
Chapter

4

Function Minimization Algorithms

In this chapter, we will look at two approaches to finding all of the prime implicants of a function and then algorithms for finding minimum sum of products solutions. We will then extend the approaches to problems with multiple outputs.

The first approach to finding prime implicants is referred to as the *Quine-McCluskey method*. It starts with minterms and uses, repeatedly, the adjacency property

$$ab + ab' = a$$

The second approach is *iterated consensus*. It starts with any set of terms that cover the function and uses the consensus operation and the absorption property

$$a + ab = a$$

Each of these methods has been computerized and is effective for a larger number of variables than the Karnaugh map, although the amount of computing becomes excessive for many practical problems.

4.1 QUINE-McCLUSKEY METHOD FOR ONE OUTPUT

In this section, we will use the Quine-McCluskey method to list all of the prime implicants of a function. In Section 4.3, we will use that list to find the minimum sum of products expression(s) for that function. We start with a list of minterms, in numerical form (that is, 1 for an

uncomplemented variable and 0 for a complemented one). If we start with minterm numbers, this is just the binary equivalent of the minterm number. We order this list by the number of 1's in each terms. We will use the function of Example 3.4:

$$f(w, x, y, z) = \Sigma m(0, 4, 5, 7, 8, 11, 12, 15)$$

Our initial list, grouped by the number of 1's, is

```

A  0 0 0 0
-----
B  0 1 0 0
C  1 0 0 0
-----
D  0 1 0 1
E  1 1 0 0
-----
F  0 1 1 1
G  1 0 1 1
-----
H  1 1 1 1

```

where we have labeled the terms for easy reference.

We now apply the adjacency property to each pair of terms. Since that property requires all the variables to be the same except for one, we need only consider terms in consecutive groups. We produce a second column of terms with one variable missing:

```

A + B = J = 0 - 0 0 (where the dash represents a missing
                    variable)
A + C = K = - 0 0 0
B + D = L = 0 1 0 -
B + E = M = - 1 0 0
C + D = none
C + E = N = 1 - 0 0
D + F = O = 0 1 - 1
D + G = none
E + F = none
E + G = none
F + H = P = - 1 1 1
G + H = Q = 1 - 1 1

```

4.1 Quine-McCluskey Method for One Output

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Whenever a term is used to produce another term, it is checked off; it is not a prime implicant. These (three literal) terms are placed in a second column as shown in Table 4.1. All of the minterms have been covered by at least one term in the second column; thus, no minterms are prime implicants.

Table 4.1 Quine-McCluskey prime implicant computation.

<i>A</i> 0 0 0 0 ✓	<i>J</i> 0-00 ✓	<i>R</i> --00
-----	<i>K</i> -000 ✓	
<i>B</i> 0 1 0 0 ✓	-----	
<i>C</i> 1 0 0 0 ✓	<i>L</i> 0 1 0-	
-----	<i>M</i> -1 0 0 ✓	
<i>D</i> 0 1 0 1 ✓	<i>N</i> 1-00 ✓	
<i>E</i> 1 1 0 0 ✓	-----	
-----	<i>O</i> 0 1-1	
<i>F</i> 0 1 1 1 ✓	-----	
<i>G</i> 1 0 1 1 ✓	<i>P</i> -1 1 1	
-----	<i>Q</i> 1-1 1	
<i>H</i> 1 1 1 1 ✓		

We now repeat the process with the second column. Again, we need only consider terms in consecutive sections of that column (number of 1's differing by only one). Also, we need only consider terms with dashes in the same position, since they are the only ones with the same three variables. Thus, we find

$$J + N = R = --00$$

$$K + M = R \quad (\text{same term})$$

There are no adjacencies between the second and third group or between the third and fourth group.

Since there is only one term in the third column, we are done. If there were more terms, we would repeat the process, forming a column with three literals missing (corresponding to a group of eight minterms). The prime implicants are

$$L \quad 0 \ 1 \ 0 \ - \quad w'xy'$$

$$O \quad 0 \ 1 \ - \ 1 \quad w'xz$$

$$P \quad - \ 1 \ 1 \ 1 \quad xyz$$

$$Q \quad 1 \ - \ 1 \ 1 \quad wyz$$

$$R \quad - \ - \ 0 \ 0 \quad y'z'$$

If there are don't cares in the problem, all of them must be included, since don't cares are part of prime implicants.

EXAMPLE 4.1

$$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$$

The process proceeds as before

$$\begin{array}{rcl}
 0000\checkmark & 000- & --00 \\
 ----- & 0-00\checkmark & \\
 0001\checkmark & -000\checkmark & \\
 0100\checkmark & ----- & \\
 1000\checkmark & 00-1 & \\
 ----- & 01-0 & \\
 0011\checkmark & -100\checkmark & \\
 0110\checkmark & 10-0 & \\
 1010\checkmark & 1-00\checkmark & \\
 1100\checkmark & ----- & \\
 ----- & -011 & \\
 1011\checkmark & 101- & \\
 1101\checkmark & 110- &
 \end{array}$$

Thus, the prime implicants are

$$\begin{array}{ll}
 w'x'y' & x'yz \\
 w'x'z & wx'y \\
 w'xz' & wxy' \\
 wx'z' & y'z'
 \end{array}$$

Although wxy' and $wx'z'$ are prime implicants, they consist of all don't cares and would never be used in a minimum solution. Indeed, as we will see in Section 4.3, only three of these are needed for a minimum solution.

[SP 1; EX 1]

This process works for larger number of variables, but the number of minterms and other implicants can increase rapidly. We will see one example with five variables in the solved problems. This process has been computerized.

4.2 ITERATED CONSENSUS FOR ONE OUTPUT

In this section, we will use the iterated consensus algorithm to list all of the prime implicants of a function. In the next section, we will use that list to find the minimum sum of products expression(s).

To simplify the discussion, we will first define the relationship *included in*.

Product term t_1 is *included in* product term t_2 (written $t_1 \leq t_2$) if t_2 is 1 whenever t_1 is 1 (and elsewhere, too, if the two terms are not equal).¹

¹The relationship *included in* is also applied to more complex functions than product terms, but that will not be important here.

All this really means for product terms is that either $t_1 = t_2$, or $t_1 = xt_2$, where x is a literal or a product of literals. From the perspective of the map, it means that t_1 is a subgroup of t_2 . If an implicant, t_1 , is included in another implicant, t_2 , then t_1 is not a prime implicant since

$$t_1 + t_2 = xt_2 + t_2 = t_2 \quad \text{[P12a]}$$

The iterated consensus algorithm for single functions is as follows:

1. Find a list of product terms (implicants) that cover the function. Make sure that no term is equal to or included in any other term on the list. (These terms could be prime implicants or minterms or any other set of implicants. However, the rest of the algorithm proceeds more quickly if we start with prime implicants.)
2. For each pair of terms, t_i and t_j (including terms added to the list in step 3), compute $t_i \not\subset t_j$.
3. If the consensus is defined, and the consensus term is not equal to or included in a term already on the list, add it to the list.
4. Delete all terms that are included in the new term added to the list.
5. The process ends when all possible consensus operations have been performed. The terms remaining on the list are ALL of the prime implicants.

Consider the following function (Example 3.4 from Chapter 3 and the function we used to describe the Quine-McCluskey method in Section 4.1).

$$f(w, x, y, z) = \Sigma m(0, 4, 5, 7, 8, 11, 12, 15)$$

We chose as a starting point a set of product terms that cover the function; they include some prime implicants and a minterm, as well as other implicants.

$$\begin{array}{l} A \quad w'x'y'z' \\ B \quad w'xy' \\ C \quad wy'z' \\ D \quad xyz \\ E \quad wyz \end{array}$$

We labeled the terms for reference and go in the order, $B \not\subset A$, $C \not\subset B$, $C \not\subset A$, $D \not\subset C$, . . . , omitting any computation when the term has been removed from the list. When a term is removed, we cross it out. The first consensus, $B \not\subset A$, produces $w'y'z'$; A is included in that term and can thus be removed. After the first step, the list becomes

$$\begin{array}{ll} \del{A} \quad \del{w'x'y'z'} & D \quad xyz \\ B \quad w'xy' & E \quad wyz \\ C \quad wy'z' & F \quad w'y'z' \end{array}$$

Table 4.2 Computing the prime implicants.

A	w'x'y'z'	
B	w'xy'	
C	wy'z'	
D	xyz	
E	wyz	
F	w'y'z'	B $\not\subset$ A \geq A (remove A)
G	xy'z'	C $\not\subset$ B
H	w'xz	D $\not\subset$ C undefined
		D $\not\subset$ B
		E $\not\subset$ D undefined
		E $\not\subset$ C undefined
		E $\not\subset$ B undefined
		F $\not\subset$ E undefined
		F $\not\subset$ D undefined
J	y'z'	F $\not\subset$ C \geq G, F, C
		(remove G, F, C)
		H $\not\subset$ E = D (do not add)
		H $\not\subset$ D undefined
		H $\not\subset$ B undefined
		J $\not\subset$ H = B (do not add)
		J $\not\subset$ E undefined
		J $\not\subset$ D undefined
		J $\not\subset$ B undefined

We next find $C \not\subset B$, which creates term $G, xy'z'$; it is not included in any other term and no other term is included in it. There is no need to compute $C \not\subset A$, since term A has already been removed from the list.

The complete computation is shown in Table 4.2, where each possible consensus is listed on a separate line.

The terms that remain, $B, D, E, H,$ and J , that is, $w'xy, xyz, wyz, w'xz,$ and $y'z'$, are all the prime implicants. The minimum sum of product expression(s) will use some of these, typically not all of them.

The process can be simplified by using a numeric representation of the terms. As in the truth table, a 0 represents a complemented variable, and a 1 represents an uncomplemented variable. If a variable is missing from a term, as we did in Quine-McCluskey, a dash (-) is used in its place so that each term has four entries. A consensus exists if there is a 1 for exactly one variable in one term and a 0 for that variable in the other. The consensus term has a 1 for a variable if one term has a 1 and the other either a 1 or a -; it has a 0 if one term has a 0 and the other a 0 or a -, and a - if one term has a 0 and the other a 1 or if both terms have a -. For the function of Table 4.2, the process becomes that of Table 4.3 (where we have not left lines for consensus operations that are undefined).

The five terms remaining in Table 4.3 are the same as those in Table 4.2.

Table 4.3 Numeric computation of prime implicants.

A	0	0	0	0	
B	0	1	0	-	
C	1	-	0	0	
D	-	1	1	1	
E	1	-	1	1	
F	0	-	0	0	B $\not\subset$ A \geq A
G	-	1	0	0	C $\not\subset$ B
H	0	1	-	1	D $\not\subset$ B (D $\not\subset$ C undefined)
					(E $\not\subset$ D, E $\not\subset$ C, E $\not\subset$ B, F $\not\subset$ E, F $\not\subset$ D undefined)
J	-	-	0	0	F $\not\subset$ C \geq G, F, C
					(H $\not\subset$ E = D; H $\not\subset$ D, H $\not\subset$ B undefined; J $\not\subset$ H = B; J $\not\subset$ E, J $\not\subset$ D, J $\not\subset$ B undefined)

If there are don't cares in the function, all of them must be included in at least one of the terms to start the process. The resulting list of prime implicants will then include all possible prime implicants (including possibly some that are made up of only don't cares). The prime implicant table will then allow us to choose the minimum cover.

$$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$$

EXAMPLE 4.2

w x \ y z	00	01	11	10
00	X	1	X	X
01	1		X	
11	1			1
10		1		X

Using the map above, we chose the following list of implicants as a starting point:

- $A \quad y'z' \quad - \quad - \quad 0 \quad 0$
 $B \quad w'x'z \quad 0 \quad 0 \quad - \quad 1$
 $C \quad w'xyz' \quad 0 \quad 1 \quad 1 \quad 0$
 $D \quad wxy' \quad 1 \quad 1 \quad 0 \quad -$
 $E \quad wx'y \quad 1 \quad 0 \quad 1 \quad -$

All of these, except the third, are prime implicants. It does not matter what set of terms we start with (as long as all of the 1's and don't cares are included in at least one term); we will get the same result. By choosing a pretty good cover, we will create few if any extraneous terms. The process then proceeds:

- $A \quad - \quad - \quad 0 \quad 0$
 $B \quad 0 \quad 0 \quad - \quad 1$
 ~~$C \quad 0 \quad 1 \quad 1 \quad 0$~~
 $D \quad 1 \quad 1 \quad 0 \quad -$
 $E \quad 1 \quad 0 \quad 1 \quad -$
 $F \quad 0 \quad 0 \quad 0 \quad - \quad B \not\subseteq A$
 $\quad \quad \quad \quad \quad \quad \quad C \not\subseteq B$ undefined
 $G \quad 0 \quad 1 \quad - \quad 0 \quad C \not\subseteq A \geq C$
 $\quad \quad \quad \quad \quad \quad \quad D \not\subseteq B, D \not\subseteq A, E \not\subseteq D$ undefined
 $H \quad - \quad 0 \quad 1 \quad 1 \quad E \not\subseteq B$
 $J \quad 1 \quad 0 \quad - \quad 0 \quad E \not\subseteq A$
 $F \not\subseteq E, F \not\subseteq D, F \not\subseteq B, F \not\subseteq A$ undefined, $G \not\subseteq F = 0 - 0 0 \leq A$;
 $G \not\subseteq E$ undefined; $G \not\subseteq D \leq A$; $G \not\subseteq B, G \not\subseteq A, H \not\subseteq G$ undefined;
 $H \not\subseteq F = B$; $H \not\subseteq E, H \not\subseteq D, H \not\subseteq B, H \not\subseteq A$, undefined; $J \not\subseteq H = E$;
 $J \not\subseteq G, J \not\subseteq E, J \not\subseteq B, J \not\subseteq A$ undefined; $J \not\subseteq F \leq A, J \not\subseteq D \leq A$

Thus, all terms but term C are prime implicants. Although there are eight prime implicants, only three are used in any minimum solution (as we will see in the next section).

[SP 2; EX 2]

4.3 PRIME IMPLICANT TABLES FOR ONE OUTPUT

Once we have a complete list of prime implicants, using either Quine-McCluskey or iterated consensus, a table is constructed with one row for each prime implicant and one column for each minterm included in the function (not don't cares). An X is entered in the column of a minterm that is covered by that prime implicant. Thus, for the prime implicants of the first function, f , in both Sections 4.1 and 4.2, the prime implicant table is shown in Table 4.4.

Table 4.4 A prime implicant (PI) table.

PI	Numeric	\$	Label	0	4	5	7	8	11	12	15
$w'xy'$	0 1 0 -	4	A		X	X					
xyz	- 1 1 1	4	B				X				X
wyz	1 - 1 1	4	C						X		X
$w'xz$	0 1 - 1	4	D			X	X				
$y'z'$	- - 0 0	3	E	X	X			X		X	

The first column is the list of prime implicants in algebraic form; the second is in numeric form.² The latter makes it easy to find a list of minterms that are covered by this term, since each - can represent either a 0 or a 1. The third column is the number of gate inputs when that term is used in a two-level circuit, that is, just one for each literal plus one for the input to the output gate (OR). The fourth column is just the label (to save writing the whole term later). We will label terms in alphabetic order. (They may differ from the labeling of these terms in Sections 4.1 and 4.2.)

Our job is to find a minimum set of rows such that using only these rows, every column has at least one X, that is, all of the minterms are included in the expression. If there is more than one set, the total number of gate inputs (\$ column) is minimized. The first step in the process is to find essential prime implicants. They correspond to rows where the X is the only one in at least one column. Those squares are shaded; the minterms covered by each of the essential prime implicants are checked

²The order of the list is not important. The two methods used to find prime implicants produced the same list but in different order.

4.3 Prime Implicant Tables for One Output

off; and an asterisk is placed next to the prime implicant as shown in Table 4.5.

Table 4.5 Finding essential prime implicants.

PI	Numeric	\$	Label	<div style="display: flex; justify-content: space-around; font-size: small;"> ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ </div>								
				0	4	5	7	8	11	12	15	
$w'xy'$	0 1 0 -	4	A		X	X						
xyz	- 1 1 1	4	B				X					X
wyz^*	1 - 1 1	4	C						X			X
$w'xz$	0 1 - 1	4	D			X	X					
$y'z'^*$	- - 0 0	3	E	X	X			X			X	

Note that all of the minterms covered by the essential prime implicants are checked, not just those columns with shaded X's. The table is now reduced to that of Table 4.6 by eliminating the essential prime implicant rows and the covered minterms.

In this simple example, the answer is apparent. Prime implicant *H* covers the remaining 1's; any other solution would require at least two more terms, for a total of four. Thus, the solution is

$$C + E + D = wyz + y'z' + w'xz$$

Before looking at some more complex examples that will require us to develop additional techniques, we will complete Examples 4.1 and 4.2 (with don't cares), for which we have already developed a list of prime implicants. The only thing that is different from the first example is that we only have columns for minterms included in the function—not for don't cares. That is really what happened in the reduced table above; the columns that were eliminated correspond to minterms that became don't cares after having chosen the essential prime implicants (as in Map Method 3).

Table 4.6 The reduced table.

\$	Label	5	7
4	A	X	
4	B		X
4	D	X	X

EXAMPLE 4.3

PI	Numeric	\$	Label	<div style="display: flex; justify-content: space-around; font-size: small;"> ✓ ✓ ✓ ✓ ✓ </div>				
				1	3	4	6	11
$y'z'$	- - 0 0	3	A			X		
$w'x'z$	0 0 - 1	4	B	X	X			
wxy'	1 1 0 -	4	C					
$wx'y$	1 0 1 -	4	D					X
$w'x'y'$	0 0 0 -	4	E	X				
$w'xz'^*$	0 1 - 0	4	F			X	X	
$x'yz$	- 0 1 1	4	G		X			X
$wx'z'$	1 0 - 0	4	H					

The first thing to note about this table is that rows *C* and *H* have no *X*'s in them; they correspond to prime implicants that cover only don't cares. *F* is essential, as indicated by the shading. We can now eliminate rows, *C*, *H*, and *F* and columns 4 and 6, producing the reduced table:

\$	Label	1	3	11
3	<i>A</i>			
4	<i>B</i>	X	X	
4	<i>D</i>			X
4	<i>E</i>	X		
4	<i>G</i>		X	X

Note that row *A* has no *X*'s; the minterm that it covered was already covered by the essential prime implicant. There are several ways to proceed from here. By looking at the table, we can see that we need at least one prime implicant that covers two minterms (either *B* or *G*). In either case, one minterm is left. There are three solutions:

$$F + B + D = w'xz' + w'x'z + wx'y$$

$$F + B + G = w'xz' + w'x'z + x'yz$$

$$F + G + E = w'xz' + x'yz + w'x'y'$$

All of these are equal cost, since each of the prime implicants used have the same number of literals. (We will see in other examples that some of the covers that use the same number of terms may have a different number of literals.)

If we are looking for only one of the minimum solutions, instead of all of them, we can often reduce a prime implicant table by removing *dominated* or equal rows. A row dominates another if the term it represents costs no more than the other and has *X*'s in every column that the dominated row does (and possibly more).

EXAMPLE 4.4

In Example 4.3, row *E* is dominated by *B*, and row *D* is dominated by *G*. Removing the dominated rows, the table reduces to

\$	Label	1	3	11
4	<i>B</i>	X	X	
4	<i>G</i>		X	X

and the only solution produced is

$$F + B + G = w'xz' + w'x'z + x'yz$$

Finally, a third approach, called *Petrick's method*, utilizes the table we have obtained after removing the essential prime implicants, but

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before removing dominated and equal rows. Create a product of sums expression by producing one term for each column. For the last example, the expression is

$$(B + E)(B + G)(D + G)$$

Minterm 1 must be covered by B or F , minterm 3 by B or H , and minterm 11 by E or H . Expanding that expression to sum of products form, we get

$$\begin{aligned} (B + EG)(D + G) &= BD + BG + DEG + EG \\ &= BD + BG + EG \end{aligned}$$

Each product term corresponds to a set of prime implicants that could be used to cover the function. These are, of course, the solutions that we found.

We are now ready to look at some more complex examples.

$$f(a, b, c, d) = \Sigma m(1, 3, 4, 6, 7, 9, 11, 12, 13, 15)$$

EXAMPLE 4.5

From the map, Quine-McCluskey, or iterated consensus,³ we could find all of the prime implicants and construct the following table:

		\$		1	3	4	6	7	9	11	12	13	15
$b'd^*$	- 0 - 1	3	A	X	X				X	X			
cd	- - 1 1	3	B		X			X		X			X
ad	1 - - 1	3	C						X	X		X	X
abc'	1 1 0 -	4	D								X	X	
$bc'd'$	- 1 0 0	4	E			X					X		
$a'bd'$	0 1 - 0	4	F			X	X						
$a'bc$	0 1 1 -	4	G				X	X					

There is one essential prime implicant, $b'd$, as shown in the table above. The table is then reduced, by eliminating that row and the terms that have been covered.

\$		4	6	7	12	13	15
3	B			X			X
3	C					X	X
4	D				X	X	
4	E	X			X		
4	F	X	X				
4	G		X	X			

³An effective approach is to map the function and find as many prime implicants as possible. Then, use iterated consensus to check that none have been left out.

The reduced table has two X's in each column and two X's in each row. Since there are six minterms to be covered, we need at least three prime implicants. Also, since B and C cost less than the other terms, we should try to use them. A careful study of the table will show that there are two covers that use three terms, each of which uses one of the less costly terms, namely,

$$A + B + D + F = b'd + cd + abc' + a'bd'$$

$$A + C + E + G = b'd + ad + bc'd' + a'bc$$

(We cannot complete the cover with three terms in addition to A by using both of the less costly rows, since they only cover three 1's between them.) The more systematic approach is to choose one of the minterms that can be covered in the fewest number of ways, for example, 4. We then recognize that we must choose either E or F in order to cover minterm 4. We will next derive a minimum solution using each of those and compare them. After we choose E , the table reduces to

\$		6	7	13	15
3	B		X		X
3	C			X	X
4	D			X	
4	F	X			
4	G	X	X		

Note that row D is dominated by C and costs more than C . It can be removed. (This row is shaded in the table above.) If that is eliminated, C is needed to cover minterm 13. (It also covers minterm 15.) Now, only minterms 6 and 7 need to be covered; the only way to do that with one term is with G . That produces the solution

$$A + C + E + G.$$

Row F is also dominated (by G); but those two terms cost the same. In general (although not in this example), we risk losing other equally good solutions if we delete dominated rows that are not more expensive.

If, instead, we chose prime implicant F to cover minterm 4, we would have

\$		7	12	13	15
3	B	X			X
3	C			X	X
4	D		X	X	
4	E		X		
4	G	X			

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Row *G* is dominated by row *B* and costs more. Thus, prime implicant *B* is needed to cover the function. With only minterms 12 and 13 left, we must choose term *D*, giving the other solution

$$A + F + B + D.$$

Finally, we could go back to the second table (with six minterms) and consider the prime implicants needed to cover each minterm. Petrick's method produces the following expression

$$\begin{aligned} &(E + F)(F + G)(B + G)(D + E)(C + D)(B + C) \\ &= (F + EG)(B + CG)(D + CE) \\ &= (BF + BEG + CFG + CEG)(D + CE) \\ &= \underline{BDF} + BDEG + CDFG + CDEG + BCEF \\ &\quad + BCEG + CEEG + \underline{CEG} \end{aligned}$$

Any of these eight combinations could be used; but only the two underlined correspond to three terms (in addition to *A*). This approach produces the same two minimum solutions.

EXAMPLE 4.6

$$f(w, x, y, z) = \Sigma m(1, 2, 3, 4, 8, 9, 10, 11, 12)$$

The prime implicants are

- $x'z$
- $x'y$
- wx'
- $xy'z'$
- $wy'z'$

The prime implicant table is

		3	1	2	3	4	8	9	10	11	12
$x'z^*$	- 0 - 1	3	A	X		X			X		X
$x'y^*$	- 0 1 -	3	B		X	X				X	X
wx'	1 0 - -	3	C				X	X	X	X	
$xy'z'^*$	- 1 0 0	4	D			X					X
$wy'z'$	1 - 0 0	4	E				X				X

There are three essential prime implicants, *A*, *B*, and *D*, which cover all but one of the 1's. The reduced table is thus

	3	8
<i>C</i>	3	X
<i>E</i>	4	X

Although either prime implicant could cover m_8 , C is less expensive. Thus, the only minimum solution is

$$f = x'z + x'y + xy'z' + wx'$$

EXAMPLE 4.7

$$g(a, b, c, d) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

From Example 3.14, we came up with the list of nine prime implicants shown in the table below. (We can check that this list is complete and that all of these are prime implicants by using them as the starting point for iterated consensus. If we do that, no new terms are produced in this example.) We do not need a cost column since all terms consist of two literals.

		0	1	3	4	6	7	8	9	11	12	13	14	15
- - 0 0	A	X			X			X			X			
- 0 0 -	B	X	X					X	X					
- 0 - 1	C		X	X					X	X				
- 1 - 0	D				X	X					X		X	
- 1 1 -	E					X	X						X	X
- - 1 1	F			X			X			X				X
1 1 - -	G										X	X	X	X
1 - 0 -	H							X	X		X	X		
1 - - 1	J								X	X		X		X

All of the minterms are covered by at least two prime implicants (some by as many as four). We will choose one of the columns that has only two X's and try to minimize the function first using one term, and then using the other. For this example, we will use either term A or term B to cover m_0 ; first we will use A and reduce the table by removing the minterms covered by A .

		1	3	6	7	9	11	13	14	15
- 0 0 -	B	X				X				
- 0 - 1*	C	X	X			X	X			
- 1 - 0	D			X					X	
- 1 1 -	E			X	X				X	X
- - 1 1	F		X		X		X			X
1 1 - -	G							X	X	X
1 - 0 -	H					X		X		
1 - - 1	J					X	X	X		X

Row B is dominated by C ; and row D is dominated by E . Although row H is dominated by J , we will leave that for now. Thus, we will choose terms C

4.3 Prime Implicant Tables for One Output

and *E*. Reducing the table once more, we get

		13
- - 1 1	<i>F</i>	
1 1 - -	<i>G</i>	X
1 - 0 -	<i>H</i>	X
1 - - 1	<i>J</i>	X

Obviously, any of *G*, *H*, or *J* could be used to cover minterm 13. Notice that row *H*, even though it was dominated, is used in one of the minimum solutions. We must now ask if that might be true of row *B* or row *D*. To be sure, we must go back to the previous table and see what happens if we don't eliminate them. We will choose *B* (rather than *C* to cover m_1 and m_4) and *E* and leave it to the reader to do it for *D* (rather than *E*) and *C*. The reduced table now becomes

		3	11	13
- 0 - 1	<i>C</i>	X	X	
- - 1 1	<i>F</i>	X	X	
1 1 - -	<i>G</i>			X
1 - 0 -	<i>H</i>			X
1 - - 1	<i>J</i>		X	X

Now, however, we need two more prime implicants to complete the cover, a total of five. Those solutions cannot be minimum since we found three (so far) with only four terms. Thus, the three minimum solutions using term *A* are

$$f = c'd' + b'd + bc + ab$$

$$f = c'd' + b'd + bc + ac'$$

$$f = c'd' + b'd + bc + ad$$

We will now go back and repeat the process, starting with term *B*. We can eliminate row *A*, since we already found all minimum solutions using row *A*.

		3	4	6	7	11	12	13	14	15
- 0 - 1	<i>C</i>	X				X				
- 1 - 0*	<i>D</i>		X	X			X		X	
- 1 1 -	<i>E</i>			X	X				X	X
- - 1 1	<i>F</i>	X			X	X				X
1 1 - -	<i>G</i>						X	X	X	X
1 - 0 -	<i>H</i>						X	X		
1 - - 1	<i>J</i>					X		X		X

Row *D* is now required. We will reduce the table one more time.

		3	7	11	13	15
- 0 - 1	<i>C</i>	X		X		
- 1 1 -	<i>E</i>		X			X
- - 1 1	<i>F</i>	X	X	X		X
1 1 - -	<i>G</i>				X	X
1 - 0 -	<i>H</i>				X	
1 - - 1	<i>J</i>			X	X	X

It is clear now that *F* is necessary, covering all the remaining minterms except m_{13} . (Otherwise, we would need both *C* and *E*, and would still leave m_{13} uncovered.) As before, prime implicants *G*, *H*, and *J* could be used to complete the function. The three solutions using term *B* are thus

$$f = b'c' + bd' + cd + ab$$

$$f = b'c' + bd' + cd + ac'$$

$$f = b'c' + bd' + cd + ad$$

giving a total of six solutions.

[SP 3; EX 3]

4.4 QUINE-McCLUSKEY FOR MULTIPLE OUTPUT PROBLEMS

The Quine-McCluskey method can be expanded to include multiple output systems by adding a tag section to each product term. The tag indicates for which functions that term can be used. We will include a bit for each function, with a – if the term is included in that function and a 0 if not. Terms can be combined if they have a common –. When combining terms (using the adjacency property), each tag is 0 if either term had a 0 and is – if both terms had a dash. We will develop the technique for finding all useful terms in this section and defer to Section 4.6 the method for finding minimum sum of products expressions.

To illustrate the process, consider the following functions:

$$f(a, b, c) = \Sigma m(2, 3, 7)$$

$$g(a, b, c) = \Sigma m(4, 5, 7)$$

4.4 Quine-McCluskey for Multiple Output Problems

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The method begins (where the letters are added for ease of identification)

$$\begin{array}{l} A \quad 010 \quad -0 \\ B \quad 011 \quad -0 \\ C \quad 100 \quad 0- \\ D \quad 101 \quad 0- \\ E \quad 111 \quad -- \end{array}$$

We now apply the adjacency property to each pair of terms that have at least one $-$ in common.

$$A + B = F = 01- \quad -0$$

$A + C, A + D$ need not be considered since they correspond to terms that are not part of the same function⁴

$$A + E = \text{none}$$

$$B + E = G = -11 \quad -0$$

$$C + D = H = 10- \quad 0-$$

$$C + E = \text{none}$$

$$D + E = J = 1-1 \quad 0-$$

When we continue to another column, terms are checked off only if they are covered in each function. Table 4.7 shows the result.

Table 4.7 Multiple output Quine-McCluskey method.

A	010	$-0\checkmark$	F	$01-$	-0
B	011	$-0\checkmark$	G	$10-$	$0-$
C	100	$0-\checkmark$	H	-11	-0
D	101	$0-\checkmark$	J	$1-1$	$0-$
E	111	$--$			

There are no adjacencies in the second column. At the end of the process, there are 2 two-literal terms for each function and 1 three-literal term that can be shared.

Before completing the solution of this problem using multiple output prime implicant tables (in Section 4.6), we will consider two additional examples.

⁴We will not show any other such pairs.

EXAMPLE 4.8

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$

$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

We begin by listing all of the minterms, with tags, including the don't cares, grouping terms by the number of 1's:

A	0 0 0 0 --√	AA	0 0 0 - --	BA	0 0 - - - 0
	-----	AB	0 0 - 0 --	BB	0 - - 0 - 0
B	0 0 0 1 --√	AC	0 - 0 0 - 0√		-----
C	0 0 1 0 --√		-----	BC	- 0 - 1 - 0
D	0 1 0 0 - 0√	AD	0 0 - 1 - 0√	BD	- - 1 0 0 -
	-----	AE	- 0 0 1 - 0√	BE	- 1 - 0 - 0
E	0 0 1 1 - 0√	AF	0 0 1 - - 0√		-----
F	0 1 1 0 --√	AG	0 - 1 0 --	BF	1 - 1 - 0 -
G	1 0 0 1 - 0√	AH	- 0 1 0 0 -√		
H	1 0 1 0 0 -√	AI	0 1 - 0 - 0√		
I	1 1 0 0 --√	AJ	- 1 0 0 - 0√		
	-----		-----		
J	1 0 1 1 --√	AK	- 0 1 1 - 0√		
K	1 1 1 0 --√	AL	- 1 1 0 --		
	-----	AM	1 0 - 1 - 0√		
L	1 1 1 1 --√	AN	1 0 1 - 0 -√		
		AO	1 - 1 0 0 -√		
		AP	1 1 - 0 --		

		AQ	1 - 1 1 --		
		AR	1 1 1 - --		

Thus, the terms that can be shared are $a'b'c'$, $a'b'd'$, $a'cd'$, bcd' , abd' , acd , and abc . The prime implicants of f are $a'b'$, $a'd'$, $b'd'$, and bd' . The prime implicants of g are cd' and ac .

Note that some sums, such as $AF + AN$ exist, but the two terms belong to different functions (and would have a tag of 00); they are not included.

EXAMPLE 4.9

Last, we will consider a small example with three outputs:

$$f(x, y, z) = \Sigma m(0, 2, 5, 6, 7)$$

$$g(x, y, z) = \Sigma m(2, 3, 5, 6, 7)$$

$$h(x, y, z) = \Sigma m(0, 2, 3, 4, 5)$$

The tag now has three bits, but otherwise the process is as before:

A	0 0 0	0-√	H	0-0	0-0	R	-1-	0-0
	-----		J	-00	00-			
B	0 1 0	---		-----				
C	1 0 0	00-√	K	0 1-	0--			
	-----		L	-10	--0			
D	0 1 1	0--√	M	1 0-	00-			
E	1 0 1	---		-----				
F	1 1 0	--0√	N	-1 1	0-0√			
	-----		P	1-1	--0			
G	1 1 1	--0√	Q	1 1-	--0			

The terms that can be used for all three functions are $x'yz'$ and $xy'z$. For f and g , we can use yz' , xz , and xy . For f and h , we can use $x'z'$. For g and h , we can use $x'y$. For h , we can use $y'z'$ and xy' . For g , we can use y .

[SP 4; EX 4]

4.5 ITERATED CONSENSUS FOR MULTIPLE OUTPUT PROBLEMS

The iterated consensus algorithm needs only minor modifications to produce all of the terms that may be used for sum of product expressions for multiple output problems. Candidates are terms that are prime implicants of any one function or prime implicants of the product of functions. (Although we did not make use of this property in other approaches, if we look back, we will find that all terms that were shared between two functions were indeed prime implicants of the product of those two functions and terms that were shared among three functions were prime implicants of the product of the three functions.) In this section, we will find all prime implicants. We will find minimum solutions in Section 4.6.

To begin the iterated consensus procedure, we must either start with minterms or include not only a cover of each function, but also a cover of all possible products of functions. We will follow the first approach in this example and use the second later. To each product term on our list for iterated consensus, we add a tag section with a dummy variable for each output. That tag contains a 0 (complemented output variable) if the term is not an implicant of that function and a blank if it is. We will illustrate the process with the functions of Example 3.29:⁵

$$f(a, b, c) = \Sigma m(2, 3, 7)$$

$$g(a, b, c) = \Sigma m(4, 5, 7)$$

⁵The three examples of this section are the same functions as those of Section 4.4.

The initial list then becomes

$a' b' c'$	g'	0 1 0	– 0
$a' b' c$	g'	0 1 1	– 0
$a b' c'$	f'	1 0 0	0 –
$a b' c$	f'	1 0 1	0 –
$a b c$		1 1 1	– –

We now proceed as before, taking the consensus of each pair of terms (including the tag), adding new terms and deleting terms included in others. The only new rule is that terms that have an all 0 tag section are also deleted. (They correspond to a grouping made of a 1 from one function with a 1 from the other function; they are not implicants of either function.) Note that the tag never affects whether or not a consensus exists, since there are no 1's in the tag section.

We now proceed, as in Table 4.8.

Table 4.8 Iterated consensus for multiple output functions.

A	0 1 0	– 0	
B	0 1 1	– 0	
C	1 0 0	0 –	
D	1 0 1	0 –	
E	1 1 1	– –	
F	0 1 –	– 0	$B \not\subset A \geq B, A$
G	1 0 –	0 –	$D \not\subset C \geq D, C$
H	– 1 1	– 0	$F \not\subset E$
J	1 – 1	0 –	$G \not\subset E$ ($G \not\subset F$ undefined)
			$H \not\subset G$ zero tag; $H \not\subset F, H \not\subset E$ undefined
			$J \not\subset H, J \not\subset F$ zero tag; $J \not\subset G, J \not\subset E$ undefined

The term that can be shared is abc ; $a'b$ and bc are prime implicants of f ; ab' and ac are prime implicants of g .

EXAMPLE 4.10

We will consider functions from Examples 3.36 and 4.8, a two-output problem with don't cares.

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$

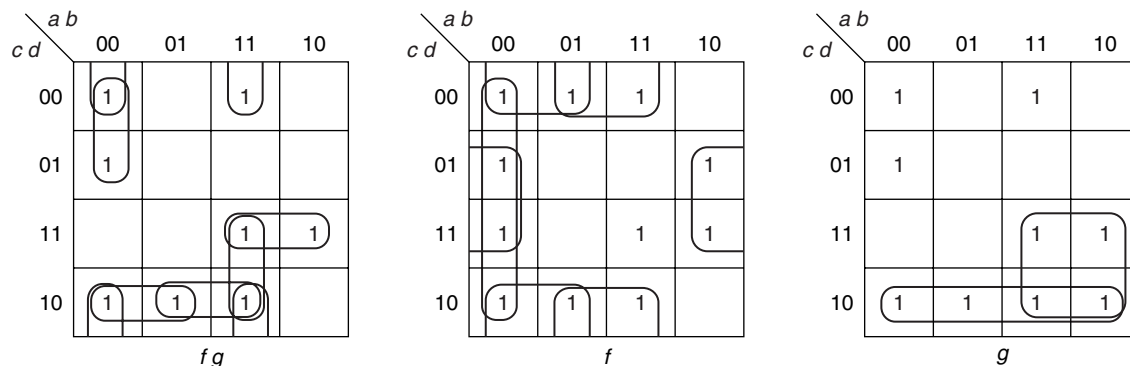
$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

To obtain the list of prime implicants to include in the prime implicant table, we can start with minterms, treating all don't cares as 1's and work the iterated consensus algorithm. It is very time-consuming and prone to error (although it would be fairly straightforward to write a computer routine to process it).⁶ The other approach is to map fg (the product of the two functions), find all of the prime implicants of that plus those terms that are only prime implicants of one of the functions. The following maps show the

⁶Another example of this approach is given in Solved Problem 5a.

4.5 Iterated Consensus for Multiple Output Problems

prime implicants of fg and those of f and g that are not prime implicants of both functions, where all don't cares have been made 1 on the maps, since we must include all prime implicants that cover don't cares, as well.



The product terms (with their tag) are

- 000- -- 00-- -0 --10 0-
- 00-0 -- 0--0 -0 1-1- 0-
- 0-10 -- -1-0 -0
- 110 -- -0-1 -0
- 111- --
- 11-0 --
- 1-11 --

We could try iterated consensus on this list but would find no new terms.

$$f(x, y, z) = \Sigma m(0, 2, 5, 6, 7)$$

$$g(x, y, z) = \Sigma m(2, 3, 5, 6, 7)$$

$$h(x, y, z) = \Sigma m(0, 2, 3, 4, 5)$$

EXAMPLE 4.11

We start by listing all of the minterms used for any of the functions, including the tag, and then perform the iterated consensus algorithm to find all of the prime implicants.

- | | | |
|---|--|------------------------------|
| A 0-00 -0- | H 0-0 -0- | B $\not\subseteq$ A \geq A |
| B 010 --- | J 01- 0-- | C $\not\subseteq$ B \geq C |
| C 011 0-- | K 10- 00- | E $\not\subseteq$ D \geq D |
| D 100 00- | L -10 --0 | F $\not\subseteq$ B \geq F |
| E 101 --- | M 1-1 --0 | G $\not\subseteq$ E \geq G |
| F 110 --0 | N -00 00- | K $\not\subseteq$ H |
| G 111 --0 | P 11- --0 | M $\not\subseteq$ L |
| | Q -11 0-0 | M $\not\subseteq$ J |
| | R -1- 0-0 | Q $\not\subseteq$ L \geq Q |

We did not show any term that produced an all 0 tag section and we did not list the consensus operations that led to undefined terms or to terms included in other terms already on the list. This leaves a total of 10 prime implicants (of one of the functions or the product of functions). Note that two of the minterms remain, since they can be used for all three functions and are not part of any one larger group in all three.

4.6 PRIME IMPLICANT TABLES FOR MULTIPLE OUTPUT PROBLEMS

Having found all of the product terms, we create a prime implicant table with a separate section for each function. The prime implicant table for the first set of functions of the last two sections

$$f(a, b, c) = \Sigma m(2, 3, 7)$$

$$g(a, b, c) = \Sigma m(4, 5, 7)$$

is shown in Table 4.9. An X is only placed in the column of a function for which the term is an implicant. (For example, there is no X in column 7 of g or for term D .) Essential prime implicants are found as before ($a'b$ for f and ab' for g).

Table 4.9 A multiple output prime implicant table.

	Σ	Term	f			g		
			2	3	7	4	5	7
1 1 1	4	A			X			X
0 1 -*	3	B	X	X				
1 0 -*	3	C				X	X	
- 1 1	3	D		X	X			
1 - 1	3	E					X	X

Table 4.10 A reduced prime implicant table.

	Σ	Term	f	g
			7	7
1 1 1	4	A	X	X
- 1 1	3	D	X	
1 - 1	3	E		X

The table is then reduced as in Table 4.10.

Now, it is clear that we can use term E to cover both functions, rather than two separate terms, even though E costs 4 and the others cost 3. Indeed, the cost to use a term in each function after the first is only 1, the input to another OR gate. (We only build one AND gate for that term.) The solution using A thus costs 5, compared to 6 for a solution that uses both D and E . (The latter solution requires an extra gate.) The solution is thus,

$$f = a'b + abc$$

$$g = ab' + abc$$

4.6 Prime Implicant Tables for Multiple Output Problems

The prime implicant table for the functions of Example 4.8 and 4.10

EXAMPLE 4.12

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$

$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

is shown below.

		f							g				
		2	3	4	6	9	11	12	2	6	10	11	12
0 0 0 -	A	4											
0 0 - 0	B	4	X						X				
0 - 1 0	C	4	X		X				X	X			
- 1 1 0	D	4			X					X			
1 - 1 1	E	4					X					X	
1 1 1 -	F	4											
1 1 - 0*	G	4						X					X
- 1 - 0	H	3		X	X			X					
0 - - 0	J	3	X		X	X							
0 0 - -	K	3	X	X									
- 0 - 1*	L	3		X			X	X					
- - 1 0	M	3							X	X	X		
1 - 1 -	N	3									X	X	

Note that the table is divided into three sections of rows. The first (A to G) includes the terms that are eligible for sharing. The second section contains the prime implicants of *f* that are not also implicants of *g*, and the last section contains those of *g* that are not implicants of *f*. Notice that rows A and F have no X's; they are prime implicants made up of only don't cares. (Of course, there are no columns corresponding to the don't cares.)

Row L, *b'd*, is an essential prime implicant of *f* and row G, *abd'* is an essential prime implicant of *g*. Although the latter is also useful for *f*, it is not essential and we may or may not want to use it. The reduced table is shown next.

		f				g			
		2	4	6	12	2	6	10	11
0 0 - 0	<i>B</i>	4	X			X			
0 - 1 0	<i>C</i>	4	X		X	X	X		
- 1 1 0	<i>D</i>	4			X		X		
1 - 1 1	<i>E</i>	4							X
1 1 - 0*	<i>G</i>	1			X				
- 1 - 0	<i>H</i>	3		X	X	X			
0 - - 0	<i>J</i>	3	X	X	X				
0 0 - -	<i>K</i>	3	X						
- - 1 0	<i>M</i>	3				X	X	X	
1 - 1 -	<i>N</i>	3						X	X

Note that the cost for term *G* has been reduced to 1, since the AND gate has already been built; we only need an input to the OR gate. Term *E* is dominated by and costs more than term *N*, and can be eliminated. (It will never be part of a minimum solution, since it is less expensive to use term *N*.) That makes term *N*, *ac*, necessary for *g*. With these two terms and the minterms they cover removed, the table reduces to

		f				g	
		2	4	6	12	2	6
0 0 - 0	<i>B</i>	4	X			X	
0 - 1 0	<i>C</i>	4	X		X	X	X
- 1 1 0	<i>D</i>	4			X		X
1 1 - 0	<i>G</i>	1			X		
- 1 - 0	<i>H</i>	3		X	X	X	
0 - - 0	<i>J</i>	3	X	X	X		
0 0 - -	<i>K</i>	3	X				
- - 1 0	<i>M</i>	3				X	X

Neither *B* nor *D* would be used for *g*, unless we used both of them. However, then we could use *C*, which would cover both minterms in *g* and also be shared with *f* (covering the same minterms in *f* that were covered by *B* and *D*). At this point, we are left with two choices. Either we choose term *G* for *f* (at a cost of 1), which would then allow us to finish covering *f* with term *J* (a total of one new gate, four inputs). Then, term *M* would be used for *g* (one new gate, three inputs). The other choice is to use term *C* to cover the 1's in both *f* and *g* and then use *H* to cover the remaining 1's in *f*. That would also require two new gates and a total of eight inputs—three for *H*, four for *C* in *f*, and one more for the OR gate in *g*. (That solution

4.6 Prime Implicant Tables for Multiple Output Problems

requires one extra gate input.) Notice that the cost column only refers to the number of inputs for one function; we need to add 1 for each additional function in which it is used.

Thus, the minimum solution is, as we found in Example 3.36,

$$f = b'd + abd' + a'd'$$

$$g = ac + abd' + cd'$$

For the functions of Example 4.9 and 4.11, we have the following prime implicant table:

EXAMPLE 4.13

		f					g					h				
		0	2	5	6	7	2	3	5	6	7	0	2	3	4	5
0 1 0	4	A	X				X						X			
1 0 1	4	B		X				X							X	
0 - 0*	3	C	X	X								X	X			
0 1 -*	3	D					X	X						X	X	
1 0 -	3	E												X	X	
- 1 0	3	F		X	X		X		X							
1 - 1	3	G		X	X			X	X							
- 0 0	3	H									X			X		
1 1 -	3	J			X	X			X	X						
- 1 -*	1	K					X	X	X	X						

We see that term *C* is an essential prime implicant of *f*, but not of *h*. (We will thus check off the terms in *f* and leave those in *h*, but reduce the cost of this term to 1 in the reduced table, since the AND gate is already accounted for; only the input to the *h* OR gate needs to be charged.) Similarly, term *D* is an essential prime implicant of *h*, but not of *g*. Finally, term *K* will be used for *g*, since it only costs 1 (the OR gate input). Even if we could cover that with two shared terms, that would cost two inputs to the OR gate. The table thus reduces to

		f			g		h		
		5	6	7	5	0	4	5	
0 1 0	4	A							
1 0 1	4	B	X		X			X	
0 - 0	1	C				X			
0 1 -	1	D							
1 0 -	3	E					X	X	
- 1 0	3	F		X					
1 - 1	3	G	X		X	X			
- 0 0	3	H				X	X		
1 1 -	3	J		X	X				

We can see that terms A and D no longer cover any terms; those rows can be eliminated. We seem to have two choices now. First, we can use B for all three functions, at a cost of 6. We would then use J for f and H for h , for a cost of 12 (on this table). This solution requires eight gates and 19 inputs.

$$\begin{aligned} f &= x'z' + xy'z + xy \\ g &= y + xy'z \\ h &= x'y + xy'z + y'z' \end{aligned}$$

The other choice is to use G for f and g (at a cost of 4). Then F or J can be used for f ; and C (since it costs only 1) and E for h . The total cost is 11 inputs and three gates (G , F or J , and E), and thus this second solution is best. (Note that the gate to create term C is not included in the gate count here, since it was already built.) The equations are

$$\begin{aligned} f &= x'z' + xz + (yz' \text{ or } xy) \\ g &= y + xz \\ h &= x'y + x'z' + xy' \end{aligned}$$

It also uses eight gates, but has only 18 inputs.

[SP 6; EX 6]

4.7 SOLVED PROBLEMS

1. For each of the following functions, find all of the prime implicants using the Quine-McCluskey method. (The first three functions have been minimized using the Karnaugh map in Solved Problems 1b, 1d, and 3b of Chapter 3.)
 - a. $f(w, x, y, z) = \sum m(2, 5, 7, 8, 10, 12, 13, 15)$
 - b. $f(a, b, c, d) = \sum m(0, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$
(2 solutions)
 - c. $F(W, X, Y, Z) = \sum m(1, 3, 5, 6, 7, 13, 14) + \sum d(8, 10, 12)$
(2 solutions)
 - d. $f(a, b, c, d, e) = \sum m(0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 21, 23, 26, 28, 29, 30, 31)$

- a. We organize the minterms by the number of 1's

A 0010√	J -010	R -1-1
B 1000√	K 10-0	
-----	L 1-00	
C 0101√	-----	
D 1010√	M 01-1√	
E 1100√	N -101√	
-----	O 110-	
F 0111√	-----	
G 1101√	P -111√	
-----	Q 11-1√	
H 1111√		

Only sums that produce a product term are shown

$$\begin{array}{ll} A + D = J & E + G = O \\ B + D = K & F + H = P \\ B + E = L & G + H = Q \\ C + F = M & M + Q = N + P = R \\ C + G = N & \end{array}$$

The prime implicants are thus $x'yz'$, $wx'z'$, $wy'z'$, wxy' , and xz .

b.

$$\begin{array}{lll} 0000\checkmark & 0-00 & 01-- \\ ----- & -000 & 10-- \\ 0100\checkmark & ----- & ----- \\ 1000\checkmark & 010-\checkmark & -1-1 \\ ----- & 01-0\checkmark & -11- \\ 0101\checkmark & 100-\checkmark & 1--1 \\ 0110\checkmark & 10-0\checkmark & 1-1- \\ 1001\checkmark & ----- & ----- \\ 1010\checkmark & 01-1\checkmark & ----- \\ ----- & -101\checkmark & ----- \\ 0111\checkmark & 011-\checkmark & ----- \\ 1011\checkmark & -110\checkmark & ----- \\ 1101\checkmark & 10-1\checkmark & ----- \\ 1110\checkmark & 1-01\checkmark & ----- \\ ----- & 101-\checkmark & ----- \\ 1111\checkmark & 1-10\checkmark & ----- \\ ----- & ----- & ----- \\ & -111\checkmark & ----- \\ & 1-11\checkmark & ----- \\ & 11-1\checkmark & ----- \\ & 111-\checkmark & ----- \end{array}$$

The prime implicants are $a'c'd'$, $b'c'd'$, $a'b$, ab' , bd , bc , ad , and ac .

c.

$$\begin{array}{lll} 0001\checkmark & 00-1\checkmark & 0--1 \\ 1000\checkmark & 0-01\checkmark & 1--0 \\ ----- & 10-0\checkmark & ----- \\ 0011\checkmark & 1-00\checkmark & ----- \\ 0101\checkmark & ----- & ----- \\ 0110\checkmark & 0-11\checkmark & ----- \\ 1010\checkmark & 01-1\checkmark & ----- \\ 1100\checkmark & -101 & ----- \\ ----- & 011- & ----- \\ 0111\checkmark & -110 & ----- \\ 1101\checkmark & 1-10\checkmark & ----- \\ 1110\checkmark & 110- & ----- \\ & 11-0\checkmark & ----- \end{array}$$

The prime implicants are $XY'Z$, $W'XY$, XYZ' , WXY' , WZ , and WZ' .

d.

00000√	000-0√	00--0	--1-1
-----	00-00√	0-0-0	
00010√	0-000√	-----	
00100√	-----	001--	
01000√	00-10√	010--	
-----	0-010√	-----	
00101√	0010-√	0-1-1√	
00110√	001-0√	--101√	
01001√	0100-√	-01-1√	
01010√	010-0√	01--1	
-----	-----	-----	
00111√	001-1√	--111√	
01011√	0-101√	-11-1√	
01101√	-0101√	1-1-1√	
10101√	0011-√	111--	
11010√	010-1√		
11100√	01-01√		
-----	0101-√		
01111√	-1010		
10111√	-----		
11101√	0-111√		
11110√	-0111√		
-----	01-11√		
11111√	011-1√		
	-1101√		
	101-1√		
	1-101√		
	11-10		
	1110-√		
	111-0√		

	-1111√		
	1-111√		
	111-1√		
	1111-√		

The prime implicants are $bc'de'$, $abde'$, $a'b'e'$, $a'c'e'$, $a'b'c$, $a'bc'$, $a'be$, abc , and ce .

2. For each of the functions of Solved Problem 1, find all of the prime implicants using iterated consensus.

a. We will start with the minterms for this solution, listing only those consensus terms that are to be added to the list.

A 0 0 1 0	J 0 1 - 1	$C \not\subset B \geq C, B$
B 0 1 0 1	K 1 0 - 0	$E \not\subset D \geq D, E$
C 0 1 1 1	L 1 1 0 -	$G \not\subset F \geq F, G$
D 1 0 0 0	M - 1 1 1	$J \not\subset H \geq H$
E 1 0 1 0	N - 0 1 0	$K \not\subset A \geq A$
F 1 1 0 0	P 1 - 0 0	$L \not\subset K$
G 1 1 0 1	Q - 1 0 1	$L \not\subset J$
H 1 1 1 1	R 1 1 - 1	$M \not\subset L$
	S - 1 - 1	$Q \not\subset M \geq J, M, Q, R$

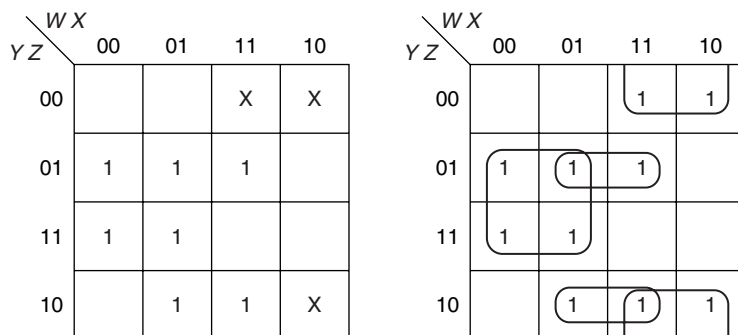
All other consensus operations are either undefined or produce a term that is already on the list. The terms remaining on the list are all the prime implicants— $wx'z'$, wxy' , $x'yz'$, $wy'z'$, and xz .

- b. We first map the function (as in Solved Problem 1d of Chapter 3) and find four prime implicants that cover the function. We then use iterated consensus to generate the rest.

A	0 - 0 0	E 0 1 0 =	$B \not\subset A$
B	- 1 - 1	F 0 1 - 0	$C \not\subset A$
C	- 1 1 -	G 1 - 1 -	$D \not\subset C$
D	1 0 - -	H 1 - - 1	$D \not\subset B$
		J - 0 0 0	$D \not\subset A$
		K 0 1 - -	$E \not\subset C \geq E, F$

No other consensus terms are formed.

- c. First, we took the map of the function and converted all of the don't cares to 1's. We then found a set of prime implicants that covered the function. (We could have used any set of product terms that covered the function, but starting with prime implicants usually reduces the amount of work.)



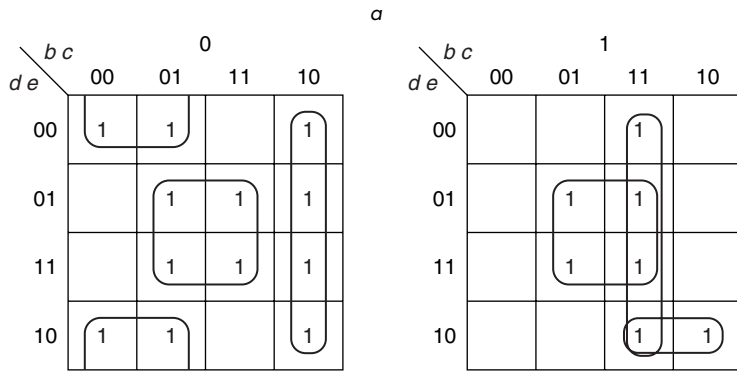
Iterated consensus then proceeds very smoothly

$$\begin{array}{r}
 A \quad 0 \quad - \quad - \quad 1 \\
 B \quad 1 \quad - \quad - \quad 0 \\
 C \quad - \quad 1 \quad 0 \quad 1 \\
 D \quad - \quad 1 \quad 1 \quad 0 \\
 \hline
 E \quad 1 \quad 1 \quad 0 \quad - \quad C \not\leq B \\
 F \quad 0 \quad 1 \quad 1 \quad - \quad D \not\leq A
 \end{array}$$

No other new terms are formed. The only other consensus terms formed are

$$\begin{array}{l}
 E \not\leq D = 1 \ 1 \ - \ 0 \leq B \\
 E \not\leq A = C \\
 F \not\leq C \leq A \\
 F \not\leq B = D
 \end{array}$$

- d. We will first map the function and cover the function with product terms on one layer.



Those product terms are shown in the first column. We then perform the consensus algorithm, which creates some new terms (in the second column) and eliminates others. There are a total of nine terms.

$$\begin{array}{ll}
 00--0 & 001-- \\
 \cancel{0-1-1} & 01--1 \\
 010-- & 0-0-0 \\
 111-- & \cancel{-11-1} \\
 \cancel{1-1-1} & --1-1 \\
 11-10 & -1010
 \end{array}$$

3. For each of the functions of Solved Problems 1 and 2, find all minimum sum of product solutions (one solution for a, two for each of the others).

a. The prime implicant table is

			\$	2	5	7	8	10	12	13	15
$wx'z'$	1 0 - 0	A	4				X	X			
wxy'	1 1 0 -	B	4						X	X	
$x'yz'*$	- 0 1 0	C	4	X				X			
$wy'z'$	1 - 0 0	D	4				X		X		
$xz*$	- 1 - 1	E	3		X	X				X	X

The 1's that make two prime implicants essential, $x'yz'$ and xz , are shaded, and the minterms covered by them are checked. The table then reduces to

		\$	8	12
$wx'z'$	A	4	X	
wxy'	B	4		X
$wy'z'$	D	4	X	X

Clearly, term D must be used; otherwise, two more terms would be necessary. The solution becomes

$$f = x'yz' + xz + wy'z'$$

b. The prime implicant table is

		\$	0	4	5	6	7	8	9	10	11	13	14	15
A	0 - 0 0	4	X	X										
B	- 1 - 1	3			X		X					X		X
C	- 1 1 -	3				X	X						X	X
D	1 0 - -	3						X	X	X	X			
E	1 - 1 -	3								X	X		X	X
F	1 - - 1	3							X		X	X		X
G	- 0 0 0	4	X					X						
H	0 1 - -	3		X	X	X	X							

There are no essential prime implicants. The starting point should be one of the columns in which there are only two X's. We will choose minterm 5, since both terms will cover four 1's (but we could have used minterm 0, 4, 5, 6, 8, 9, 10, 13, or 14). We will first try prime implicant *B*; then we will try prime implicant *H*. If we choose *B*, the table reduces to

		\$	0	4	6	8	9	10	11	14
<i>A</i>	0 - 0 0	4	X	X						
<i>C</i>	- 1 1 -	3			X					X
<i>D</i>	1 0 - -*	3				X	X	X	X	
<i>E</i>	1 - 1 -	3						X	X	X
<i>F</i>	1 - - 1	3					X		X	
<i>G</i>	- 0 0 0	4	X			X				
<i>H</i>	0 1 - -	3		X	X					

Row *F* is dominated by row *D*. If row *D* is chosen, the table reduces to

		\$	0	4	6	14
<i>A</i>	0 - 0 0	4	X	X		
<i>C</i>	- 1 1 -	3			X	X
<i>E</i>	1 - 1 -	3				X
<i>G</i>	- 0 0 0	4	X			
<i>H</i>	0 1 - -	3		X	X	

At this point, the only way to cover the function with two terms is to choose *A* and *C*, giving a solution of

$$f = bd + ab' + a'c'd' + bc$$

Notice that if the dominated term *F* had been chosen instead of *D*, three additional terms would be required to cover the function, since minterms 8 and 10 are not covered by *F*.

Now, we must consider what happens if we choose term *H* instead of *B*. The resulting table is

		\$	0	8	9	10	11	13	14	15
<i>A</i>	0 - 0 0	4	X							
<i>B</i>	- 1 - 1	3						X		X
<i>C</i>	- 1 1 -	3							X	X
<i>D</i>	1 0 - -	3		X	X	X	X			
<i>E</i>	1 - 1 -*	3				X	X		X	X
<i>F</i>	1 - - 1	3			X		X	X		X
<i>G</i>	- 0 0 0*	4	X	X						

Prime implicant *A* is dominated by *G*, and *C* is dominated by *E*. Eliminating them, we must choose *G* and *E*, leaving only minterms 9 and 13 uncovered. They can both be covered by term *F*. No other solution (that used *H*) requires as few as four terms (even those using one of the dominated terms, *A* or *C*). The resulting function, the second equally good solution, is

$$f = a'b + ac + b'c'd' + ad$$

c. The prime implicant table is

		\$	1	3	5	6	7	13	14
0 - - 1*	<i>A</i>	3	X	X	X		X		
1 - - 0	<i>B</i>	3							X
- 1 0 1	<i>C</i>	4			X			X	
- 1 1 0	<i>D</i>	4				X			X
1 1 0 -	<i>E</i>	4						X	
0 1 1 -	<i>F</i>	4				X	X		

Note that there are no columns for the don't cares; they do not need to be covered. There is one essential prime implicant,

$A(W'Z)$, and the table can then be reduced to

		\$	6	13	14
1 - - 0	<i>B</i>	3			X
- 1 0 1	<i>C</i>	4		X	
- 1 1 0	<i>D</i>	4	X		X
1 1 0 -	<i>E</i>	4		X	
0 1 1 -	<i>F</i>	4	X		

A study of the reduced table reveals that row *D* must be chosen; otherwise, it would take both terms *B* and *F* to cover minterms 6 and 14. That leaves us two choices to conclude the cover, *C* or *E*. Thus, the two solutions to the problem are

$$F = W'Z + XYZ' + XY'Z$$

$$F = W'Z + XYZ' + WXY'$$

d. The prime implicant table is

			0	2	4	5	6	7	8	9	10	11	13	15	21	23	26	28	29	30	31
- 1 0 1 0	<i>A</i>	4									X						X				
1 1 - 1 0	<i>B</i>	4															X			X	
0 0 - - 0	<i>C</i>	3	X	X	X		X														
0 - 0 - 0	<i>D</i>	3	X	X					X		X										
0 0 1 - -	<i>E</i>	3			X	X	X	X													
0 1 0 - -	<i>F</i>	3							X	X	X	X									
0 1 - - 1	<i>G</i>	3								X		X	X	X							
1 1 1 - -	<i>H*</i>	3																X	X	X	X
- - 1 - 1	<i>J*</i>	2				X		X					X	X	X	X			X		X

The two essential prime implicants are *H* and *J*. The table is then reduced to

			0	2	4	6	8	9	10	11	26
- 1 0 1 0	<i>A</i>	4							X		X
1 1 - 1 0	<i>B</i>	4									X
0 0 - - 0	<i>C</i>	3	X	X	X	X					
0 - 0 - 0	<i>D</i>	3	X	X			X		X		
0 0 1 - -	<i>E</i>	3			X	X					
0 1 0 - -	<i>F</i>	3					X	X	X	X	
0 1 - - 1	<i>G</i>	3						X		X	

At this point, there are nine minterms left to be covered. Either A or B must be used for m_{26} . C and F must be used to cover the remaining 1's, producing the solutions

$$f = abc + ce + a'b'e' + a'bc' + bc'de'$$

$$= abc + ce + a'b'e' + a'bc' + abde'$$

4. For each of the sets of functions, find all terms that may be used in a minimum two-level AND/OR gate (or NAND gate) solution using the Quine-McCluskey method.
- a. $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 5, 7, 8, 10, 11, 13)$
 $g(a, b, c, d) = \Sigma m(0, 2, 5, 8, 10, 11, 13, 15)$
- b. $f(w, x, y, z) = \Sigma m(5, 7, 9, 11, 13, 15)$
 $g(w, x, y, z) = \Sigma m(1, 5, 7, 9, 10, 11, 14)$
- c. $f(a, b, c, d) = \Sigma m(0, 3, 5, 7) + \Sigma d(10, 11, 12, 13, 14, 15)$
 $g(a, b, c, d) = \Sigma m(0, 5, 6, 7, 8) + \Sigma d(10, 11, 12, 13, 14, 15)$
- d. $f(a, b, c, d) = \Sigma m(0, 2, 3, 8, 9, 10, 11, 12, 13, 15)$
 $g(a, b, c, d) = \Sigma m(3, 5, 7, 12, 13, 15)$
 $h(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 8, 10, 14)$

- a. We first form a column of minterms, organized by the number of 1's in each term. We then produce a second column of three-literal terms and a third of two-literal terms.

0 0 0 0	--√	0 0 0 -	- 0 √	0 0 - -	- 0
-----		0 0 - 0	--√	- 0 - 0	--
0 0 0 1	- 0 √	- 0 0 0	--√	0 - - 1	- 0
0 0 1 0	--√	-----		- 0 1 -	- 0
1 0 0 0	--√	0 0 - 1	- 0 √		
-----		0 - 0 1	- 0 √		
0 0 1 1	- 0 √	0 0 1 -	- 0 √		
0 1 0 1	--√	- 0 1 0	--√		
1 0 1 0	--√	1 0 - 0	--√		
-----		-----			
0 1 1 1	- 0 √	0 - 1 1	- 0 √		
1 0 1 1	--√	- 0 1 1	- 0 √		
1 1 0 1	--√	0 1 - 1	- 0 √		
-----		- 1 0 1	--		
1 1 1 1	0 - √	1 0 1 -	--		
-----		-----			
		1 - 1 1	0 -		
		1 1 - 1	0 -		

The unshared prime implicants of f are $a'b'$, $a'd$, and $b'c$; those of g are acd and abd . The shared terms are $b'd'$, $bc'd$, and $ab'c$.

b.

0001 0-√	0-01 0-	-1-1 -0
-----	-001 0-	1--1 -0
0101 --√	-----	
1001 --√	01-1 --	
1010 0-√	-101 -0√	
-----	10-1 --	
0111 --√	1-01 -0√	
1011 --√	101- 0-	
1101 -0√	1-10 0-	
1110 0-√	-----	
-----	-111 -0√	
1111 -0√	1-11 -0√	
	11-1 -0√	

The prime implicants of f are xz and wz ; those of g are $w'y'z$, $x'y'z$, $wx'y$, and wyz' . The terms that can be shared are $w'xz$ and $wx'z$.

c. We must include all of the don't cares

0000 --	-000 0-	1--0 0-
-----	-----	-----
1000 0-√	10-0 0-√	--11 -0
-----	1-00 0-√	-1-1 --
0011 -0√	-----	-11- 0-
0101 --√	0-11 -0√	1-1- --
0110 0-√	-011 -0√	11-- --
1010 --√	01-1 --√	
1100 --√	-101 --√	
-----	011- 0-√	
0111 --√	-110 0-√	
1011 --√	101- --√	
1101 --√	1-10 --√	
1110 --√	110- --√	
-----	11-0 --√	
1111 --√	-----	
	-111 --√	
	1-11 --√	
	11-1 --√	
	111- --√	

The unshared prime implicant of f is cd ; those of g are $b'c'd'$, ad' , and bc . The shared terms are $a'b'c'd'$, bd , ac , and ab .

d. The tag has three terms

0000 -0-√	00-0 -0-√	0--0 00-
-----	0-00 00-√	-0-0 -0-
0010 -0-√	-000 -0-√	-----
0100 00-√	-----	-01- -00
1000 -0-√	001- -0-	--10 00-
-----	0-10 00-√	10-- -00
0011 ---	-010 -0-√	1-0- -00
0101 0-0√	01-0 00-√	-----
0110 00-√	100- -00√	-1-1 0-0
1001 -00√	10-0 -0-√	1--1 -00
1010 -0-√	1-00 -00√	
1100 --0√	-----	
-----	0-11 0-0	
0111 0-0√	-011 -00√	
1011 -00√	01-1 0-0√	
1101 --0√	-101 0-0√	
1110 00-√	-110 00-√	
-----	10-1 -00√	
1111 --0√	1-01 -00√	
	101- -00√	
	1-10 00-√	
	110- --0	

	-111 0-0√	
	1-11 -00√	
	11-1 --0	

The unshared prime implicants of f are $b'c$, ab' , ac' , and ad ; those of g are $a'bc$ and bd ; those of h are $a'd'$ and cd' . Terms shared by f and g are abc' and abd ; those shared by f and h are $a'b'c$ and $b'd'$; the one shared by all three functions is $a'b'cd$.

5. Repeat Solved Problem 4 using iterated consensus.

a. The maps of f , g , and fg are shown below

	$a\ b$	00	01	11	10
$c\ d$	00	1			1
	01	1	1	1	
	11	1	1		1
	10	1			1

f

	$a\ b$	00	01	11	10
$c\ d$	00	1			1
	01		1	1	
	11			1	1
	10	1			1

g

	$a\ b$	00	01	11	10
$c\ d$	00	1			1
	01		1	1	
	11				1
	10	1			1

fg

We will start by finding all of the prime implicants of the product, fg , and then those that are prime implicants of the individual functions but not of the product. (Of course, we will add the appropriate tag section, – if it is included in the function, 0 if it is not.)

- A – 0 – 0 – –
- B 1 0 1 – – –
- C – 1 0 1 – –
- D 0 0 – – – 0
- E 0 – – 1 – 0
- F 1 1 – 1 0 –
- G 1 – 1 1 0 –

After “completing” the list, it is a good idea to try the consensus of all pairs of terms, in case we missed one. In this case, we did.

$$H \quad - 0 1 - \quad - 0 \quad D \not\subset B$$

Note that in trying the consensus, there is no need to take the consensus of F or G with D or E (or H) since the tag would be 00, indicating that the term is in neither function.

The unshared prime implicants of f are $a'b'$, $a'd$, and $b'c$; those of g are abd and acd . The terms that can be shared are $b'd'$, $ab'c$, and $bc'd$.

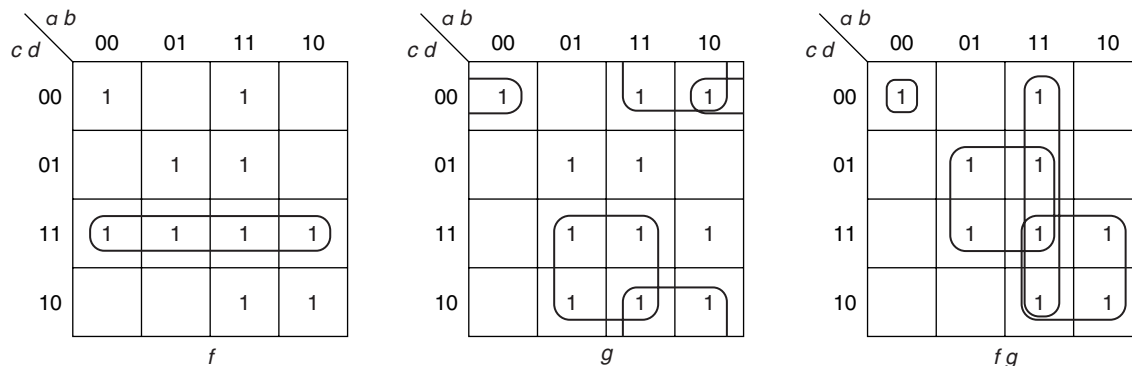
b. We will solve this problem by starting with minterms and finding all of the prime implicants.

1 0 0 0 1 0	A	0 - 0 1 0 -	5 \notin 1 \geq 1
5 0 1 0 1	B	0 1 - 1 - -	7 \notin 5 \geq 7, 5
7 0 1 1 1	C	1 0 1 - 0 -	11 \notin 10 \geq 10
9 1 0 0 1	D	1 0 - 1 - -	11 \notin 9 \geq 11, 9
10 1 0 1 0 0	E	1 1 - 1 - 0	15 \notin 13 \geq 15, 13
11 1 0 1 1	F	1 - 1 0 0 -	C \notin 14 \geq 14
13 1 1 0 1 0	G	- 0 0 1 0 -	D \notin A
14 1 1 1 0 0	H	1 - - 1 - 0	E \notin D \geq E
15 1 1 1 1 0	J	- 1 - 1 - 0	H \notin B

Each of the new terms that is created by consensus is shown; all of the original terms and one of the groups of 2 are included in a larger prime implicant.

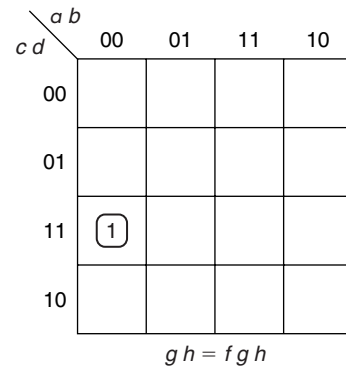
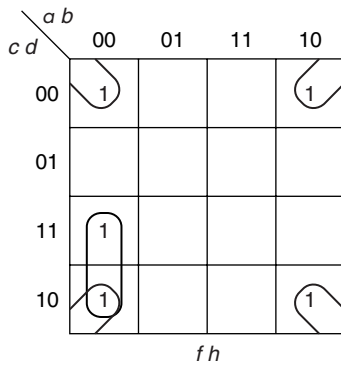
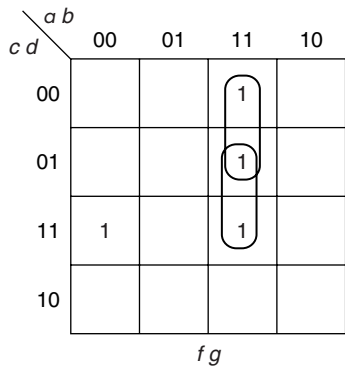
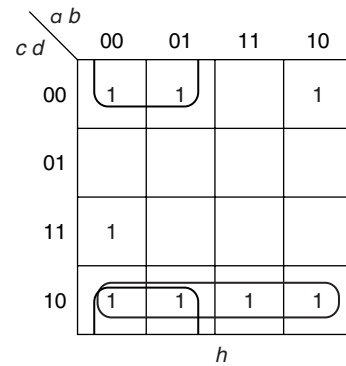
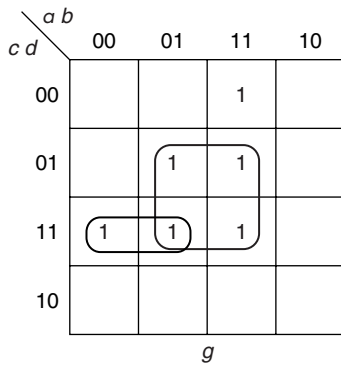
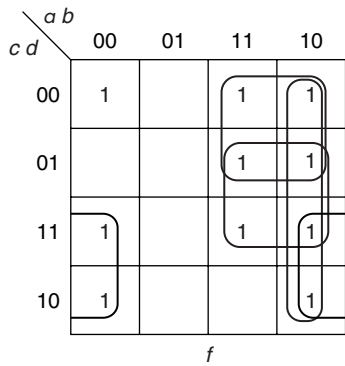
The unshared prime implicants of f are wz and xz ; those of g are $w'y'z$, $wx'y$, wyz' , and $x'y'z$. The product terms that can be shared are $w'xz$ and $wx'z$.

- c. In finding the prime implicants, we must treat all don't cares as 1's. We first map f , g , and fg , converting all X's to 1's to find the prime implicants. (Once again, it is a good idea to check that none have been missed by using the iterated consensus algorithm on the result.)



The unshared prime implicant of f is cd ; those of g are bc , ad' , and $b'c'd'$. The shared terms are $a'b'c'd'$, bd , ab , and ac .

- d. We first map the functions and all the products of pairs of functions. (We do not need a separate map for fgh , since it equals gh .)



We started with the products and listed the circled terms (*A* through *E*) and then listed the prime implicants of the individual functions (*F* through *M*). When we finished, we applied the iterated consensus algorithm and found that we had missed term *N* (circled in green),

- A* 0011 ---
- B* 110- --0
- C* 11-1 --0
- D* 001- -0-
- E* -0-0 -0-
- F* -01- -00
- G* 10-- -00
- H* 1-0- -00
- J* 1--1 -00
- K* -1-1 0-0
- L* --10 00-
- M* 0--0 00-
- N* 0-11 0-0

6. For each of the sets of functions of Solved Problems 4 and 5, find a set of minimum sum of products expressions, corresponding to a two-level AND/OR gate (or NAND gate) system (a. 1 solution, b. 6 solutions, c. 2 solutions, d. 2 solutions).

a. The prime implicant table is

		<i>f</i>											<i>g</i>						
		0	1	2	3	5	7	8	10	11	13	0	2	5	8	10	11	13	15
- 0 - 0*	3	A	X		X				X	X		X	X		X	X			
1 0 1 -	4	B							X	X						X	X		
- 1 0 1*	4	C				X					X			X				X	
0 0 - -	3	D	X	X	X	X													
0 - - 1*	3	E		X		X	X	X											
1 1 - 1	4	F																X	X
1 - 1 1	4	G															X		X
- 0 1 -	3	H			X	X			X	X									

Terms *A* and *C* are essential prime implicants of both *f* and *g*, and *E* is an essential prime implicant of *f*. The reduced table thus becomes

		<i>f</i>	<i>g</i>	
		11	11	15
1 0 1 -	4	B	X	X
0 0 - -	3	D		
1 1 - 1	4	F		X
1 - 1 1	4	G	X	X
- 0 1 -	3	H	X	

Term *G* completes the cover of *g* and term *H* would then be used for *f* since it is less expensive than *B*. The other option, using *B* for both *f* and *g*, and then using either *F* or *G* to cover m_{15} in *g*, would cost an extra input. The solution

thus becomes

$$f = b'd' + bc'd + a'd + b'c$$

$$g = b'd' + bc'd + acd$$

b. The prime implicant table is

		<i>f</i>							<i>g</i>						
			5	7	9	11	13	15	1	5	7	9	10	11	14
0 - 0 1	<i>A</i>	4							X	X					
0 1 - 1*	<i>B</i>	4	X	X						X	X				
1 0 1 -	<i>C</i>	4											X	X	
1 0 - 1	<i>D</i>	4			X	X						X		X	
1 - 1 0*	<i>E</i>	4											X		X
- 0 0 1	<i>F</i>	4							X			X			
1 - - 1	<i>G</i>	3			X	X	X	X							
- 1 - 1	<i>H</i>	3	X	X			X	X							

The two essential prime implicants of *g*, $w'xz$ and wyz' , are shown. The table is then reduced (and the cost of *B* is made 1, since the AND gate was already built).

		<i>f</i>							<i>g</i>		
			5	7	9	11	13	15	1	9	11
0 - 0 1	<i>A</i>	4							X		
0 1 - 1	<i>B</i>	1	X	X							
1 0 1 -	<i>C</i>	4									X
1 0 - 1	<i>D</i>	4			X	X				X	X
- 0 0 1	<i>F</i>	4							X	X	
1 - - 1	<i>G</i>	3			X	X	X	X			
- 1 - 1	<i>H</i>	3	X	X			X	X			

If we are looking for just one of the minimum solutions, we can eliminate rows *A* and *C*, because they are dominated by *D* and *F*, respectively, and cost the same to implement. If we do that, we would choose *D* and *F* to cover function *g*,

leaving the following reduced table:

			f					
			5	7	9	11	13	15
0 1 - 1	B	1	X	X				
1 0 1 -	C	4						
1 0 - 1	D	1			X	X		
1 - - 1	G	3			X	X	X	X
- 1 - 1	H	3	X	X			X	X

We can cover function f with either B and G (at a cost of 4) or D and H (at a cost of 4). (Note that H and J would cover the function also, but the cost would be 6.) Thus, two of the minimum solutions (at a cost of seven gates and 20 inputs) are

$$f = w'xz + wz$$

$$g = w'xz + wyz' + wx'z + x'y'z$$

$$f = wx'z + xz$$

$$g = w'xz + wyz' + wx'z + x'y'z$$

However, there are other equally good solutions, using the two dominated rows we eliminated for the last table. In order to achieve the minimum cost, we must share either B or D between the two functions. If we share B , then we must use H for f (as in the first solution above). But, we can now use one of three solutions for the remainder of g (in addition to the essential prime implicants):

$$A + D$$

$$C + F$$

$$D + F$$

The third is the solution already found. Thus, the three solutions that share $w'xz$ are

$$f_1 = w'xz + wz$$

$$g_1 = w'xz + wyz' + w'y'z + wx'z$$

$$g_2 = w'xz + wyz' + wx'y + x'y'z$$

$$g_3 = w'xz + wyz' + wx'z + x'y'z$$

If we share $wx'z$ (term D), then we can use either A or F to complete the cover of g , giving the two solutions below (one of which we found before), for a total of five equally good solutions.

$$f_2 = wx'z + xz$$

$$g_4 = w'xz + wyz' + wx'z + w'y'z$$

$$g_5 = w'xz + wyz' + wx'z + x'y'z$$

c. This produces the following prime implicant table.

		<i>f</i>				<i>g</i>				
		0	3	5	7	0	5	6	7	8
1 1 - -	<i>A</i>	3								
- 1 - 1*	<i>B</i>	3		X	X		X		X	
1 - 1 -	<i>C</i>	3								
0 0 0 0*	<i>D</i>	5	X			X				
- - 1 1*	<i>E</i>	3		X	X					
- 0 0 0	<i>F</i>	4				X				X
- 1 1 -*	<i>G</i>	3						X	X	
1 - - 0	<i>H</i>	3								X

Notice that prime implicants A and C cover no minterms; they are both groups of four don't cares. The function f is covered by essential prime implicants; only minterms 0 and 8 of g are left. The reduced prime implicant table (for g) becomes

		<i>g</i>	
		0	8
0 0 0 0	<i>D</i>	1	X
- 0 0 0	<i>F</i>	4	X
1 - - 0	<i>H</i>	3	X

There are two equally good solutions. Prime implicant F covers both minterms, but requires one AND gate and four inputs. Prime implicant D was an essential prime implicant of

			<i>g</i>							<i>h</i>						
			3	5	7	12	13	15	0	2	3	4	6	8	10	14
0 0 1 1	<i>A</i>	5	X								X					
1 1 0 -*	<i>B</i>	4				X	X									
1 1 - 1	<i>C</i>	4					X	X								
- 0 - 0*	<i>D</i>	3							X	X				X	X	
0 0 1 -	<i>E</i>	4								X	X					
1 0 - -	<i>F</i>	3														
1 - 0 -	<i>G</i>	3														
1 - - 1	<i>H</i>	3														
- 0 1 -	<i>J</i>	3														
- 1 - 1*	<i>K</i>	3		X	X		X	X								
0 - 1 1	<i>L</i>	4	X		X											
0 - - 0*	<i>M</i>	3							X	X		X	X			
- - 1 0*	<i>N</i>	3								X			X		X	

The table can be reduced and the two halves combined as shown below. Note that all of *g* and *h* other than minterm 3 have already been covered and that the cost of prime implicant *B* has been reduced to 1, since it is an essential prime implicant of *g*.

			<i>f</i>					<i>g</i> <i>h</i>		
			3	9	11	12	13	15	3	3
0 0 1 1	<i>A</i>	5	X						X	X
1 1 0 -	<i>B</i>	1				X	X			
1 1 - 1	<i>C</i>	4					X	X		
0 0 1 -	<i>E</i>	4	X							X
1 0 - -	<i>F</i>	3		X	X					
1 - 0 -	<i>G</i>	3		X		X	X			
1 - - 1	<i>H</i>	3		X	X		X	X		
- 0 1 -	<i>J</i>	3	X		X					
0 - 1 1	<i>L</i>	4							X	

Clearly, prime implicant *A* should be used to cover m_3 in both *g* and *h* (at a cost of $5 + 1 = 6$), since otherwise we would

need both E and L at a cost of 8. For f , we can eliminate prime implicant C , since that row is dominated by row H and costs more. That requires us to choose H to cover m_{15} . Once H is chosen, all that remains to be covered are m_3 and m_{12} , which can be covered by A and B (respectively), each at a cost of 1. (J or G could have been used, but they would cost 3 each.) The final functions are

$$f = b'd' + ad + a'b'cd + abc'$$

$$g = abc' + bd + a'b'cd$$

$$h = b'd' + a'd' + cd' + a'b'cd$$

4.8 EXERCISES⁷

- For each of the following functions, find all prime implicants using the Quine-McCluskey method.
 - $f(a, b, c) = \Sigma m(1, 2, 3, 6, 7)$
 - *b. $g(w, x, y) = \Sigma m(0, 1, 5, 6, 7)$
 - c. $g(w, x, y, z) = \Sigma m(2, 3, 6, 7, 8, 10, 11, 12, 13, 15)$
 - *d. $h(p, q, r, s) = \Sigma m(0, 2, 3, 4, 5, 8, 11, 12, 13, 14, 15)$
 - e. $f(a, b, c, d) = \Sigma m(5, 7, 9, 11, 13, 14) + \Sigma d(2, 6, 10, 12, 15)$
 - *f. $f(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 7, 8, 9, 10, 14) + \Sigma d(3, 13)$
 - g. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 8, 9, 10, 15, 16, 18, 19, 20, 24, 26, 28, 31)$
 - *h. $H(V, W, X, Y, Z) = \Sigma m(0, 1, 2, 3, 5, 7, 10, 11, 14, 15, 16, 18, 24, 25, 28, 29, 31)$
- For the functions of Exercise 1, find all prime implicants using iterated consensus.
- For the functions of Exercises 1 and 2, find all minimum sum of product expression (a. 2 solutions, d. 3 solutions, e. 4 solutions, f. 3 solutions, h. 2 solutions, all others, 1 solution).
- For the following sets of functions, find all product terms that could be used in a minimum two-level AND/OR system using the Quine-McCluskey algorithm.
 - a. $f(a, b, c, d) = \Sigma m(5, 8, 9, 12, 13, 14)$
 $g(a, b, c, d) = \Sigma m(1, 3, 5, 8, 9, 10)$
 - *b. $F(W, X, Y, Z) = \Sigma m(1, 5, 7, 8, 10, 11, 12, 14, 15)$
 $G(W, X, Y, Z) = \Sigma m(0, 1, 4, 6, 7, 8, 12)$

⁷Each of the functions and sets of functions was included in the exercises of Chapter 3. Other exercises from that chapter could also be used here.

- c. $f(a, b, c, d) = \Sigma m(1, 3, 5, 7, 8, 9, 10)$
 $g(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 8, 10, 11, 12)$
 $h(a, b, c, d) = \Sigma m(1, 2, 3, 5, 7, 10, 12, 13, 14, 15)$
- *d. $f(a, b, c, d) = \Sigma m(0, 3, 4, 5, 7, 8, 12, 13, 15)$
 $g(a, b, c, d) = \Sigma m(1, 5, 7, 8, 9, 10, 11, 13, 14, 15)$
 $h(a, b, c, d) = \Sigma m(1, 2, 4, 5, 7, 10, 13, 14, 15)$
5. For each of the sets of functions of Solved Problem 4, find all product terms that could be used in a minimum two-level AND/OR system using iterated consensus.
6. For each of the sets of functions of Solved Problems 4 and 5, find a set of minimum sum of products expressions, corresponding to a two-level AND/OR gate(or NAND gate) system.
- 3 solutions, 8 gates, 25 inputs
 - 8 gates, 23 inputs
 - 2 solutions, 12 gates, 33 inputs
 - 2 solutions, 11 gates, 33 inputs

4.9 CHAPTER 4 TEST (50-MINUTES)⁸

1. For the following function, find all of the prime implicants using
- the Quine-McCluskey method.
 - iterated consensus.

$$f(w, x, y, z) = \Sigma m(0, 2, 3, 6, 8, 12, 15) + \Sigma d(1, 5)$$

2. For the following function,

$$g(a, b, c, d) = \Sigma m(3, 4, 5, 6, 7, 8, 9, 12, 13, 14)$$

we have found the complete list of prime implicants

$$\begin{array}{ll} a'cd & bd' \\ a'b & ac' \\ bc' & \end{array}$$

Find both of the minimum sum of products solutions.

3. For the following set of functions, find all terms that can be used in a minimum two-level AND/OR system using
- the Quine-McCluskey method.
 - iterated consensus.

$$f(w, x, y, z) = \Sigma m(1, 2, 5, 7, 10, 11, 13, 15)$$

$$g(w, x, y, z) = \Sigma m(0, 2, 3, 4, 5, 7, 8, 10, 11, 12)$$

⁸The timing assumes that the student will solve either 1a. or 1b. and either 3a. or 3b.

4. For the following set of functions,

$$f(a, b, c, d) = \sum m(2, 3, 4, 6, 7) + \sum d(0, 1, 14, 15)$$

$$g(a, b, c, d) = \sum m(2, 3, 5, 7, 8, 10, 13) + \sum d(0, 1, 14, 15)$$

We found the possible shared terms: $a'b'$, $a'cd$, bcd , abc .

Other prime implicants of f are $a'd'$, $a'c$, bc .

Other prime implicants of g are $a'd$, $b'd'$, bd , acd' .

Find a set of minimum sum of products expressions, corresponding to a two-level AND/OR gate (or NAND gate) system.

