Piecewise defined functions are defined using the unit step function. Dirac delta functions in the driver can be handled via Laplace transforms.

Use *Mathematica* to do the following problems.

1. Obtain the Laplace transform of the following functions.

$$t\cos(3\,t)\,,\quad e^{t}\,\sin(2\,t)\,,\quad \text{UnitStep}(t-\pi)\,(t+\cos(t))\,,\quad \sin(t)\;+\;\text{DiracDelta}(t-3\,\pi)$$

2. Obtain the inverse Laplace transform of the following functions.

$$\frac{s}{s^2-4}\,,\quad \frac{s\,e^{-2\,s}}{s^2-4}\,,\quad \frac{1}{s^2+4\,s-5}\,,\quad \frac{1}{s^2+4\,s-4}\,,\quad \frac{e^{-3\,s}}{s^4}\,,\quad \text{UnitStep}(t-3)+(t-4)\,\text{UnitStep}(t-5)$$

3. Consider the following initial value problem

$$y'' + y' + 9\,y = \text{UnitStep}(t - 2\,\pi)\;\;,\;\; y(0) = 0\;,\; y'(0) = 1$$

a. Obtain the solution using **DSolve**. Plot it for $0 \le t \le 40$ and explain the behavior of the solution curve.

b. Obtain the solution using the method of Laplace transforms. How does the solution formula compare to the formula obtained in part a?

4. Consider the following initial value problem

$$y'' + y' + 9\,y = \text{UnitStep}(t - 2\,\pi) - \text{UnitStep}(t - 5\,\pi)\;\;,\;\; y(0) = 0\;,\; y'(0) = 1$$

a. Obtain the solution using DSolve. Plot it for $0 \le t \le 40$ and explain the behavior of the solution curve.

b. Obtain the solution using the method of Laplace transforms. How does the solution formula compare to the formula obtained in part a?

5.* Consider the following initial value problem

$$y'' + y' + 9\,y = \text{UnitStep}(t - 2\,\pi) - \text{UnitStep}(t - 5\,\pi) + 2\,\text{DiracDelta}(t - 9\,\pi)\;\;,\;\; y(0) = 0\;,\; y'(0) = 1$$

a. Obtain the solution using **DSolve**. Plot it for $0 \le t \le 40$ and explain the behavior of the solution curve.

b. Obtain the solution using the method of Laplace transforms. How does the solution formula compare to the formula obtained in part a?

6.* Use the solution to 5 a to make a function, g. Use g to plot the phase plane trajectory and as well as the state space trajectory for the system.

7.* Continuing 6. Plot an animation of the phase space trajectory.