

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 2.3—Functions



— Page references correspond to locations of Extra Examples icons in the textbook.

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#1. Determine if the following describes a function with the given domain and codomain.

$f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n)$ is equal to the sum of the digits in n .

Solution:

For each input value n (a nonnegative integer), there is one number that is the sum of the digits of n . Thus, this is a function.

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#2. Determine if each of the following describes a function with the given domain and codomain.

(a) $f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = 7 - n$.

(b) $f: \mathbf{N} \rightarrow \mathbf{Z}$ where $f(n) = 7 - n$.

Solution:

(a) This is not a function with codomain \mathbf{N} because $f(8) = 7 - 8 = -1$, which is not an element of \mathbf{N} .

(b) (Note that we have taken part (a) and changed the codomain.) If we take any natural number and subtract it from 7, we have an integer. Therefore, this is a function.

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#3. Determine if each of the following describes a function with the given domain and codomain.

(a) $f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = \frac{1}{n-\pi}$.

(b) $f: \mathbf{N} \rightarrow \mathbf{R}$ where $f(n) = \frac{1}{n-\pi}$.

(c) $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(n) = \frac{1}{n-\pi}$.

Solution:

(a) This is not a function because $f(0) = -1/\pi$, which is not a natural number.

(b) This is a function because every input integer produces a real number as output. Note that no integer will produce a 0 in the denominator.

(c) This is not a function because $f(\pi)$ is not defined. (It yields a denominator of 0.)

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#4. Determine if the following describes a function with the given domain and codomain.

$$f: \mathbf{N} \rightarrow \mathbf{N} \text{ where } f(n) = \begin{cases} x + 4, & \text{if } n < 7 \\ x^2, & \text{if } n > 11. \end{cases}$$

Solution:

This is not a function because $f(7)$ is not defined (neither case covers the values $n = 7, 8, 9, 10, 11$).

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#5. Determine if the following describes a function with the given domain and codomain.

$$f: \mathbf{N} \rightarrow \mathbf{N} \text{ where } f(n) = \begin{cases} x + 4, & \text{if } n < 7 \\ x^2, & \text{if } n > 4. \end{cases}$$

Solution:

This is not a function because $f(5)$ is equal to both 9 (using the first case) and 25 (using the second case). Note: some programming languages will accept this as a function by using the first applicable case to define the function; in this case the programming language would give $f(5) = 5 + 4 = 9$.

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#1. Let $f: \mathbf{N} \rightarrow \mathbf{Z}$ be defined by the two-part rule $f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$

Determine whether f is one-to-one.

Solution:

Suppose $m \neq n$ are two integers. There are three cases to consider, depending on whether m or n are even or odd.

Case 1: m and n are even. Then $f(m) = m/2$ and $f(n) = n/2$. But $m/2 \neq n/2$ (because $m \neq n$). Therefore $f(m) \neq f(n)$.

Case 2: m and n are odd. Then $f(m) = -(m+1)/2$ and $f(n) = -(n+1)/2$. Because $m \neq n$, $-m \neq -n$. Therefore $-m-1 \neq -n-1$, and $-(m+1)/2 \neq -(n+1)/2$. Therefore $f(m) \neq f(n)$.

Case 3: one of m and n is even and the other is odd. (Say m is even and n is odd.) Therefore $f(m) \geq 0$ and $f(n) < 0$, and hence $f(m) \neq f(n)$.

These are the only three possibilities. Therefore f is one-to-one.

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#1. Let $f: \mathbf{N} \rightarrow \mathbf{Z}$ be defined by the two-part rule $f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ -(n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$

Determine whether f is onto \mathbf{Z} .

Solution:

Let $y \in \mathbf{Z}$. We need to try to find an $n \in \mathbf{N}$ such that $f(n) = y$. There are two cases to consider, depending on whether $y \geq 0$ or $y < 0$.

Case 1: $y \geq 0$. Let $n = 2y$. Because n is even we use the first case in the definition of f . We have $f(2y) = (2y)/2 = y$.

Case 2: $y < 0$. Let $n = -2y - 1$. Then $f(-2y - 1) = -(-2y - 1 + 1)/2 = -(-2y)/2 = y$.

Therefore, for each $y \in \mathbf{Z}$ there is an $n \in \mathbf{N}$ such that $f(n) = y$. Hence f is onto \mathbf{Z} .

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#2. Find a function $f: \mathbf{Z} \rightarrow \mathbf{N}$ that is one-to-one but not onto.

Solution:

We can take $f(n) = \begin{cases} n^2 & n < 0 \\ n^2 + 2 & n \geq 0. \end{cases}$

The function is not onto because there is no n such that $f(n) = 5$ (there is no integer n such that either n^2 or $n^2 + 2$ is equal to 5).

The function is one-to-one. If m and n are nonequal nonnegative integers, then $m^2 + 2$ cannot be equal to $n^2 + 2$. Likewise, if m and n are nonequal negative integers, then m^2 cannot be equal to n^2 . Finally, suppose $m < 0$ and $n \geq 0$. Then the function value m^2 cannot equal the function value $n^2 + 2$ (because no two squares differ by 2).

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#3. Find a function $f: \mathbf{Z} \rightarrow \mathbf{N}$ that is one-to-one and onto.

Solution:

For example, we can take the function $f(n) = \begin{cases} -2n & n \leq 0 \\ 2n - 1 & n > 0. \end{cases}$

The function is one-to-one because no two function values of the form $-2n$ ($n \leq 0$) can be equal, no two function values of the form $2n - 1$ ($n > 0$) can be equal, and no function value of the form $2n$ (which is an even integer) can equal a function value of the form $2n - 1$ (which is an odd integer).

The function is onto. If $n \in \mathbf{N}$ is even, then $f(-n/2) = n$; if $n \in \mathbf{N}$ is odd, then $f((n + 1)/2) = n$.

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#1. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $f(x) = [3x] - 1$. Find $f(S)$ where $S = [1, 3]$.

Solution:

Because we have the expression $[3x]$, we need to examine x -values where x has the form $k/3$ (that is $x = \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3}, \frac{9}{3}$), because at these numbers $f(x) = [3x] - 1$ changes value. We obtain $\{2, 3, 4, 5, 6, 7, 8\}$.

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#2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $f(x) = \lfloor 3x \rfloor - 1$. Find $f^{-1}(S)$ where $S = \{0\}$

Solution:

$f^{-1}(\{0\})$ is equal to the set of all x such that $\lfloor 3x \rfloor - 1 = 0$, or $\lfloor 3x \rfloor = 1$. Any number $x \in [1/3, 2/3)$ will work.

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#3. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $f(x) = \lfloor 3x \rfloor - 1$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $g(x) = x/3$. Find $f \circ g(T)$ where $T = [-3, 3.5]$.

Solution:

We first find $g(T)$. The function g takes each number and divides it by 3. When we divide the numbers in the interval $[-3, 3.5]$ by 3 we have the interval $[-1, \frac{3.5}{3}]$. Now we apply the function f to each of these numbers. Applying $f(x) = \lfloor 3x \rfloor - 1$ to each number in the interval $[-1, \frac{3.5}{3}]$ we have $\{-4, -3, -2, -1, 0, 1, 2\}$.

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#4. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $f(x) = \lfloor 3x \rfloor - 1$.

(a) Find $(f \circ f)(1)$.

(b) Find $(f \circ f)(U)$ where $U = [2, 3]$.

Solution:

(a) $(f \circ f)(1) = f(f(1)) = f(\lfloor 3 \cdot 1 \rfloor - 1) = f(2) = \lfloor 3 \cdot 2 \rfloor - 1 = 5$.

(b) Paying careful attention to the numbers $2, \frac{7}{3}, \frac{8}{3}, 3$ (because there are the values of x at which the graph of f jumps), we have $(f \circ f)(U) = f(f(U)) = f(\{5, 6, 7, 8\}) = \{14, 17, 20, 23\}$.

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#5. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $f(x) = \lfloor 3x \rfloor - 1$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ have the rule $g(x) = x/3$.

(a) Find $(f \circ g)^{-1}(\{2.5\})$.

(b) Find $(f \circ g)^{-1}(\{2\})$.

Solution:

(a) We are looking for numbers x such that $(f \circ g)(x) = 2.5$. But $(f \circ g)(x) = f(g(x))$ and the range of f consists only of integers, and hence cannot include 2.5. Therefore we cannot have such an x , so $(f \circ g)^{-1}(V) = \emptyset$. That is, 2.5 is not an element of the range of $f \circ g$.

(b) Note that $(f \circ g)^{-1}(\{2\}) = g^{-1} \circ f^{-1}(\{2\}) = g^{-1}(f^{-1}(\{2\}))$. But $f^{-1}(\{2\}) = [1, 4/3)$. Therefore $g^{-1}([1, 4/3)) = [3, 4)$. Hence, $(f \circ g)^{-1}(\{2\}) = [3, 4)$.

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#6. Find all solutions to $\lceil x \rceil + \lfloor x \rfloor = 2x$.

Solution:

$\lceil x \rceil + \lfloor x \rfloor$, the sum of two integers, must be an integer. Hence $2x$ must be an integer, which means that either x itself is an integer, or $x + 0.5$ is an integer.

If x is an integer, then $\lceil x \rceil + \lfloor x \rfloor = x + x = 2x$.

If $x + 0.5$ is an integer, then $\lceil x \rceil + \lfloor x \rfloor = (x + 0.5) + (x - 0.5) = 2x$.

Thus, the solution set is $\{x : x \text{ or } x + 0.5 \text{ is an integer}\} = \{\frac{k}{2} : k \text{ an integer}\}$.

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#7. Find all solutions to $\lfloor x \rfloor \lceil x \rceil = x^2$.

Solution:

We first observe that every integer is a solution because in this case $\lfloor x \rfloor = x$ and $\lceil x \rceil = x$.

Now suppose that x is not an integer. Therefore, there is an integer n such that $\lfloor x \rfloor = n$ and $\lceil x \rceil = n + 1$. Hence, in this case the original equation becomes $n(n + 1) = x^2$, or $x = \pm\sqrt{n(n + 1)} = \pm\sqrt{2}, \pm\sqrt{6}, \pm\sqrt{12}, \pm\sqrt{20}$, etc. Therefore, the solutions to the equation $\lfloor x \rfloor \lceil x \rceil = x^2$ are all integers and all numbers of the form $\pm\sqrt{n(n + 1)}$.

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#8. Use the floor and/or ceiling function to find a formula for computing the units' digit of a positive integer n .

Solution:

For example, the units' digit of 547 is 7, and can be obtained as follows: $547 - 540 = 7$. This indicates that the units' digit of n can be obtained by rounding down n to the nearest multiple of 10 and subtracting this rounded-down number from n . The expression $10 \lfloor \frac{n}{10} \rfloor$ rounds n down to the nearest multiple of 10. Hence

$$n - 10 \lfloor \frac{n}{10} \rfloor = \text{units' digit of } n.$$
