

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 5.2—The Pigeonhole Principle



— Page references correspond to locations of Extra Examples icons in the textbook.

p.348, icon at Example 4

#1. Prove that in any group of three positive integers, there are at least two whose sum is even.

Solution:

Consider two pigeonholes, labeled EVEN and ODD. If three positive integers are placed in these pigeonholes, one of the pigeonholes must have at least two integers (say a and b) in it. Thus, a and b are either both even or both odd. In either case, $a + b$ is even.

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#2. If positive integers are chosen at random, what is the minimum number you must have in order to guarantee that two of the chosen numbers are congruent modulo 6.

Solution:

In order for a and b to be congruent modulo 6, we must have $a \bmod 6 = b \bmod 6$. But there are six possible values for $x \bmod 6$: 0, 1, 2, 3, 4, or 5. Therefore seven positive integers must be chosen in order to guarantee that at least two are congruent modulo 6.

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#3. Suppose you have a group of n people ($n \geq 2$). Use the Pigeonhole Principle to prove that there are at least two in the group with the same number of friends in the group.

Solution:

Suppose you have a group of n people, and suppose that no two persons have the same number of friends in the group. There are n numbers of possible friends that a person can have: $0, 1, 2, \dots, n - 1$, where 0 means that the person has no friends in the group, 1 means that the person has exactly one friend in the group, \dots , $n - 1$ means that the person is friends with all other people in the group.

It is impossible for the numbers 0 and $n - 1$ to both occur as “friendship numbers” in a group of n people. To see this, suppose person A has 0 friends and person B has $n - 1$ friends. If the friendship number for B is $n - 1$, then B must be friends with everyone, including A . But this contradicts the fact that the friendship number for A is 0. Therefore, of the n possible friendship numbers, only $n - 1$ friendship numbers are available in any group of n people.

Because there are n people in the group and only $n - 1$ available friendship numbers, by the Pigeonhole Principle at least two people have the same friendship number. That is, at least two people in the group have the same number of friends in the group.

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#4. Prove that in any set of 700 English words, there must be at least two that begin with the same pair of letters (in the same order), for example, STOP and STANDARD.

Solution:

The number of possible pairs of letters that can appear in the first two positions is $26 \cdot 26 = 676$. Thus, by the Pigeonhole Principle, any set of 677 or more words must have at least two words with the same pair of letters at the beginning of the word.

(Note: In reality, the number 700 can be replaced with a much smaller number, because many combinations of letters do not appear as the two beginning letters of a word. For example, there are no English words that begin with NQ, RR, or TZ.)

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#1. Each type of machine part made in a factory is stamped with a code of the form “letter-digit-digit”, where the digits can be repeated. Prove that if 8000 parts are made, then at least four must have the same code stamped on them.

Solution:

The maximum number of possible codes (pigeonholes) is $26 \cdot 10 \cdot 10 = 2600$. But $8000 > 3 \cdot 2600$. Therefore at least four parts must have the same code.

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#2. Each student is classified as a member of one of the following classes: Freshman, Sophomore, Junior, Senior. Find the minimum number of students who must be chosen in order to guarantee that at least eight belong to the same class.

Solution:

The four classes are the pigeonholes. A group of 28 students could have 7 belonging to each class. But if there are 29 students, at least 8 must be members of the same class. Therefore, the minimum number of students who must be chosen is 29.

In other words, we are looking for the minimum number N such that $\lceil \frac{N}{4} \rceil = 8$. The minimum number is 29.

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#3. What is the minimum number of cards that must be drawn from an ordinary deck of cards to guarantee that you have been dealt

- (a) at least three of at least one rank?
- (b) at least three aces?
- (c) the ace of diamonds?

Solution:

- (a) There are thirteen ranks. If you are dealt 26 cards, it is possible that they consist of two cards of each rank — two aces, two twos, two threes, . . . , two queens, two kings. The 27th card dealt must give you a hand with three of some rank. (The pigeonholes are the thirteen ranks and the cards are the pigeons.)
- (b) In the worst-case scenario, you are dealt 48 cards that include no aces. In order to get three aces, you would need to be dealt 51 cards.
- (c) It is possible that the ace of diamonds could be the last card dealt. Thus, you would need to have all 52 cards dealt to obtain the ace of diamonds.
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#4. What is the minimum number of cards that must be drawn from an ordinary deck of cards to guarantee that you have been dealt

- (a) at least three of at least one suit?
(b) at least three clubs?

Solution:

- (a) There are four suits. If you are dealt 8 cards, it is possible that they consist of two cards of each suit. The ninth card dealt must give you three of at least one suit. (The pigeonholes are the four ranks and the cards are the pigeons.)
- (b) In the worst-case scenario, you are dealt 39 cards that include only diamonds, hearts, and spades. In order to get three clubs, you would need to be dealt three additional cards, making 42 cards.
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