

Mathematica Technology Manual

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to accompany

Differential Equations: Theory, Technique, and Practice

by

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Preface

This manual provides a straightforward and practical introduction to the computer algebra system *Mathematica* for students and instructors who wish to use it in the study of ordinary differential equations. Readers will find what they need to get a quick start with the program in the context of simple numerical calculations, fundamental calculus computations, and elementary graphing techniques. Applications are interwoven with explanations of how *Mathematica* works and examples illustrating the important features of this important software tool.

Once a sufficient level of confidence has been attained, the reader is guided through *Mathematica* procedures designed to solve differential equations, analyze the behavior of solutions, and use solutions, both exact and approximate, in the mathematical modeling process. The exercises can be used to clarify the reader's understanding of how the procedures work. Annotated examples guide the reader through the more challenging problems.

Prerequisites and Corequisites

Instructions in the manual are written with the assumption that the reader knows how to start a computer and make a document using a word processor. The ability to use a mouse: point and click, select and drag, pull down a menu, etc. will be taken for granted. The only mathematics prerequisite is the successful completion of a university level calculus course or its equivalent. As a corequisite it is hoped that the reader is actively engaged in the learning of the fundamentals of ordinary differential equations. Please note that this is not a text in differential equations. Indeed, the reader will often be referred to a differential equations textbook (Simmons and Krantz) for statements of definitions, algorithms, solution techniques, and solution formulas, as well as background material for some of the examples and exercises.

Part I. The *Mathematica* Notebook

This is a brief overview of *Mathematica*'s notebook interface. The essential components of a notebook are described and simple examples illustrate how to enter and process mathematics and text.

Part II. Calculations and Calculus with *Mathematica*

The second part contains a more detailed introduction to *Mathematica* and how to use it to "do mathematics." The input/output paradigm is stressed as the reader learns how to use *Mathematica* as a calculator. The very useful `Table` function is discussed. *Mathematica* output can be used as subsequent input by assigning the output a name, and then referring to it by that name just as one might do using paper and pencil. The notion of assignments are important for the successful use of any mathematics software to a problem that requires more than one calculation. This idea is illustrated several times using familiar examples from calculus. Graphing examples introduce plotting procedures.

Part III. First Order Ordinary Differential Equations

The discussion of differential equations in *Mathematica* starts here in the context of mathematical models requiring first order equations in their analysis. Readers who are already familiar with *Mathematica* are encouraged to skim over Parts I and II and begin in Part III. Direction fields and exact solution curves are plotted. Implementation of Euler's method provides the reader with an opportunity to learn how to write a *Mathematica* program that can easily be converted into a user-defined procedure.

Part IV. Linear Differential Equations

Linear differential equations, linear systems, and Laplace transforms are featured in this part of the manual. Procedures in the `LinearAlgebra` package are introduced to handle vectors and matrices gracefully.

Exercises

Exercise sets are divided by Part and Section. Solutions to asterisked exercises appear in the solution section.

Appendix

The appendix, which itself is divided into parts 1 - 5 contains miscellaneous items that did not fit nicely into Part I, II, III, or IV: Power series and special functions, Picard iterates, Partial differential equations, and some Sound advice and encouraging words.

Solutions to Selected Exercises

A list of solutions to the exercises that appear with an asterisk.

The Table of Contents that follows is detailed enough to be used as a complete outline of the topics that are treated in the manual.

Table of Contents

Part I. The *Mathematica* Notebook

Section 1. Cells: Input/Output.....	1
Input comments: Use a hash mark #.....	1
The pull down menus.....	2
Section 2. Entering Text.....	4
Annotating input and output: An example.....	4
Undo it: Command-Z.....	5
Delete all of it: Select the cell bracket and press [delete].....	5

Part II. Calculations and Calculus with *Mathematica*

Section 1. Getting Started: <i>Mathematica</i> as a Calculator.....	6
Parentheses for grouping must be the round kind: ().....	8
Lists in <i>Mathematica</i>	9
The Table function.....	9
Using the Table function.....	10
The Table function is very versatile.....	11
Moving towards symbolic calculations: Factoring in <i>Mathematica</i>	12
Section 2. Symbolics: Equations and Assignments.....	14
Variables, equations, and assignments.....	14
Solving equations exactly: The Solve function	16
Approximate solutions using NSolve and FindRoot.....	17
More about Solve and FindRoot.....	18
Using variables as functions.....	20
Section 3. Functions as Transformations.....	23
Using a transformation to define a function.....	23
Plot several tangent lines.....	25

Find the length of the curve.....	26
An area calculation.....	28
Higher derivatives.....	29
Unevaluated derivatives.....	31
The chain rule.....	31
Just a little bit about differential equations.....	32
Working with the solution.....	33
Getting at the solution formula.....	33
 Part III. First Order Ordinary Differential Equations	
Section 1. Entering and Solving.....	35
Solve this: DSolve.....	35
Checking the solution.....	35
Plotting solution curves.....	36
Solving an initial value problem.....	36
Getting information out of the solution formula.....	37
A solution formula in terms of the initial condition.....	38
Look before you leap.....	40
Section 2. Working with Solutions: Modeling.....	45
Exponential Models.....	45
A more interesting problem.....	46
Monthly payments.....	48
Heating and cooling.....	48
Variable ambient temperataure.....	50
Section 3. Slope Fields.....	54
The tool you need: PlotVectorField.....	54
Keep your eye on $f(t,y)$	57

Section 4. Approximate Solutions.....	61
One step at a time: Euler's algorithm.....	61
Make it yours: User defined procedures.....	63
Modify it: The Euler two step algorithm.....	65
Using NDSolve to generate numeric solutions.....	66
A preview of second order equations: An aging spring.....	68
Part IV. Linear Differential Equations	
Section 1. Linear Oscillators.....	70
Solve this too: DSolve and the unforced equation.....	70
The phase plane trajectory.....	72
Simple harmonic motion.....	74
The phase plane trajectory for underdamped oscillations.....	76
Phase plane trajectories and vector fields.....	78
Section 2. State Space.....	81
Autonomous equations.....	81
Force it: A non-autonomous equation.....	83
State space trajectories: The ParametricPlot3D function.....	86
Numerical solutions: NDSolve.....	88
Section 3. Two Dimensional Systems.....	91
Fields and flows.....	91
Estimating the period: Numeric data.....	93
Nullclines, critical points.....	95
Symbolic solutions, integral curves.....	96
Linear systems, linear flows.....	98
Section 4. Matrix Methods.....	101
Matrices and Vectors.....	101

Eigenvectors and eigenvalues.....	105
Eigenvector solutions to $v' = Av$	107
The matrix exponential.....	109
A three dimensional example, approximate solutions.....	109
Section 5. The Laplace Transform.....	112
Laplace transforms and the inverse Laplace transform.....	112
Typical driver: Off, then on, then off again.....	114
The Dirac delta: DiracDelta.....	116
Applying the Laplace transform method to a linear system.....	117

Exercises

Appendices

- A1. Power Series and Special Functions
- A2. Picard Iteration
- A3. Partial Differential Equations
- A4. Sound advice and encouraging words

Selected Solutions