

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 3.2—The Growth of Functions

 — Page references correspond to locations of Extra Examples icons in the textbook.

p.181, icon at Example 1

#1. Give a big- O estimate for each of these functions. Use a simple function in the big- O estimate.

(a) $3n + n^3 + 4$.

(b) $1 + 2 + 3 + \cdots + n$.

(c) $\log_{10}(2^n) + 10^{10}n^2$.

Solution:

(a) $3n + n^3 + 4 \leq 3n^3 + n^3 + 4n^3 = 8n^3$ for $n > 1$. Therefore $3n + n^3 + 4$ is $O(n^3)$. (It is also $O(n^4)$, $O(n^5)$, etc.)

(b) We have $1 + 2 + 3 + \cdots + n \leq n + n + n + \cdots + n = n \cdot n$. Therefore $1 + 2 + 3 + \cdots + n$ is $O(n^2)$. (It is also $O(n^3)$, $O(n^4)$, etc.)

(c) $\log_{10}(2^n) + 10^{10}n^2 = n \log_{10} 2 + 10^{10}n^2 \leq (\log_{10} 2 + 10^{10})n^2$ if $n \geq 1$. But $\log_{10} 2 + 10^{10}$ is a constant. Therefore $\log_{10}(2^n) + 10^{10}n^2$ is $O(n^2)$. (It is also $O(n^3)$, $O(n^4)$, etc.)

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#2. Use the definition of big- O to prove that $5x^4 - 37x^3 + 13x - 4 = O(x^4)$

Solution:

We must find integers C and k such that

$$5x^4 - 37x^3 + 13x - 4 \leq C|x^4|$$

for all $x \geq k$. We can proceed as follows:

$$|5x^4 - 37x^3 + 13x - 4| \leq |5x^4 + 37x^3 + 13x + 4| \leq |5x^4 + 37x^4 + 13x^4 + 4x^4| = 59|x^4|,$$

where the first inequality is satisfied if $x \geq 0$ and the second inequality is satisfied if $x \geq 1$. Therefore

$$|5x^4 - 37x^3 + 13x - 4| \leq 59|x^4|$$

if $x \geq 1$, so we have $C = 59$ and $k = 1$.

Note that the solution we have given is by no means the only possible one. Here is a second solution. It makes the value C smaller, but requires us to make the value k larger:

$$|5x^4 - 37x^3 + 13x - 4| \leq |5x^4 + 37x^3 + 13x + 4| \leq |5x^4 + 4x^4 + x^4 + x^4| = 11|x^4|$$

In the first inequality we changed from subtraction to addition of two terms (which is valid if $x \geq 0$). In the second inequality we replaced the term $37x^3$ by $4x^4$ (which is valid if $x \geq 10$), replaced $13x$ by x^4 (which is valid if $x \geq 3$) and replaced 4 by x^4 (which is valid if $x \geq 2$). Therefore,

$$|5x^4 - 37x^3 + 13x - 4| \leq 11|x^4|,$$

if $x \geq 10$. Hence we can use $C = 11$ and $k = 10$.

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#3. Suppose we wish to prove that $f(x) = 2x^2 + 5x + 9$ is big- O of $g(x) = x^2$ and want to use $C = 3$ in the big- O definition. Find a value k such that $|f(x)| \leq 3|g(x)|$ for all $x > k$.

Solution:

We need a value k such that $|2x^2 + 5x + 9| \leq 3x^2$ for all $x > k$. The expression $2x^2 + 5x + 9$ is nonnegative, so we can omit the absolute value bars. But $2x^2 + 5x + 9 \leq 3x^2$ if and only if $5x + 9 \leq x^2$, which is true if and only if $x \geq 7$. Therefore, we can take $k = 7$ (or any larger integer).

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#4. Use the definition of big- O to prove that $\frac{3x^4 - 2x}{5x - 1}$ is $O(x^3)$.

Solution:

We must find positive integers C and k such that

$$\left| \frac{3x^4 - 2x}{5x - 1} \right| \leq C|x^3|$$

for all $x \geq k$. To make the fraction $\left| \frac{3x^4 - 2x}{5x - 1} \right|$ larger, we can do two things: make the numerator larger or make the denominator smaller:

$$\left| \frac{3x^4 - 2x}{5x - 1} \right| \leq \left| \frac{3x^4}{5x - 1} \right| \leq \left| \frac{3x^4}{5x - x} \right| = \left| \frac{3x^4}{4x} \right| = \frac{3}{4}|x^3|.$$

In the first step we made the numerator larger (by not subtracting $2x$) and in the second step we made the denominator smaller by subtracting x , not 1 . Note that the first inequality requires $x \geq 0$ and the second inequality requires $x \geq 1$.

Therefore, if $x > 0$, $\left| \frac{3x^4 - 2x}{5x - 1} \right| \leq \frac{3}{4}|x^3|$, and hence $\left| \frac{3x^4 - 2x}{5x - 1} \right|$ is $O(x^3)$.

p.190, icon at Example 11

#1. Show that the sum of the squares of the first n odd positive integers is of order n^3 .

Solution:

First note that $1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 \leq n(2n + 1)^2 = 4n^3 + 4n^2 + n \leq 9n^3$ for all positive integers n . It follows that the sum of the squares of the first n odd positive integers is $O(n^3)$. Note also

that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 \geq (2\lceil n/2 \rceil + 1)^2 + \cdots + (2n+1)^3 \geq (n - \lceil n/2 \rceil + 1)(2\lceil n/2 \rceil + 1)^2 \geq (n/2)(n+1)^2 \geq (n/2)n^2 = n^3/2$. Consequently, $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \Theta(n^3)$. It follows that the sum of the squares of the first n odd integers is of order n^3 .

p.190, icon at Example 12

#1. Use the definition of big-theta to prove that $7x^2 + 1$ is $\Theta(x^2)$.

Solution:

We have

$$7x^2 \leq 7x^2 + 1 \leq 7x^2 + x^2 \leq 8x^2$$

(where we need $x \geq 1$ to obtain the second inequality).

Therefore,

$$7x^2 \leq 7x^2 + 1 \leq 8x^2 \quad \text{if } x \geq 1.$$

This says that $7x^2 + 1$ is $\Theta(x^2)$.
