

Rosen, Discrete Mathematics and Its Applications, 6th edition  
Extra Examples

Section 2.2—Set Operations



— Page references correspond to locations of Extra Examples icons in the textbook.

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**p.125, icon at Example 10**

#1. Prove that the following is true for all sets  $A$ ,  $B$ , and  $C$ : if  $A \cap C \subseteq B \cap C$  and  $A \cap \overline{C} \subseteq B \cap \overline{C}$ , then  $A \subseteq B$ .

**Solution:**

Let  $x \in A$ . We need to show that  $x \in B$ . We will construct a proof by cases, depending on whether  $x \in C$  or  $x \notin C$ .

Case 1:  $x \in C$ . If  $x \in C$ , then by the original hypothesis ( $x \in A$ ) we know that  $x \in A \cap C$ . But it is given that  $A \cap C \subseteq B \cap C$ . Therefore  $x \in B \cap C$ , and hence  $x \in B$ .

Case 2:  $x \notin C$ . Then  $x \in \overline{C}$ . Then by the original hypothesis ( $x \in A$ ) we know that  $x \in A \cap \overline{C}$ . But it is given that  $A \cap \overline{C} \subseteq B \cap \overline{C}$ . Therefore  $x \in B \cap \overline{C}$ , and hence  $x \in B$ .

Therefore, in either case, if  $x \in A$ , then  $x \in B$ . Therefore  $A \subseteq B$ .

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**p.125, icon at Example 10**

#2. Prove that the following is true for all sets  $A$ ,  $B$ , and  $C$ : if  $A \cap C = B \cap C$  and  $A \cup C = B \cup C$ , then  $A = B$ .

**Solution:**

We will show  $A \subseteq B$  and  $B \subseteq A$ .

Proof that  $A \subseteq B$ : Let  $x \in A$ . We need to show that  $x \in B$ . We will give a proof by cases, depending on whether or not  $x \in C$ .

Case 1:  $x \in C$ . In this case  $x \in A \cap C$ . Because  $A \cap C = B \cap C$ , we have  $x \in B \cap C$ , and hence  $x \in B$ .

Case 2:  $x \notin C$ . In this case  $x \in A \cup C$  (because  $x \in A$ ). Because  $A \cup C = B \cup C$ , we have  $x \in B \cup C$ . But  $x \notin C$ . Therefore we must have  $x \in B$ .

Cases 1 and 2 show that if  $x \in A$ , then  $x \in B$ , or  $A \subseteq B$ .

A similar proof can be given to show that  $B \subseteq A$ .

Because  $A \subseteq B$  and  $B \subseteq A$ ,  $A = B$ .

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**p.125, icon at Example 10**

#3. Use logical equivalence to show that  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Solution:**

We begin with  $A \cap (B \cup C)$  and show that this is the same as  $(A \cap B) \cup (A \cap C)$ .

$$\begin{aligned}
A \cap (B \cup C) &= \{x \mid x \in A \wedge x \in B \cup C\} && \text{definition of intersection} \\
&= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} && \text{definition of union} \\
&= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} && \text{distributive law} \\
&= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\} && \text{definition of intersection} \\
&= (A \cap B) \cup (A \cap C) && \text{definition of union}
\end{aligned}$$


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#4. Prove: if  $A \subseteq \overline{B}$ , then  $B \subseteq \overline{A}$ .

**Solution:**

We will use a direct proof. Suppose  $A \subseteq \overline{B}$ . We must show that  $B \subseteq \overline{A}$ . To show that  $B \subseteq \overline{A}$ , assume that  $x \in B$  and show that  $x \in \overline{A}$ .

Suppose  $x \in B$ .  
Therefore  $x \notin \overline{B}$ .  
Therefore  $x \notin \overline{A}$  (because  $A \subseteq \overline{B}$ ).  
Therefore  $x \in \overline{A}$ .

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**p.125, icon at Example 10**

#5. Prove: If  $A \subseteq B$  and  $C \subseteq D$ , then  $A \cap C \subseteq B \cap D$ .

**Solution:**

We assume that  $A \subseteq B$  and  $C \subseteq D$ , and we must show that  $A \cap C \subseteq B \cap D$ . In terms of predicates and quantifiers, the statement we need to prove has the form

$$\forall x (x \in A \cap C \rightarrow x \in B \cap D)$$

where the universe for  $x$  is the universal set  $U$  (any set containing  $A$ ,  $B$ ,  $C$ , and  $D$ ). To show that this statement is true, suppose that  $x \in A \cap C$ . (Note that the only thing we know about  $x$  is that it is an arbitrary element of  $A \cap C$ .) Therefore,  $x \in A$  and  $x \in C$ , by definition of intersection of sets. Therefore,  $x \in B$  and  $x \in D$  (because  $A \subseteq B$  and  $C \subseteq D$ ). This says that  $x \in B \cap D$ . Hence, if  $x$  is any element of  $A \cap C$ , then  $x$  is also an element of  $B \cap D$ .

Therefore,  $A \cap C \subseteq B \cap D$ .

We can write the proof more briefly as:

Let  $x \in A \cap C$ .  
 $\therefore x \in A$  and  $x \in C$ .  
 $\therefore x \in B$  and  $x \in D$ .  
 $\therefore x \in B \cap D$ .

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