

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 7.4—Generating Functions



— Page references correspond to locations of Extra Examples icons in the textbook.

p.484, icon at Example 1

#1. Find a formula for a generating function for $1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \dots$

Solution:

$$1 - x + x^2 - x^3 + x^4 - x^5 + \dots = \frac{1}{1+x}.$$

p.484, icon at Example 1

#2. Find a formula for a generating function for:

(a) $1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, \dots$

(b) $0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, \dots$ (Note that this is the same sequence as in (a), except that it begins with an extra pair of 0's.)

Solution:

$$(a) 1 + x^3 + x^6 + x^9 + x^{12} + \dots = \frac{1}{1-x^3}.$$

(b) The sequence in (b) is the sequence in (a) with two 0's placed at the beginning, thus shifting the sequence in (a) two places to the right. This shift can be accomplished by multiplying the function in (a) by x^2 :

$$\begin{aligned} x^2 + x^5 + x^8 + x^{11} + x^{14} + \dots &= x^2(1 + x^3 + x^6 + x^9 + x^{12} + \dots) \\ &= x^2 \left(\frac{1}{1-x^3} \right) \\ &= \frac{x^2}{1-x^3}. \end{aligned}$$

p.484, icon at Example 1

#3. Find a formula for a generating function for $4, 3, 2, 1, 0, -1, -2, -3, \dots$

Solution:

We need to obtain a formula for $4+3x+2x^2+x^3+0x^4-x^5-2x^6-3x^7-\dots$. First obtain a generating function for $-1, -2, -3, -4, -5, \dots$ and then manipulate this generating function to obtain the desired generating function.

$$\begin{aligned} -1 - x - x^2 - x^3 - x^4 - \dots &= \frac{-1}{1-x} \\ &= (x-1)^{-1}. \end{aligned}$$

Taking the derivative of both sides yields

$$-1 - 2x - 3x^2 - 4x^3 - 5x^4 - \dots = \frac{-1}{(1-x)^2}.$$

Multiplying by x^5 gives

$$-x^5 - 2x^6 - 3x^7 - 4x^8 - \dots = \frac{-x^5}{(1-x)^2},$$

which is a generating function for the sequence $0, 0, 0, 0, 0, -1, -2, -3, -4, \dots$. If we add $4+3x+2x^2+x^3+0x^4$ we will change the five 0's at the beginning of the sequence to $4, 3, 2, 1, 0$. Hence

$$4 + 3x + 2x^2 + x^3 + 0x^4 - x^5 - 2x^6 - 3x^7 - \dots = 4 + 3x + 2x^2 + x^3 + \frac{-x^5}{(1-x)^2}.$$

p.493, icon at Example 16

#1. Solve $a_n = 3a_{n-1} - 2$, $a_0 = 4$, using generating functions.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Beginning with $a_n = 3a_{n-1} - 2$, a formula for $G(x)$ can be obtained, and from this a formula for a_n :

$$a_n = 3a_{n-1} - 2 \quad (\text{recurrence relation to be solved})$$

$$a_n x^n = 3a_{n-1} x^n - 2x^n \quad (\text{multiply by } x^n)$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n - \sum_{n=1}^{\infty} 2x^n \quad (\text{sum from } n = 1 \text{ to } \infty)$$

$$G(x) - a_0 = 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} - 2x \sum_{n=1}^{\infty} x^{n-1} \quad (\text{substitute for } G(x); \text{ rewrite sums})$$

$$G(x) - 4 = 3x \sum_{n=0}^{\infty} a_n x^n - 2x \sum_{n=0}^{\infty} x^n \quad (\text{change indices of summation})$$

$$G(x) - 4 = 3xG(x) - \frac{2x}{1-x} \quad (\text{evaluate sums})$$

$$G(x)(1-3x) = 4 - \frac{2x}{1-x} \quad (\text{combine like terms})$$

$$G(x) = \frac{4-6x}{(1-x)(1-3x)} \quad (\text{solve for } G(x))$$

$$G(x) = \frac{1}{1-x} + \frac{3}{1-3x} \quad (\text{rewrite as sum of two fractions})$$

$$G(x) = \sum_{n=0}^{\infty} x^n + 3 \sum_{n=0}^{\infty} 3^n x^n \quad (\text{rewrite as infinite sums})$$

$$G(x) = \sum_{n=0}^{\infty} (1 + 3^{n+1}) x^n. \quad (\text{combine sums})$$

Hence, a formula for a_n is $a_n = 1 + 3^{n+1}$.